where \( \mathbf{v}_{k+1} \) is the right singular vector of \( \mathbf{G}(k_1, k_2) \) associated with the \((I + 1)\)th singular value.

The choice of \( k_1 \) and \( k_2 \) may affect the performance of the algorithm. However, it is not difficult to combine the cyclic statistics for some or all (distinct) \( k_1 \)'s and \( k_2 \)'s. In doing so, the criterion in (51) can be modified by replacing \( \mathbf{G}(k_1, k_2) \) by

\[
\mathbf{G} = [\mathbf{G}'(0, 1, \ldots, \mathbf{G}'(0, T - 1), \mathbf{G}'(1, 2), \ldots, \mathbf{G}'(1, T - 1), \ldots, \mathbf{G}'(T - 2, T - 1)]^T.
\]

(53)

Such modification may improve the performance.

V. CONCLUDING REMARKS

We established several different necessary and sufficient conditions for the identifiability of a possibly nonminimum phase channel from its output cyclic autocorrelation functions. In comparison to the time-domain approach presented earlier in [12], the frequency-domain approach via the channel identification problem gives more insight into the issue of channel identifiability. It also provides the basis for new channel identification algorithms.

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solved by standard recursive algorithms which are described in the literature for Gauss–Markov processes (GMP’s) [2]. Furthermore, the solutions of the estimation problems derived for pinned-to-zero boundary conditions could be extended to solve the corresponding problems for general Dirichlet boundary conditions. Some preliminary results about these more general problems are also reported in [5].

The solution for the fixed-interval smoothing (often considered as the natural problem for RGP’s) has been given in [1, 3]. However, to the authors’ knowledge no explicit formulas are available for its application. Regarding this point, we observe that in some real-time applications the delay associated to the fixed-interval smoother cannot be tolerated so that the filter may constitute an effective alternative if the resulting accuracy loss is small enough.

The aim of this correspondence is then to give recursive expressions for the filter, the smoothers, and their mean square performance in a unified approach based on the innovations theory.

II. THE FILTER

In order to derive the minimum mean square error (MMSE) filtering formulas for the pinned RGP’s, let us denote by \(Y^k = \{y(k), y(k+1), \ldots, y(k+m)\}\) the observed sequence until step \(k\); by \(F_k \triangleq Y^k\), \(k \geq M + 1\), the \(\sigma\)-algebra generated by \(Y^k\) (with \(F_M \triangleq \{\emptyset, \Omega\}\)); by \(\hat{x}(k \mid k-m) = E\{X(k) \mid F_{k-m}\}\), the filtered sequence (for \(m = 0\)) and the one-step predictions sequence (for \(m = 1\)); by \(S(k \mid k-m)\), with \(m = 0, 1\), their corresponding error covariance matrices.

On the basis of the observation model in (2) and using the well-known Martingale Difference representation theorem, the filter equation is given by

\[
\hat{x}(k \mid k) = \hat{x}(k \mid k-1) + G(k)[y(k) - \Gamma(k)\hat{x}(k \mid k-1)], \quad M + 1 \leq k \leq N - 1.
\]

From the assumptions of Section I it follows that the gains sequence \(G(k)\) in (3) is necessarily \(F_k\)-predictable, so that it can be expressed as

\[
G(k) = [S(k \mid k-1)\Gamma(k)][\Gamma(k)S(k \mid k-1)\Gamma(k) + R(k)]^{-1}, \quad M + 1 \leq k \leq N - 1.
\]

Starting from the integral form of the difference model in (1) (reported, for example, in [1, eq. (3.13)]) and on the basis of (2), after some algebra the one-step predictions sequence in (3) can be recursively calculated as

\[
\hat{x}(k \mid k-1) = M_0^{-1}(k)[M_0^T(k-1) + M_A(k)H(k-1, N)]^{-1}\hat{x}(k-1 \mid k-1), \quad M + 1 \leq k \leq N - 1
\]

where

\[
H(k, N) \triangleq G(k+1, k-1, N)M_0^T(k-1), \quad M + 1 \leq k \leq N - 1
\]

and \(\{G(r, s; k-1, N), (k-1) \leq r \leq N, k \leq s \leq N - 1\}\) is the Green’s function associated to (1) over the subinterval \([k-1, N]\) of the whole assigned index-space \([M, N]\) (it is defined here according to [1, eq. (3.10)]). It is worthwhile observing that the recursive computation of the one-step prediction sequence in (5) requires the use of the integral form of the model in (1) (and then, the use of the Green’s functions of (6)), so that the effect of the assigned boundary values is taken into account in the solution of the problem. This is a direct consequence of the noncausal structure of the model in (1).

In order to express the gains sequence of (4) in a recursive form, first of all the second-order statistics of the estimator must be calculated. Regarding the covariance matrix of the one-step prediction error, from (1)-(5) and from the bi-orthogonality property mentioned in Section I we obtain

\[
S(k \mid k-1)
\]

\[
= M_0^{-1}(k) + M_0^{-1}(k)[C(k)S(k-1 \mid k-1)C^T(k)
+ M_A(k)G(k+1, k-1, k-1, N)M_0^T(k)
\cdot [1 - \delta(k, N - 1)] + 2V_0(k, k-1; N)]M_0^{-1}(k), \quad M + 1 \leq k \leq N - 1
\]

with

\[
C(k) \triangleq M_0^T(k-1) + M_A(k)H(k-1, N), \quad M + 1 \leq k \leq N - 1
\]

and where \(V_0(k, k-1; N)\) is the symmetric part of the matrix

\[
M_0^T(k-1)S(k-1 \mid k-1)H(k-1, N)M_0^T(k).
\]
Finally, on the basis of (2)-(4), the mean square filter performance is given by

$$S(k | k) = (I - G(k)\Gamma(k))S(k | k-1),$$

$$M + 1 \leq k \leq N - 1. \quad (9)$$

The equations of this section, initialized with zero values (i.e.,
$$\tilde{x}(M | M) = 0, S(M | M) = 0$$), allow the recursive evaluation of the desired filtered estimates. More details about their derivations can be found in [4, 7].

The proposed filter in (3) is finite-dimensional and linear. As a direct consequence of the followed approach, it exhibits a recursive structure formally identical to the classic Kalman filter for GMP's and its implementation requires an equal real-time computational effort. In fact, the Green's functions do not depend on the observed sequence, so that they can be precomputed off line on the basis of the model in (1) only. Furthermore, for any subinterval $$[k-1, k]$$ of the index space only two samples of the Green's function (those appearing in (6) and (7)) must be computed, and this can be carried out by means of computationally efficient (and numerically stable) algorithms proposed in the literature for the solution of block-tridiagonal systems.

Remark 1: The filtering formulas of this section do not change when $$X(M)$$ is a random variable with nonsingular Gaussian distribution and $$\sigma(X) = 0$$. Of course, in this case the initial conditions $$\{\tilde{x}(M | M); S(M | M)\}$$ must be calculated on the basis of the available information (i.e., the a priori statistics of $$X(M)$$ and the initial observation $$y(M)$$, generally noisy).

III. THE SMOOTHERS

Recursive solutions for the fixed-point, fixed-interval, and fixed-lag smoothing problems can be derived on the basis of the available filtered estimates only. First of all, let us consider the following general smoothing formulas, easily deduced starting from the observation model in (2) by applying the innovations method [6]

$$\hat{x}(k | b) = \hat{x}(k | k) + \sum_{m=k+1}^{b} S(k | m)\Gamma(m)$$

$$\cdot \{[\Gamma(m)S(m | m - 1)\Gamma(m) + R(m)]^{-1}$$

$$\cdot (p(m) - \Gamma(m)\hat{x}(m | m - 1)), \quad M + 1 \leq k < b \quad (10)$$

The same as in Fig. 1, with $$M_0 = 0.67, M_+ = 0.33$$, and noise variance $$R(k) = 5 \{a\}, 20 \{b\}, 100 \{c\}$. 

$$S(k | b) = S(k | k) - \sum_{m=k+1}^{b} S(k | m)\Gamma(m) \cdot [\Gamma(m)S(m | m - 1)\Gamma(m) + R(m)]^{-1}$$

$$\cdot \Gamma(m)S(k | m), \quad M + 1 \leq k < b \quad (11)$$

where $$b$$ is an assigned index-value (i.e., $$k < b \leq N - 1$$), $$\hat{x}(k | b)$$ is the smoothed estimate, $$S(k | b)$$ its corresponding error covariance matrix, and

$$S(k | m) = S(k | k - 1)\Phi(k | m - 1), \quad m \geq k + 1$$

with

$$S(k | k) = S(k | k - 1) \quad (12)$$

where the state-transition matrices $$\Phi(k | m - 1)$$ are recursively calculated as

$$\Phi(k | m - 1) = \Phi(k | m - 2)\{I - \Gamma(\Gamma(m - 1)S(m - 1)\Gamma(m - 1))$$

$$\cdot [M_x(m - 1) + H^T(m - 1, N)]$$

$$\cdot M_x^T(m)\{[M_x(m)]^{-1}, \quad m \geq k + 1$$

with

$$\Phi(k | k - 1) = I. \quad (13)$$

Finally, from (3)-(5) the general smoothing formula in (10) can be rewritten as

$$\hat{x}(k | b) = \hat{x}(k | k) + \sum_{m=k+1}^{b} S(k | m)S(m | m - 1)^{-1}$$

$$\cdot \{\hat{x}(m | m) - \beta(m - 1)\hat{x}(m - 1 | m - 1)\}$$

$$M + 1 \leq k < b \quad (14)$$

where the sequence $$\{\beta(m)\}$$ is defined as

$$\beta(m) \triangleq [M_x(m) + H^T(m, N)M_x^T(m - 1)\{M_x(m + 1)]^{-1}.$$
A. Fixed-Point Smoothing

Directly from (11)-(14) the following recursive formulas are derived:

\[
\hat{x}(k + b + 1) = \hat{x}(k + b) + S(k; b + 1)|S(b + 1; b)|^{-1} \cdot \{\hat{x}(b + 1; b + 1) - \beta^T(b)\hat{z}(b; b)\}, \quad b \geq k, \text{ fixed} \tag{16}
\]

\[
S(k; b + 1) = S(k; b) - S(k; b + 1)\Gamma^T(b + 1)|\Gamma(b + 1) \cdot S(b + 1; b)\Gamma^T(b + 1) + R(b + 1)|^{-1} \cdot \Gamma(b + 1)S^T(k; b + 1), \quad b \geq k, \text{ fixed} \tag{17}
\]

with

\[
S(k; b + 1) = S(k; b)[I - \Gamma^T(b)G^T(b)]\beta(b), \quad b \geq k, \text{ fixed}. \tag{18}
\]

The recursions in (16)+(18) must be initialised by \(\hat{x}(k | k)\), \(S(k | k)\), and \(S(k | k - 1)\), respectively.

B. Fixed-Interval Smoothing

On the basis of (12) and (13), from (11)-(14) and after some algebra the following backward recursions can be obtained for the fixed-interval smoother and for the corresponding error covariance matrix:

\[
\hat{x}(k - 1 | b) = \hat{x}(k - 1 | k - 1) + S(k - 1 | k - 1)\beta(k - 1) = \{S(k - 1 | k - 1)^{-1} \cdot \hat{z}(k | b) - \beta^T(k - 1)\}, \quad b \geq k \tag{19}
\]

\[
S(k - 1 | b) = S(k - 1 | k - 1) - S(k - 1 | k - 1)\beta(k - 1) = \{S(k - 1 | k - 1)^{-1} \cdot \{S(k - 1 | k - 1) - S(k | b)\}\}, \quad b \geq k \tag{20}
\]

The above equations, initialised by \(\hat{x}(b | b)\) and \(S(b | b)\), allow to recursively compute the fixed-interval smoothed estimates on the basis of the available filtered estimates only.

Remark 2: Let us consider now the case considered in [1], [3] of Dirichlet boundary conditions and noise-free observations at the boundary points of the index space, i.e., \(y(M) = x(M), y(N) = x(N)\). Starting from the integral form of the difference model in (1), the fixed-interval smoothed sequence \(\tilde{x}(k | N)\) in this case can be expressed as

\[
\tilde{x}(k | N) = \tilde{x}(k | N - 1) + G(k, M + 1; M, N)M_x^T(M)x(M) + G(k, N - 1; M, N)M_y^T(M)y(N). \tag{21}
\]

C. Fixed-Lag Smoothing

In this case, from (11)-(14) the following forward recursion can be derived:

\[
\hat{x}(k + 1 | k + 1 + \Delta) = \beta^T(k)\hat{x}(k | k + S(k + 1 | k) \cdot \{S(k | k)\beta(k)\}^{-1} \cdot \hat{x}(k | k + \Delta) + \phi(k + 1; k + \Delta)|S(k + 1 + \Delta | k + \Delta)^{-1} \cdot \hat{x}(k + 1 + \Delta | k + \Delta) - \beta^T(k + \Delta) \cdot \hat{x}(k + \Delta | k + \Delta)), \tag{22}
\]

\[
S(k + 1 + \Delta | k + \Delta) = S(k + 1 + \Delta | k - S(k + 1 + \Delta | k + \Delta)] \cdot \phi(k + 1 + k + \Delta) \cdot S(k + 1 + \Delta | k + \Delta)]^{-1} \cdot G(k + 1 + \Delta)\Gamma(k + 1 + \Delta) \cdot \phi(k + 1 + k + \Delta)]S(k + 1 + \Delta), \tag{23}
\]

The above formulas must be initialised by \(\hat{x}(M + 1 | M + 1 + \Delta)\) and \(S(M + 1 | M + 1 + \Delta)\), available from the fixed-point smoothed solutions in (16)-(17).

In Figs. 1, 2 the filtering and the fixed-interval smoothing MSE’s are reported for two RGP’s with different parameters and for some different channel noise levels. The full agreement between the theoretical curves and the simulation results is appreciated. From the reported plots the accuracy loss of the filter with respect to the fixed-interval smoother can be evaluated. It results negligible in the case of weakly correlated RGP’s, such as that of Fig. 1, while it is larger for the highly correlated RGP of Fig. 2.

References