The existence of court balanced tournament designs

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Abstract

Let $V$ be a set of $n$ elements. A tournament design, $TD(n, c)$, is a $c$-row array of the $\binom{n}{2}$ pairs of elements from $V$ such that every element appears at most once in each column. A court balanced tournament design, $CBTD(n, c)$, has the added property that every element appears the same number of times in each row. We show that $CBTD(n, c)$ exist for all $n$ and $c$ satisfying $c \mid \binom{n}{2}$ and $c \mid n - 1$ and discuss the application of $CBTD$s to scheduling round robin tournaments fairly with respect to a given number of courts.

1. Introduction

We generalize the definitions [6] of round robin tournament and tournament design as follows. A round robin tournament consists of rounds of matches between a set of teams on a collection of courts such that

- the rounds are played one at a time,
- every pair of teams meets in exactly one match during the tournament,
- each court is used in every round.

Let $V$ be a set of $n$ elements and $c$ a positive integer. We defined a tournament design, $TD(n, c)$, to be a $c$-row array of the $\binom{n}{2}$ distinct unordered pairs of elements from $V$ such that every element of $V$ appears at most once in each column.

If the courts are of unequal attractiveness, this may benefit some teams over others. An example of this is in football where some players prefer grass, and others prefer artificial turf. In order to eliminate this factor from the tournament we require that the schedule be designed such that every team plays exactly the same number of times in each available court.

Define a court balanced tournament design $CBTD(n, c)$, to be a $TD(n, c)$ such that every element of $V$ appears the same number of times in each row. If we let $t$ be the...
number of columns and \( \alpha \) be the replication number of element \( i \) in row \( j \) \((1 \leq i \leq n, 1 \leq j \leq c)\) then the necessary conditions for the existence of a CBTD\((n,c)\) are

\[
ct = \binom{n}{2},
\]

\(1 \leq c \leq \left\lfloor \frac{n}{2} \right\rfloor, \tag{2}\)

\(c\alpha = n - 1, \tag{3}\)

with the third condition being a result of the court balance property. Figure 1 is an example of a CBTD\((10,3)\).

The following bounds on \( t \) and \( \alpha \) can be easily deduced.

\[
n - 1 + (n \mod 2) \leq t \leq \binom{n}{2}, \quad 1 \leq \alpha \leq n - 1.
\]

If \( c = 1 \) the design is trivial; simply write out all \( \binom{n}{2} \) unordered pairs in one row. We will therefore assume that \( c > 1 \).

2. Balanced tournament designs and odd balanced tournament designs

Although when \( n = 2c \), condition (3) is not satisfied, this case has been well studied. The optimum balance that can be achieved in this case is when every element of \( V \) appears twice in \( c - 1 \) rows and once in the remaining row. The design is then called a balanced tournament design of side \( c \), BTD\((c)\). BTDs were first investigated in \([2,4]\) and their existence was later settled by Schellenberg, van Ress and Vanstone \([6]\) who showed that a BTD\((c)\) exists for all \( c \neq 2 \).

In the next section, our constructions make use of the fact that there exists a CBTD\((2c+1,c)\) for all positive \( c \). These designs are more commonly known as odd balanced tournament designs of side \( c \), OBTD\((c)\), and have also been studied \([5]\).

Let \( V \) be a set of \( 2c+1 \) elements. Then an OBTD\((c)\) defined on \( V \) is a \( c \times (2c+1) \) array of the \( \binom{2c+1}{2} \) distinct unordered pairs of elements from \( V \) such that

- every column of the array is a near resolution class,
- every element of \( V \) appears at most twice in each row.
Since there are $2(2c+1)$ elements of $V$ in a row, the second condition implies every element of $V$ appears exactly twice in each row of the array.

Given a positive integer $c$ it is not hard to construct an $OBTD(c)$ by expanding a patterned starter [8]. Let $S$ be an additive abelian group of odd cardinality. A patterned starter for $S$ is the set of pairs $\{ [x, x \in S\}$ (we use a space-saving vertical notation in order to simplify the diagrams). To construct an $OBTD(c)$, place a patterned starter for $\mathbb{Z}_{2c+1}$ in the first column. If cell $(i, 1)$ contains the pair $[z]$ then the remaining cells of row $i$ are filled by placing the pair 

$$\begin{bmatrix}
  a + j - 1 \\ b + j - 1
\end{bmatrix} \mod (2c+1)$$

in cell $(i, j)$. We will assume that the OBTDs we use are derived from patterned starters. Fig. 2 demonstrates the patterned starter construction for an $OBTD(3)$. In the diagrams we omit the $[ ]$, and just write $\xi$.

Note that if a patterned starter is used to construct an $OBTD$ there is a further element of balance within the design. If the pairs are considered to be ordered, then in each row of the $OBTD$, every element appears once in the first coordinate of some pair and once in the second coordinate of another pair.

3. Court balanced tournament designs

We will show that the necessary conditions for the existence of a $CBTD(n, c)$ are also sufficient. The proof will be split into two parts based on the parity of $n$. For the case $n$ odd the following lemma will be required.

**Lemma 3.1.** If $n$ is odd then $\alpha$ is even.

**Proof.** From conditions (1) and (3) we have

$$2t = \frac{n(n-1)}{c} = n\alpha.$$

Thus $\alpha$ is even since $n$ is odd. $\square$
**Theorem 3.2.** Let \( n \) and \( c \) be positive integers satisfying the necessary conditions (1)–(3), with \( n \) odd. Then there exists a CBTD\((n, c)\).

**Proof.** Let \( V \) be a set of \( n = 2u + 1 \) elements. Construct \( A \), an OBTD\((u)\), by expanding a patterned starter for \( Z_{2u+1} \).

By Lemma 3.1, \( x \) is even, say, \( 2s \). Thus
\[
\frac{n-1}{2} = \frac{cx}{2} = cs.
\]

Now construct the \( c \times ns \) array \( B \), pictured below, by concatenating the rows of \( A \), in order, \( s \) at a time.

<table>
<thead>
<tr>
<th>1</th>
<th>2cs+1</th>
<th>s</th>
<th>s-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2cs</td>
<td>2cs-1</td>
<td>(2c-1)s+1</td>
<td>(2c-1)s</td>
</tr>
<tr>
<td>s+1</td>
<td>s</td>
<td>2s</td>
<td>2s-1</td>
</tr>
<tr>
<td>(2c-1)s</td>
<td>(2c-1)s-1</td>
<td>(2c-2)s+1</td>
<td>(2c-2)s</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>(i-1)s+1</td>
<td>(i-1)s</td>
<td>is</td>
<td>is-1</td>
</tr>
<tr>
<td>(2c-i+1)s</td>
<td>(2c-i+1)s-1</td>
<td>(2c-i)s+1</td>
<td>(2c-i)s</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>(c-1)s+1</td>
<td>(c-1)s</td>
<td>cs</td>
<td>cs-1</td>
</tr>
<tr>
<td>(c+1)s</td>
<td>(c+1)s-1</td>
<td>cs+1</td>
<td>cs</td>
</tr>
</tbody>
</table>

1st \( s \) rows
2nd \( s \) rows
ith \( s \) rows
last \( s \) rows

We will show that \( B \) is a CBTD\((n, c)\). Let \( v \in V \). Every column of \( B \) consists of some subset of a column of \( A \) and \( v \) appears at most once in a given column of \( A \). Thus every element of \( V \) appears at most once in each column of \( B \). Every row of \( B \) is some \( s \)-subset of the rows of \( A \) and each row of \( A \) contains \( v \) exactly twice. Therefore, each row of \( B \) contains \( v \) exactly \( 2s = x \) times. \( \square \)

**Theorem 3.3.** Let \( n \) and \( c \) be positive integers satisfying the necessary conditions (1)–(3), with \( n \) even, \( c \neq 3 \). Then there exists a CBTD\((n, c)\).

**Proof.** We will work on the set \( Z_n \otimes Z_c \), denoting the element \((u, s)\) by \( u_s \) for convenience. Let \( V = \{ u_s : 1 \leq u \leq c, 1 \leq s \leq c \} \) and \( V_\infty = V \cup \{ \infty \} \). Then \( |V_\infty| = cx + 1 = n \).

**Step 1.** We will construct a \( c \times (c(\frac{1}{2}) + 1) \) array \( G \), satisfying the following properties:

1. every unordered pair of the form \( \{u_s, v_t, u \neq v\} \) appears in exactly one cell of \( G \);
2. every unordered pair of the form \( \{u_s, v_t : u_t \in V, u \neq v\} \) appears in exactly one cell of \( G \);
3. every element of \( V_\infty \) appears at most once in each column of \( G \);
4. every element of the form \( \{u_s : u_t \in V\} \) appears \( \alpha \) times in row \( s \) and \( \alpha - 1 \) times in each of the remaining rows;
5. \( \infty \) appears exactly \( \alpha \) times in each row of \( G \).
Since \( n \) is even, condition (3) implies \( \alpha \) must be odd. \( \alpha = 1 \) only occurs in the trivial case of a CBTD(2, 1). So we may assume \( \alpha > 1 \). Construct an OBTD((\( \alpha - 1 \))/2) using a patterned starter and consider the pairs to be ordered. Concatenate the rows of an OBTD((\( \alpha - 1 \))/2), in order, to get the \( 1 \times (\frac{\alpha}{2}) \) array

\[
C = \begin{bmatrix}
1 & 2 & \ldots & \alpha & \frac{\alpha-1}{2} & \frac{\alpha+1}{2} & \ldots & \frac{\alpha-3}{2} \\
\alpha-1 & \alpha & \ldots & \alpha-2 & \frac{\alpha+1}{2} & \frac{\alpha+3}{2} & \ldots & \frac{\alpha-1}{2}
\end{bmatrix}
\]

Since \( n \) is even, condition (3) implies \( c \) must be odd and since \( c \neq 3 \) we can use the following theorem.

**Theorem 3.4.** For all positive integers \( c \neq 2, 3, 6 \) there exists two mutually orthogonal idempotent Latin squares of side \( c \) [7, 9].

Superimpose two mutually orthogonal idempotent Latin squares of side \( c \). Then the resulting \( c \times c \) array \( D \) contains each of the ordered pairs \( \{i, j\} \) (1 ≤ \( i, j \) ≤ \( c \)) exactly once. To obtain \( D \otimes C \), in every cell of \( D \), replace the ordered pair \( \{i, j\} \) with the array \( C(i, j) \). where \( C(i, j) \) denotes the array \( C \) with the first component of every pair subscripted by \( i \) and the second by \( j \). \( D \otimes C \) is now a \( c \times c \) array with every unordered pair of the form \( \{u, v\} : u, v \in V, u \neq v \} \) appearing in exactly one cell. Note that \( D \otimes C \) is just the Kronecker product of \( D \) and \( C \), where

\[
\begin{bmatrix}
S \\
I
\end{bmatrix}
\begin{bmatrix}
u_s \\
v_i
\end{bmatrix} =
\begin{bmatrix}
u_s \\
v_i
\end{bmatrix}.
\]

For 1 ≤ \( i \leq c \), define

\[
E(i) = \begin{bmatrix}
\infty & \infty & \infty & \ldots & \infty \\
\alpha-1_i & \alpha_i & 1_i & \ldots & \alpha-2_i
\end{bmatrix}
\]

and let \( E = \begin{bmatrix}
E(1) \\
E(2) \\
\vdots \\
E(c)
\end{bmatrix} \).

\( E \) is a \( c \times \alpha \) array with every unordered pair of the form \( \{u, v\} : u, v \in V \} \) appearing in exactly one cell. Form the \( c \times (c(\frac{\alpha}{2}) + \alpha) \) array \( F = (D \otimes C \mid E) \). \( F \) is pictured in Fig. 3.

![Fig. 3. The array F.](image-url)
Fig. 4. The array $G$.

$F$ satisfies almost all of the desired properties outlined at the beginning of step 1. Property 1 follows from the construction of $D \otimes C$ and properties 2 and 5 from the construction of $E$. $D$ contains $x - 1$ appearances of every element from $V$ in each of its rows and $E$ contains one appearance of $u_s$ in row $s$. Thus $F$ satisfies property 4. Property 3 is not satisfied since the element $\infty$ always appears in the last $x$ columns of each row of $F$. We now show how to rearrange the pairs in $F$ so that property 3 is satisfied.

For $1 \leq i \leq c$, split $C(i, i)$ into two subarrays. Let $C'(i, i)$ denote the first $x$ pairs of $C(i, i)$ and $C''(i, i)$ the remaining pairs. So that

$$C(i, i) = (C'(i, i) \cup C''(i, i)).$$

Construct the array $G$ by exchanging the positions of $C'(i, i)$ and $E(i)$ for $1 \leq i \leq c$ in $F$. $G$ is pictured in Fig. 4. The idempotent $MOLS(c)$ guarantee that, after this exchange, $\infty$ will appear at most once in each column of $F$. $G$ now satisfies property 3 as well as the other stated properties.

Step 2. We will construct a $c \times x((c - 1)/2)$ array $J$, satisfying the following properties:

1. every unordered pair of the form $\{u_1, u_2 \in V\}$ appears in exactly one cell of $J$,
2. every element of $V_x$ appears at most once in each column of $J$,
3. every element of the form $\{u_s, u_t \in V\}$ does not appear in row $s$ and appears exactly once in each of the remaining rows,
4. $\infty$ appears nowhere.

One important characteristic of an $OBTD(n)$ is that the unions of the edges in each row are 2-factors of $K_{2n+1}$. The following result is immediate.

**Theorem 3.5.** Every row of an $OBTD$ is a 3-edge-colourable graph.
an example of the 3-edge-colouring on the first column shown by varying the font type.

\[
\begin{array}{cccccc}
2 & 3 & c+1 \\
c & c-1 & \cdots & \frac{c+3}{2} \\
3 & 4 & \cdots & \frac{c+5}{2} \\
1 & c & \cdots & \frac{c+7}{2} \\
4 & 5 & \cdots & \frac{c+9}{2} \\
2 & 1 & \cdots & \frac{c+11}{2} \\
   &   &   & \vdots \\
c-1 & c & \cdots & \frac{c-5}{2} \\
c-3 & c-4 & \cdots & \frac{c-3}{2} \\
c & 1 & \cdots & \frac{c-1}{2} \\
c-2 & c-3 & \cdots & \frac{c+1}{2} \\
1 & 2 & \cdots & \frac{c+1}{2} \\
\end{array}
\]

Corresponding to the set of colours \{1, 2, 3\}, define the following three arrays of pairs. The latter two arrays are cyclic permutations of the first by one and two pairs, respectively.

\[
K^1 = \begin{array}{ccc}
1 & 2 & \alpha \\
2 & 1 & \alpha \\
\end{array} \quad K^2 = \begin{array}{ccc}
2 & 3 & 1 \\
2 & 3 & \cdots & 1 \\
\end{array} \quad K^3 = \begin{array}{ccc}
3 & 4 & 2 \\
3 & 4 & \cdots & 2 \\
\end{array}
\]

For \(1 \leq p \leq 3\), let \(K^p(i, j)\) denote the \(1 \times \alpha\) array of unordered pairs given by \(K^p\) with the first component of every pair subscripted by \(i\), and the second by \(j\). To obtain \(J = H \otimes K^p\), for every cell of \(H\), replace the unordered pair \([j]\) with \(K^p(i, j)\) if \([j]\) is of colour \(p\). \(J\) is now a \(c \times \alpha(c-1)/2\) array of distinct unordered pairs. Fig. 5 shows \(J\) with the superscripts on the first column shown.

We now check that \(J\) satisfies all the properties stated at the beginning of step 2. Properties 1 and 4 are clear. Property 3 follows since column \(s\) of the \(OBTD\) has deficient element \(s\). By the \(OBTD\) construction we have that in any column of \(J\) the subscripts on the pairs correspond to a 2-factor of the complete graph on \(c\) vertices. The 3-edge-colouring guarantees that the two appearances of each element will correspond to different colours. Thus, in any column, the two cells corresponding to these appearances will not contain the same \(K^p\). Therefore, no element will appear twice in the same column and property 2 is satisfied.

**Step 3.** From the \(c \times (c+\alpha + \alpha(c-1)/2)\) array \(L = (G|J)\).

We check that \(L\) is a \(CBTD(n, c)\). By properties 1 and 2 of step 1 and property 1 of step 2, every unordered pair appears in exactly one cell of \(L\). Property 3 of step 1 and
Fig. 5. The array $J$. 

property 2 of step 2 guarantee that the column condition is satisfied, and properties 4 and 5 of step 1 and property 4 of step 2 guarantee that the row condition is satisfied.

Theorem 3.6. Let $n$ and $c$ be positive integers satisfying the necessary conditions 1–3, with $n$ even, $c = 3$. Then there exists a CBTD($n, c$).

Proof. Since $n$ is even, condition 3 implies $x$ must be odd, $2s + 1$ say. So

$$n = 3x + 1 = 3(2s + 1) + 1 = 6s + 4.$$ 

We will show that given a CBTD($n, 3$) with $n$ even, we can recursively construct a CBTD($n+6, 3$). For the base case we will use the CBTD(10, 3) given in Fig. 1.

Partition $V = \{1, \ldots, n\}$ into $A \cup B \cup C \cup \{z\}$, where

$$A = \{a_1, a_2, \ldots, a_{2s+1}\}, \quad B = \{b_1, b_2, \ldots, b_{2s+1}\}, \quad C = \{c_1, c_2, \ldots, c_{2s+1}\}$$

and let the set of points to be added be $W = \{\infty_i: 1 \leq i \leq 6\}$.

Let

$$M = \begin{bmatrix}
A & A & B & B & C & C \\
\infty_1 & \infty_2 & \infty_3 & \infty_4 & \infty_5 & \infty_6 \\
C & C & A & A & B & B \\
\infty_3 & \infty_4 & \infty_5 & \infty_6 & \infty_1 & \infty_2 \\
B & B & C & C & A & A \\
\infty_5 & \infty_6 & \infty_1 & \infty_2 & \infty_3 & \infty_4
\end{bmatrix},$$

where $[\frac{a}{b}]$ denotes the $1 \times (2s+1)$ array $\begin{bmatrix}
a_1 & a_2 & \ldots & a_{2s+1}
\end{bmatrix}$. 

Using a patterned starter construct $N$, an OBTD(3) defined on the set of points $W \cup \{z\}$. Then if $L$ is the CBTD($n$, 3), the CBTD($n+6$, 3), $P$ is given $(L|M|N)$, where $(M|N)$ is

\[
\begin{array}{cccccccccccc}
A & A & B & B & C & C & \infty_{1} & \infty_{2} & \infty_{3} & \infty_{4} & \infty_{5} & \infty_{6} & \infty_{7} \\
\infty_{1} & \infty_{2} & \infty_{3} & \infty_{4} & \infty_{5} & \infty_{6} & \infty_{7} & \infty_{2} & \infty_{3} & \infty_{4} & \infty_{5} & \infty_{6} \\
C & A & A & B & B & \infty_{1} & \infty_{2} & \infty_{3} & \infty_{4} & \infty_{5} & \infty_{6} & \infty_{7} \\
\infty_{3} & \infty_{4} & \infty_{5} & \infty_{6} & \infty_{2} & \infty_{3} & \infty_{4} & \infty_{5} & \infty_{6} & \infty_{2} & \infty_{3} \\
B & B & C & C & A & A & \infty_{1} & \infty_{2} & \infty_{3} & \infty_{4} & \infty_{5} & \infty_{6} \\
\infty_{5} & \infty_{6} & \infty_{1} & \infty_{2} & \infty_{3} & \infty_{4} & \infty_{5} & \infty_{2} & \infty_{3} & \infty_{4} \\
\end{array}
\]

Since $L$ is a CBTD($n$, 3), to verify that $P$ is a CBTD($n+6$, 3) it is sufficient to check that the following properties hold:

1. every unordered pair of the form $\{[x_i]: v \in V, \infty_i \in W\}$ appears in exactly one cell of $(M|N)$,
2. every unordered pair of the form $\{[\infty_i]: \infty_i, \infty_j \in W, i \neq j\}$ appears in exactly one cell of $(M|N)$,
3. every element of $V \cup W$ appears at most once in each column of $(M|N)$,
4. every element of $V \cup W$ appears the same number of times in each row of $P$.

$M$ consists of every unordered pair of the form

\[
\left\{ \left[ \begin{array}{c}
v \\
\infty_i \\
\end{array} \right] : v \in V \setminus \{z\}, \infty_i \in W \right\}
\]

exactly once and $N$ consists of every unordered pair of the forms

\[
\left\{ \left[ \begin{array}{c}
z \\
\infty_i \\
\end{array} \right] : \infty_i \in W \right\} \quad \text{and} \quad \left\{ \left[ \begin{array}{c}
\infty_i \\
\infty_j \\
\end{array} \right] : \infty_i, \infty_j \in W, i \neq j \right\}
\]

exactly once. Thus, the first two properties are satisfied. The third property is satisfied since an OBTD has no repeated elements in its columns and the columns of $M$ can readily be seen to have no repeated elements. Let $v \in V \setminus \{z\}$. Then $v$ appears $(n-1)/3$ times in each row of $L$, 2 times in each row of $M$, and does not appear in $N$. $z$ appears $(n-1)/3$ times in each row of $L$, nowhere in $M$, and 2 times in each row of $N$. Let $\infty_i \in W$. Then $\infty_i$ appears nowhere in $L$, 2s+1 = $(n-1)/3$ times in each row of $M$, and 2 times in each row of $N$. Thus, every element of $V \cup W$ appear $(n-1)/3 + 2$ times in each row of $P$ so that the fourth property holds.

4. Conclusion

In this paper we have shown the following theorem.

Theorem 4.1. A CBTD($n$, c) exists for all $n$ and $c$ satisfying the necessary conditions.

We will conclude with a discussion of a question raised by Grieg [3]. Grieg asked the following question.
Assuming the necessary conditions are satisfied, can every tournament design be transformed into a court balanced tournament design by some permutation of the pairs within their respective columns?

Using a computer search, it was found that, by using the above method of permuting pairs, it is not possible to transform the \( TD(9,2) \), shown in Fig. 6, into a \( CBTD(9,2) \).

**Note added in proof.** Dinitz has pointed out that Theorem 3.6 holds for all odd integers \( c \) \[1\].

### References


