On the oriented chromatic number of Halin graphs

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Abstract

An oriented k-coloring of an oriented graph \( G \) is a mapping \( c : V(G) \rightarrow \{1, 2, \ldots, k\} \) such that (i) if \( xy \in E(G) \) then \( c(x) \neq c(y) \) and (ii) if \( xy, zt \in E(G) \) then \( c(x) = c(t) \Rightarrow c(y) \neq c(z) \). The oriented chromatic number \( \vec{\chi}(G) \) of an oriented graph \( G \) is defined as the smallest \( k \) such that \( G \) admits an oriented \( k \)-coloring. We prove in this paper that every Halin graph has oriented chromatic number at most 9, improving a previous bound proposed by Vignal.

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1 Introduction

We consider oriented graphs, that is digraphs having no loops and no opposite arcs. If \( G \) is an oriented graph, we denote by \( V(G) \) its set of vertices and by \( E(G) \) its set of arcs. If \( xy \) is an arc in \( E(G) \), we say that \( y \) is a successor of \( x \) and that \( x \) is a predecessor of \( y \).

An oriented \( k \)-coloring of an oriented graph \( G \) is a mapping \( c : V(G) \rightarrow \{1, 2, \ldots, k\} \) such that (i) if \( xy \in E(G) \) then \( c(x) \neq c(y) \) and (ii) if \( xy, zt \in E(G) \) then \( c(x) = c(t) \Rightarrow c(y) \neq c(z) \).

With every oriented \( k \)-coloring \( c \) of \( G \) one can associate a digraph \( H_c \), called the colour-graph of \( c \), with vertex set \( V(H_c) = \{c(x) \mid x \in V(G)\} \) and arc set \( E(H_c) = \{c(x)c(y) \mid xy \in E(G)\} \). Thanks to conditions (i) and (ii), \( H_c \) is an oriented graph. The
oriented $k$-coloring $c$ can then be viewed as a homomorphism (that is an arc-preserving vertex mapping) from $G$ to $H_c$. Similarly, every homomorphism of $G$ to an oriented graph $H$ on $k$ vertices can be viewed as a $k$-coloring of $G$, using the vertices of $H$ as colours. The oriented coloring problem has been extensively studied these last years [1, 2, 3, 4, 5, 6].

The oriented chromatic number $\bar{\chi}(G)$ of an oriented graph $G$ is defined as the smallest $k$ such that $G$ admits an oriented $k$-coloring or, equivalently, as the minimum number of vertices in an oriented graph $H$ such that $G$ has a homomorphism to $H$.

From the definition of oriented $k$-coloring, we get that if $xyz$ is a directed 2-path in $G$ ($xy, yz \in E(G)$) then $c(x) \neq c(y) \neq c(z) \neq c(x)$ for every oriented $k$-coloring of $G$. In other words, any two vertices that are linked in $G$ by a directed path of length 1 or 2 must be assigned distinct colours.

Let $H$ be a planar graph and $F$ be its face set. If all the edges on the boundary of some face $F_0$ (whose vertices are all of degree 3) of $F$ are deleted and a tree with at least three leaves is obtained, then the graph $H$ is called a Halin graph. The vertices on $F_0$ are called exterior vertices of $H$, and the remaining vertices are called interior vertices of $H$.

In [8] Vignal proved that every oriented Halin graph has oriented chromatic number at most 11. She conjectured that the oriented chromatic number of every oriented Halin graph is at most 8. We are going to prove that every oriented Halin graph has oriented chromatic number at most 9, which improves the upper bound obtained by Vignal.

We now introduce some oriented graphs that are used as target graphs in the proof of our main result: the tournament $QR_7$ constructed from the non zero quadratic residues of 7, defined by $V(QR_7) = \{0, 1, \ldots, 6\}$ and $E(QR_7) = \{ij \mid j-i \equiv 1, 2 \text{ or } 4 \text{(mod 7)}\}$, the tournament $T_5$ defined by $V(T_5) = \{0, 1, \ldots, 4\}$ and $E(T_5) = \{ij \mid j-i \equiv 1 \text{ or } 2 \text{(mod 5)}\}$, the circuit on three vertices $C_3$ with vertices 1, 2, 3 and arcs 12, 23, 31 and the oriented graph $G_9$ constructed as follows. Let $C_x$, $C_y$ and $C_z$ be three circuits on 3 vertices with vertex sets $\{x_1, x_2, x_3\}$, $\{y_1, y_2, y_3\}$, $\{z_1, z_2, z_3\}$ and with arc sets $\{x_1x_2, x_2x_3, x_3x_1\}$, $\{y_1y_2, y_2y_3, y_3y_1\}$, $\{z_1z_2, z_2z_3, z_3z_1\}$ respectively. The graph $G_9$ is obtained from $C_x$, $C_y$ and $C_z$ by adding all arcs from every vertex of $C_x$ towards all vertices of $C_y$, all arcs from every vertex of $C_z$ towards all vertices of $C_x$, the arc $x_1y_1$ (mod 3) and the arc $y_1x_{i+1}$ (mod 3) for every $i = 1, 2, 3$ (see Figure 1).

**Definition 1** An orientation vector of size $n$ is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\{0, 1\}^n$; let $G$ be an oriented graph and $X = (x_1, x_2, \ldots, x_n)$ be a sequence of pairwise distinct vertices of $G$. A vertex $y$ of $G$ is said to be an $\alpha$-successor of $X$ if for every $i$, $1 \leq i \leq n$, we have $\alpha_i = 1 \implies x_iy \in E(G)$ and $\alpha_i = 0 \implies yx_i \in E(G)$.

**Definition 2** We say that a color-graph $C$ satisfies property $P_k$ for some $k > 0$ if for every oriented $n$-clique subgraph $(c_1, c_2, \ldots, c_n)$ in $C$ with $1 \leq n \leq k$, and every orientation vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of size $n$, there exists a color $c$ in $V(C)$ which is an $\alpha$-successor of $(c_1, c_2, \ldots, c_n)$.

Note that every color-graph satisfying property $P_k$ also satisfies property $P_{k'}$ for every $k' < k$. Then we have:

**Proposition 3** [7] The tournament $QR_7$ satisfies property $P_2$. 

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Figure 1: The graph $G_9$

Proposition 4 [7] Every oriented tree is $C_3$-colorable.

Proposition 5 [7] Every oriented cycle has a homomorphism to the tournament $T_5$.

2 Halin graphs

In this section, we prove that the oriented chromatic number of every oriented Halin graph is at most 9, and that there exists oriented Halin graphs with oriented chromatic number 8.

Theorem 6 For every oriented Halin graph $H$, $\chi(H) \leq 9$.

Proof. Let $H$ be an oriented Halin graph with exterior face $F_0$. We denote by $C(H) = (f_1, f_2, \ldots, f_n)$, $n \geq 3$, the cycle induced by the exterior vertices of $H$, and by $T$ the tree induced by the interior vertices of $H$. Let $p_i$ denote the father of $f_i$ for every $i$, $1 \leq i \leq n$.

Thanks to Proposition 4, $T$ is $C_x$-colorable. Let $h'$ be a homomorphism from $T$ to $C_x$. We are going to extend $h'$ to a homomorphism $h$ from $H$ to an oriented graph $G$ having at most 9 vertices, containing $C_x$ as an induced subgraph.

We consider two cases according to $n$ :

1. $n \leq 6$.

Let $G$ be the oriented graph obtained from $C_x$ with $V(G) = \{x_1, x_2, x_3, y_1, y_2, \ldots, y_n\}$ and $A(G) = A(C_x) \cup \{y_iy_j \mid f_if_j \in A(H), 1 \leq i, j \leq n\} \cup \{h'(p_i)y_i \mid p_if_i \in A(H)\} \cup \{y_jh'(p_j) \mid f_jp_j \in A(H)\}$.

It is easy to check that the mapping $h : V(H) \to V(G)$ defined by

$$
\begin{align*}
  h(x) &= h'(x) \quad \text{for all } x \in V(T), \\
  h(f_i) &= y_i \quad \text{for all } i, 1 \leq i \leq n
\end{align*}
$$

is a homomorphism from $H$ to $G$. Therefore, $\chi(H) \leq 9$.

For $n > 6$, we can construct $G$ by adding $2(n-6)$ vertices $z_1, z_2, \ldots, z_{n-6}$ to $C_x$, and connecting $z_i$ to $y_i$ and $y_{i+1}$ by undirected edges, where $i$ is odd. Then $\chi(H) \leq 9$ by a similar argument.

Proof completed.
2. \( n \geq 7 \).

We consider three sub-cases:

(a) For every \( i, 1 \leq i \leq n, p_i f_i \in A(H) \) or for every \( i, 1 \leq i \leq n, f_i p_i \in A(H) \).

Since \( C(H) \) is a cycle, by Proposition 4 \( C(H) \) is 5-colorable. Let \( g \) be a homomorphism from \( C(H) \) to \( T_5 \). We consider the mapping \( h : V(H) \rightarrow V(G) \), defined by:

\[
h(x) = \begin{cases} 
  h'(x) & \text{if } x \in V(T), \\
  g(x) & \text{if } x \in V(C(H))
\end{cases}
\]

where \( G \) is the oriented graph obtained from \( C_x \) and \( T_5 \) by adding all the arcs from every vertex of \( C_x \) towards every vertex of \( T_5 \), if \( p_i f_i \in A(H) \) for every \( i, 1 \leq i \leq n, \) and all the arcs from every vertex of \( T_5 \) towards every vertex of \( C_x \), if \( f_i p_i \in A(H) \) for every \( i, 1 \leq i \leq n \).

In both cases \( h \) is clearly a homomorphism from \( H \) to \( G \). Therefore, \( \chi(H) \leq |G| = 8 \).

(b) \( H \) contains one of the configurations depicted in Figure 2 (in the configurations (3) and (4) of Figure 2 the edge \( p_i f_i \) can be oriented in any direction).

Since \( C(H) \setminus f_i \) is a path, according to Proposition 4 there exists a homomorphism \( g \) from \( C(H) \setminus f_i \) to the circuit \( C_3 \). We consider the mapping \( h : V(H \setminus f_i) \rightarrow V(G_9) \) defined by:

\[
h(x) = \begin{cases} 
  h'(x) & \text{if } x \in V(T), \\
  f_j & \text{if } j \neq i \text{ and } p_j f_j \in A(H), \\
  y_{g(f_i)} & \text{if } j \neq i \text{ and } f_j p_j \in A(H).
\end{cases}
\]

It is easy to check that \( h \) is a homomorphism from \( H \setminus f_i \) to \( G_9 \). Let \( x_i \) and \( x'_i \) be the unique successor and the unique predecessor of \( h'(p_i) \) in \( C_x \) respectively. By setting \( h(f_i) = x_i \) (respectively \( h(f_i) = x'_i \)) if \( p_i f_i \in A(H) \) (respectively if \( f_i p_i \in A(H) \)) \( h \) can be extended to a homomorphism from \( H \) to \( G_9 \).

(c) \( n \) is even and for every \( i, 1 \leq i \leq n/2, p_{2i-1} f_{2i-1}, f_{2i} p_{2i} \in A(H) \).

Two subcases arise:

i. There exists an \( i \) such that \( f_{i-1} f_i, f_{i+1} f_i \in A(H) \) or \( f_i f_{i-1}, f_i f_{i+1} \in A(H) \).

We can suppose without loss of generality that \( p_i f_i \in A(H) \).
Since \( C(H) \setminus f_i \) is a path, there exists a homomorphism \( g \) of \( C(H) \setminus f_i \) to \( \bar{C}_3 \) such that \( g(f_{i+1}) = 1 \). We consider the mapping \( h : V(H \setminus f_i) \to V(G_9) \) defined by

\[
\begin{align*}
  h(x) &= h'(x) & \text{if } x \in V(T), \\
  h(f_j) &= y_{g(f_j)} & \text{if } j \neq i \text{ and } f_j \in A(H), \\
  h(f_j) &= z_{g(f_j)} & \text{if } j \neq i \text{ and } f_j p_j \in A(H).
\end{align*}
\]

It is easy to check that \( h \) is a homomorphism \( H \setminus f_i \to G_9 \).

We suppose first that \( f_i f_{i-1}, f_i f_{i+1} \in A(H) \). Let \( x_i \) be the unique successor of \( h(p_i) \) in \( C_x \). By setting \( h(f_i) = x_i \), \( h \) can be extended to a homomorphism from \( H \) to \( G_9 \).

We now suppose that \( f_i f_{i-1}, f_i f_{i+1} \in A(H) \). In this case, since we have \( h(f_{i+1}) = z_3 \). If \( h(f_{i-1}) = z_1 \) by setting \( h(f_i) = y_3 \), \( h \) can be extended to a homomorphism from \( H \) to \( G_9 \). If \( h(f_{i-1}) = z_2 \) by setting \( h(f_i) = y_1 \), \( h \) can be extended to a homomorphism from \( H \) to \( G_9 + y_1 z_1 \). If \( h(f_{i-1}) = z_3 \) by setting \( h(f_i) = y_3 \), \( h \) can be extended to a homomorphism from \( H \) to \( G_9 + y_3 z_3 \).

ii. \( C(H) \) is a circuit.

Let us suppose that \( f_i f_{i+1} \in A(H) \) for every \( i, 1 \leq i \leq n \). We consider three subcases according to \( n \):

A. \( n \equiv 0 \pmod{3} \). The mapping \( g : V(C(H)) \to V(\bar{C}_3) \) defined by \( g(f_i) = i \pmod{3} \) for all \( f_i, 1 \leq i \leq n, \) is clearly a homomorphism and it is easy to check that the mapping \( h : H \to G_9 \) defined by

\[
\begin{align*}
  h(x) &= h'(x) & \text{if } x \in V(T), \\
  h(f_i) &= y_{g(f_i)} & \text{if } p_i f_i \in A(H), \\
  h(f_i) &= z_{g(f_i)} & \text{if } f_i p_i \in A(H),
\end{align*}
\]

is a homomorphism.

B. \( n \equiv 1 \pmod{3} \). In this case, the mapping defined by \( g(f_i) = i \pmod{3} \) for all \( f_i, 1 \leq i \leq n-1, \) is a homomorphism from \( C(H) \setminus f_n \) to \( \bar{C}_3 \) and it is easy to check that the mapping \( h \) defined by

\[
\begin{align*}
  h(x) &= h'(x) & \text{if } x \in V(T), \\
  h(f_i) &= y_{g(f_i)} & \text{if } i \neq n \text{ and } p_j f_j \in A(H), \\
  h(f_j) &= z_{g(f_j)} & \text{if } i \neq n \text{ and } f_j p_j \in A(H),
\end{align*}
\]

is a homomorphism from \( H \setminus f_n \) to \( G_9 \) such that \( h(f_1) = y_1 \) and \( h(f_{n-1}) = y_3 \). By setting \( h(f_n) = z_1 \), \( h \) can be extended to a homomorphism from \( H \) to \( G_9 + z_1 y_1 \).

C. \( n \equiv 2 \pmod{3} \). In this case, the mapping \( g \) defined by \( g(f_i) = i \pmod{3} \) for every \( f_i, 1 \leq i \leq n-5, \) is a homomorphism from \( C(H) \setminus \{f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n\} \) to \( \bar{C}_3 \) and it is easy to checked that the mapping \( h \) defined by

\[
\begin{align*}
  h(x) &= h'(x) & \text{if } x \in V(T), \\
  h(f_i) &= y_{g(f_i)} & \text{if } p_j f_j \in A(H) \text{ and } 1 \leq i \leq n-5, \\
  h(f_j) &= z_{g(f_j)} & \text{if } f_j p_j \in A(H) \text{ and } 1 \leq i \leq n-5,
\end{align*}
\]

is a homomorphism from \( H \setminus f_n \) to \( G_9 \) such that \( h(f_1) = y_1 \) and \( h(f_{n-1}) = y_3 \). By setting \( h(f_n) = z_1 \), \( h \) can be extended to a homomorphism from \( H \) to \( G_9 + z_1 y_1 \).
is a homomorphism from $H \setminus \{f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n\}$ to $G_9$ such that $h(f_1) = y_1$ and $h(f_{n-5}) = y_3$. By setting $h(f_{n-4}) = z_1$, $h(f_{n-3}) = y_1$, $h(f_{n-2}) = z_2$, $h(f_{n-1}) = y_2$ and $h(f_n) = z_3$, $h$ can be extended to a homomorphism from $H$ to $G_9 + \{z_1y_1, z_2y_2\}$.

(d) There exists an $i$, $1 \leq i \leq n$, such that $f_{i-1}p_{i-1}$, $p_ifi$, $p_{i+1}f_{i+1} \in A(H)$ or $p_{i-1}fi - 1$, $f_ipi$, $f_{i+1}p_{i+1} \in A(H)$.

We can suppose without loss of generality that for some $i$, $1 \leq i \leq n$, $f_{i-1}p_{i-1}$, $p_ifi$, $p_{i+1}f_{i+1} \in A(H)$ and we get $i = 1$. So, we suppose $f_np_n$, $p_1f_1$, $p_2f_2 \in A(H)$.

Since $C(H) \setminus f_1$ is a path, there exists a homomorphism $g$ from $C(H) \setminus f_1$ to $C_3$ such that $g(f_2) = 1$. We consider the mapping $h : V(H \setminus f_1) \rightarrow G_9$ defined by

$$h(x) = h'(x) \quad \text{if} \ x \in V(T),$$
$$h(f_j) = y_{g(f_j)} \quad \text{if} \ j \neq 1 \text{ and } p_jf_j \in A(H),$$
$$h(f_j) = z_{g(f_j)} \quad \text{if} \ j \neq 1 \text{ and } f_jp_j \in A(H).$$

It is easy to check that $h$ is a homomorphism $H \setminus f_1$ to $G_9$ such that $h(f_2) = y_1$.

Since $f_np_n \in A(H)$, $h(f_n) \in V(C_3)$.

We have four subcases according to the orientation of the edges $f_nf_1$ and $f_1f_2$:

i. $f_nf_1, f_1f_2 \in A(H)$.
   In this case, $H$ contains the configuration (1) of Figure 2.

ii. $f_1f_n, f_2f_1 \in A(H)$.
    If $h(f_n) = z_3$; by setting $h(f_1) = y_2$, $h$ can be extended to a homomorphism from $H$ to $G_9$.
    If $h(f_n) = z_2$; by setting $h(f_1) = y_2$, $h$ can be extended to a homomorphism from $H$ to $G_9 + y_2z_2$.
    If $h(f_n) = z_1$; let $j$, $2 \leq j \leq n - 1$, be the largest integer such that $p_if_i \in A(H)$ for every $i$, $1 \leq i \leq j$. In this case, we can suppose that for every $i$, $2 \leq i \leq j$, $f_if_{i-1} \in A(H)$ (otherwise $H$ contains the configuration (3) of Figure 2). We consider now the edge $f_jf_{j+1}$; if $f_{j+1}f_j \in A(H)$, we have the configuration (1) of Figure 2 (take $f_i = f_j$, $f_{i+1} = f_{j+1}$ and $f_{i-1} = f_{j-1}$), which corresponds to case 2.(b). If $f_jf_{j+1} \in A(H)$, let $h(f_j) = y_k$; by setting $h(f_{j+1}) = z_k$ and $h(f_m) = y_{g(f_m)-1}$ (respectively $h(f_m) = z_{g(f_m)-1}$) if $p_mf_m \in A(H)$ (respectively, if $f_mpp_m \in A(H)$), $j + 2 \leq m \leq n$, we can have $h(f_n) = z_3$. By setting $h(f_1) = y_2$, $h$ can be extended to a homomorphism from $H$ to $G_9 + y_kz_k$.

iii. $f_1f_n, f_1f_2 \in A(H)$.
    If $h(f_n) = z_1$; by setting $h(f_1) = y_3$, $h$ can be extended to a homomorphism from $H$ to $G_9$.
    If $h(f_n) = z_3$; by setting $h(f_1) = y_3$, $h$ can be extended to a homomorphism from $H$ to $G_9 + y_3z_3$.
    If $h(f_n) = z_2$; let $j$, $2 \leq j \leq n - 1$, the largest integer such that $p_if_i \in A(H)$ for every $i$, $1 \leq i \leq j$. We consider two subcases:
   A. There exists an integer $r$, $3 \leq r \leq j$, such that $f_rf_{r-1} \in A(H)$.
      In this case, we have $f_if_{i-1} \in A(H)$ for every $i$, $r+1 \leq i \leq j$ (otherwise $H$ contains the configuration (3) of Figure 2). We consider now the
edge $f_j f_{j+1}$; if $f_{j+1} f_j \in A(H)$, we have the configuration (1) of Figure 2 (take $f_i = f_j$, $f_{i+1} = f_{j-1}$ and $f_{i+1} = f_{j+1}$), which corresponds to case 2.(b). If $f_j f_{j+1} \in A(H)$, let $h(f_j) = y_k$; by setting $h(f_{j+1}) = z_k$ and $h(f_m) = y_g(f_m) - 1$ (respectively $h(f_m) = z_g(f_m) - 1$) if $p_m f_m \in A(H)$ (respectively, if $f_m p_m \in A(H)$), $j + 2 \leq m \leq n$, we get $h(f_n) = z_1$. By setting $h(f_1) = y_3$, $h$ can be extended to a homomorphism from $H$ to $G_9 + y_k z_k$.

B. For all $r$, $2 \leq r \leq j$, $f_{r-1} f_r \in A(H)$.

We consider the edge $f_j f_{j+1}$; if $f_j f_{j+1} \in A(H)$, let $h(f_j) = y_k$; by setting $h(f_{j+1}) = z_k$ and $h(f_m) = y_g(f_m) - 1$ (respectively $h(f_m) = z_g(f_m) - 1$) if $p_m f_m \in A(H)$ (respectively, if $f_m p_m \in A(H)$), $s + 1 \leq m \leq n$ we get $h(f_n) = z_1$. By setting $h(f_1) = y_3$, $h$ can be extended to a homomorphism from $H$ to $G_9 + \{y_k z_k\}$. If $f_s f_{s-1} \in A(H)$, by setting $h(f_m) = y_g(f_m) + 1$ (respectively $h(f_m) = z_g(f_m) + 1$) if $p_m f_m \in A(H)$ (respectively, if $f_m p_m \in A(H)$), $s + 1 \leq m \leq n$ we get $h(f_n) = z_3$. By setting $h(f_1) = y_3$, $h$ can be extended to a homomorphism from $H$ to $G_9 + \{y_k z_k, y_3 z_3\}$.

- There exists an integer $s$, $j + 2 \leq s \leq n - 1$ such that $f_s p_s \in A(H)$.

We consider the edge $f_{j+1} f_{j+2}$. We suppose first that $f_{j+2} f_{j+1} \in A(H)$; in that case, we have the configuration (2) of Figure 2 (take $f_i = f_{j+1}$, $f_{i-1} = f_{j+2}$ and $f_{i+1} = f_j$), which corresponds to case 2.(b). We suppose now that $f_{j+1} f_{j+2} \in A(H)$. We have $f_m f_{m+1} \in A(H)$ for every $m$, $j + 1 \leq m \leq n - 1$ (otherwise, $H$ contains the configuration (4) of Figure 2). In fact, we have in that case: $p_i f_i \in A(H)$ for $i = 1, 2, \ldots, j$, $f_i p_i \in A(H)$ for $i = j+1, \ldots, n$, $f_i f_{i+1} \in A(H)$ for $i \in \{1, \ldots, n-1\} \setminus j$ and $f_{j+1} f_j, f_1 f_n \in A(H)$ (see Figure 3).

We first show that the graph $H' = H \setminus \{f_j\}$ is $QR_7$-colorable. Since the tournament $QR_7$ contains the cycle $C_7$ as a subgraph, the tree $T + \{f_s p_s\}$ is $QR_7$-colorable. Let $h$ be a homomorphism from $T + \{f_n p_n\}$ to $QR_7$. Since every vertex of $QR_7$ has three predecessors and three successors, we can suppose that $h(f_n) \neq h(p_1)$. Since $h(f_n) \neq h(p_1)$, thanks to Proposition 3 there exists a color in $V(QR_7)$ for $f_1$. We now color successively the vertices $f_2, \ldots, f_j$. Every vertex $f_i$, $1 \leq i \leq j$, has two neighbors, $f_{i-1}$ and $p_i$, which are already $QR_7$-colored (they can have the same color). Since $f_{i-1} f_i$ and $p_i f_i \in A(H)$, thanks to Proposition 3, there always exists a color
in $QR_7$ for $f_i$. We now color successively the vertices $f_{n-1}, \ldots, f_{j+2}$. Every vertex $f_i$, $j+2 \leq i \leq n-1$, has two neighbors, $f_{i+1}$ et $p_i$, which are already $QR_7$-colored (they can have the same color). Since $f_if_{i+1}$ and $f_ip_i \in A(H)$, thanks to Proposition 3, there exists a color in $QR_7$ for $f_i$. Hence, there exists a $QR_7$-coloring $h$ of $H'$. By setting $h(f_{j+1}) = c_8$ for an additional color $c_8$, we get $\chi(H) \leq 9$.

iv. $f_1f_n, f_2f_1 \in A(H)$.
   - If $h(f_n) = z_1$, by setting $h(f_1) = y_2$, $h$ can be extended to a homomorphism from $H$ to $G_9$.
   - If $h(f_n) = z_2$, by setting $h(f_1) = y_2$, $h$ can be extended to a homomorphism from $H$ to $G_9 + z_2y_2$.
   - If $h(f_n) = z_3$, let $j$, $2 \leq j \leq n-1$, be the largest integer such that for every $i$, $1 \leq i \leq j$, $p_if_i \in A(H)$. In this case, we can suppose that $f_if_{i-1} \in A(H)$ for all $i$, $2 \leq i \leq j$ (otherwise $H$ contains the configuration (3) of Figure 2). We consider now the edge $f_jf_{j+1}$; if $f_jf_{j+1} \in A(H)$, we have the configuration (1) of Figure 2 (take $f_i = f_j$, $f_{i+1} = f_{j-1}$ and $f_{i-1} = f_{j+1}$), which corresponds to case 2.(b) and if $f_jf_{j+1} \in A(H)$, we have the case 2.(d).iii (take $f_1 = f_j$, $f_2 = f_{j-1}$ and $f_n = f_{j+1}$).

We thus found in all cases a homomorphism from $H$ to some oriented graph with at most 9 vertices. Thus, $\chi(H) \leq 9$.

Concerning the lower bound of the oriented chromatic number of the family of Halin graphs, we have:

**Proposition 7** There exists oriented Halin graphs with oriented chromatic number at least 8.
Figure 4: A Halin graph with oriented chromatic number 8

Proof. We consider the graph $H$ depicted on Figure 4. We show that $\overrightarrow{\chi}(H) = 8$.

Let $c$ be an oriented $k$-coloring of $H$. Let us first notice that for all $i, j, 1 \leq i < j \leq 7$, the two vertices $x_i$ and $x_j$ are linked by a directed path of length at most 2. Thus, for all $i, j, 1 \leq i < j \leq 7$, the vertices $x_i$ and $x_j$ must be colored with two distinct colors. So, we have $k \geq 7$. Let $i = c(x_i), i = 1, 2, \ldots, 7$. If $k = 7$, we necessarily have $c(u) = 7$ (there is a directed path of length at most 2 linking $u$ and each of $x_1, x_2, \ldots, x_6$). Similarly there is a directed path of length at most 2 linking $v$ to each of $x_2, x_3, \ldots, x_7$ and thus $c(v) = 1$. By considering the arcs $vx_7$ and $ux_1$, we get a contradiction and thus $k \geq 8$. ■

References


