Invertibility in $\lambda\eta$

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Abstract

This paper investigates invertibility properties of surjective and bijective terms in the closed term model of $\lambda\eta$. With the help of unsolvable terms, it will be shown that some special surjective terms are right-invertible and that all bijective terms are invertible.

1 Introduction

This paper is concerned with pure $\lambda$-calculus. A closed $\lambda$-term $F$ is a syntactical object, whose intended meaning is a function $\text{FUN}_{A}(F)$ in an applicative structure $A$. Define $\text{FUN}_{A}(F)(e) := (Fe)_{A}$. In the introduction of [Ba] the question What is the correspondence between the applicative behavior and the syntactical form of a term? is declared to be one of the main topics in $\lambda$-calculus. Answers to this question (will) show to what degree the algorithmic concept of $\lambda$-term can fulfill the intended meaning of a function. The properties surjectivity, injectivity and bijectivity are fundamental in the study of set theoretical functions. They imply right-invertibility, left-invertibility and invertibility. Does the same hold for $\lambda$-calculus? We say, that the invertibility correspondence holds in $A$, if the invertibility (bijectivity) of $\text{FUN}_{A}(F)$ implies the invertibility of $F$ in the $\lambda$-theory of $A$ and similar for the one sided invertibility. The invertibility of $\text{FUN}_{A}(F)$ is a set theoretic concept and the invertibility of $F$ is defined via the notion $F \circ G := \lambda x. F(Gx)$ in a $\lambda$-theory. In fact, mainly term-models have been considered. It has been proven, that the invertibility correspondence holds for $M(\lambda)$, $M^0(\lambda)$ and $M(\lambda\eta)$. The proofs for $M(\lambda)$ and $M^0(\lambda)$ can be found in [MZ], only $I$ is invertible here. The proof of the (right)-invertibility correspondence for $M(\lambda\eta)$ relies on the existence of free variables. A proof will be given in section 4. The left-invertibility correspondence does not hold in any of these models. $\Omega$ serves as counterexample. The right-invertibility correspondence holds also in $M(\lambda)$ and $M^0(\lambda)$. Indeed the set of right-invertible terms in $\lambda$ is the set of terms possessing a head normal form $\lambda x.xM_{1}\ldots M_{r}$. The right-invertibility question was raised for $M^0(\lambda\eta)$ in the open problems section on page 368 of [Bö] and we shall show, that the right-invertibility correspondence does not hold in $M^0(\lambda\eta)$. The invertibility question for $M^0(\lambda\eta)$ was raised in [Ba], 21.4.9. We shall show, that the invertibility correspondence holds in $M^0(\lambda\eta)$. Besides these invertibility correspondence questions there are many other papers dealing with invertibility and one sided invertibility. [DC] and [BK] have characterized the invertible terms in $\lambda\eta$ and $K^{*}$ via their Böhm tree - or rather their normal form for the $\lambda\eta$ case. The $\lambda\eta$-invertible terms are those lying in the set of hereditary finite permutations (FHP). Furthermore there are many - and more recent - papers on the connection between solving equations and one sided invertibility. [BP1], [BP2], [BPT], [BT1], [BT2] and [PT] are only some of them.

There are no variables in $M^0(\lambda\eta)$. But there are unsolvable terms of order zero in $M^0(\lambda\eta)$, and those will help to prove the right-invertibility of some special surjective terms. As bijective terms are such special surjective terms, one obtains their invertibility. This use of unsolvable terms is the basic idea of the paper presented.

This paper is divided into three technical sections:
- Standard Reduction
- Unsolvability
- Invertibility

2 Standard Reduction

We refer to [Ba] as source of basic definitions and fundamental theorems about $\lambda$-calculus and to [Fe] for further details. In particular we shall use the combinators $I := \lambda x.x$, $K := \lambda y. x$, $F := \lambda y. y$ and $Y_{T} := (\lambda xy.y(xx)y)((\lambda xy.y(xx)y))$. Furthermore we shall use the encoding $[M, N] := \lambda x.xMN$, the decoding combinators $\pi_{n} := \lambda x.x_{n}F\ldots FN$ and the numerals $0^{n} := I$ and $n+1^{n} := [F, n^{n}]$. For $z_{1}, \ldots, z_{n}; A_{1}, \ldots, A_{n}$ let $\text{sub}(z_{1}, \ldots, z_{n}; A_{1}, \ldots, A_{n})$ denote the function $M \mapsto M[z_{1} := A_{1}, \ldots, z_{n} := A_{n}]$. We call $\text{sub}(z_{1}, \ldots, z_{n}; A_{1}, \ldots, A_{n})$ a substitution.
\textbf{2.1 Definition.} app and abs are defined by recursion.

1. \textbf{app}(x) := 0; abs(x) := 0
2. app(MN) := app(M) + 1; abs(MN) := 0
3. app(\lambda x.M) := 0; abs(\lambda x.M) := abs(M) + 1

Neither app nor abs are invariant under \(\lambda\)-equality. The function \(\text{ord}(M) := \sup\{\text{abs}(N) \mid \lambda + M = N\}\) measures the functionality degree of \(M\). We say that \(M\) is of degree zero, if and only if \(\text{ord}(M) = 0\).

Now consider a \(\lambda\)-theory \(T\).

\textbf{2.2 Definition.} A \(\lambda\)-term \(M\) is solvable in \(T\), if there are a substitution \(\ast\) and \(\lambda\)-terms \(N_1, \ldots, N_l\) such that \(T \vdash M^\ast \cdot N_1 \cdots N_l = I\).

\(\lambda\)-terms of the form \(\lambda y_1 \cdots y_m.xM_1 \cdots M_l\) are called head normal forms (hnfs). A term possesses a hnf if there is a hnf \(N\) with \(T \vdash M = N\). \(\lambda\)-terms, which are not a hnf, have a head redex, they are of the form \(\lambda y_1 \cdots y_m.(\lambda x.M_0)M_1 \cdots M_l\) with \(l > 0\). Contracting the head redex leads to \(\lambda y_1 \cdots y_m.M_0[x := M_1] M_2 \cdots M_l\). We write \(M \overset{h}{\rightarrow} N\), if \(N\) is the result of contracting the head redex of \(M\) and \(M \overset{h}{\rightarrow} N\) for the reflexive and transitive closure of \(M \overset{h}{\rightarrow} N\) and call this head-reduction. An iteration of this leads to standard-reductions \(M \overset{\beta}{\rightarrow} N\). The standardization theorem [Ba],11.4.7 says, that every reduction \(M \overset{\beta}{\rightarrow} N\) gives rise to a standard-reduction \(M \overset{\beta}{\rightarrow} N\). The standardization theorem is proved by splitting a \(\beta\)-reduction into a head and an inner \(\beta\)-reduction. Inner \(\beta\)-reductions do not change the external structure of a term. So, head-reductions are the interesting part of \(\beta\), if we are only interested in the external structure. We need a similar result for \(\beta\eta\)-reduction. By postponement of \(\eta\) ([Ba], 15.1.6), every reduction \(M \overset{\eta}{\rightarrow} N\) can be split into reductions \(M \overset{\eta}{\rightarrow} K \overset{\beta}{\rightarrow} N\). But even the simple \(\eta\)-contraction \((\lambda x.yx)I \overset{\eta}{\rightarrow} yI\) changes the structure drastically. An \(\eta\)-redex like the one above is called critical in [Kl]. It is also a \(\beta\)-redex. We give a precise definition:

\textbf{2.3 Definition.} The notion \(\eta\) is defined by recursion. We write \(\overset{\eta}{\rightarrow}\) if the \(r\) does not matter and call \(M \overset{\eta}{\rightarrow} N\) an uncritical \(\eta\)-reduction.

1. If \(E_i \overset{\eta}{\rightarrow} u_i\) then \(u_i \cdots u_r.xE_1 \cdots E_r \overset{\eta}{\rightarrow} x\).
2. If \(E_i \overset{\eta}{\rightarrow} u_i, A_1 \overset{\eta}{\rightarrow} B_1\) and \(A_2 \overset{\eta}{\rightarrow} B_2\) then \(\lambda u_i \cdots u_r.A_1.A_2E_1 \cdots E_r \overset{\eta}{\rightarrow} B_1B_2\).
3. If \(A \overset{\eta}{\rightarrow} B\) then \(\lambda x.A. \overset{\eta}{\rightarrow} \lambda x.B\).

We always assume, that \(u_1, \ldots, u_r\) are fresh and pair-wise disjoint.

Uncritical \(\eta\)-reductions avoid the critical redexes \((\lambda x.A)x \rightarrow AB\) and \((\lambda x.(\lambda x.A)x) \rightarrow \lambda x.A\) defined on page 252 of [Kl].

\textbf{2.4 Lemma.} Every reduction \(M \overset{\beta}{\rightarrow} N\) can split into \(M \overset{\beta}{\rightarrow} K \overset{\beta}{\rightarrow} L\).

This lemma is proved by reducing all critical \(\eta\)-redexes first. Those redexes produce a \(\beta\)-reduction. The complete proof is more difficult, than it might seem. The length of the reduction may increase considerably. For details we refer to [Fo]. A full analysis of uncritical \(\eta\)-reductions will also be given in a separate paper.

\textbf{2.5 Corollary.} (Standardization for \(\overset{\beta}{\rightarrow}\))

1. If \(M \overset{\beta}{\rightarrow} \lambda y_1 \cdots y_m.xM_1 \cdots M_l\), then there are \(E_1, \ldots, E_r, N_1, \ldots, N_l\) such that \(N_i \overset{\beta}{\rightarrow} M_i, E_i \overset{\beta}{\rightarrow} u_i\) and \(M \overset{\beta}{\rightarrow} \lambda y_1 \cdots y_m.u_i \cdots u_r.xN_1 \cdots N_lE_1 \cdots E_r\).
2. If \(M \overset{\beta}{\rightarrow} \lambda y_1 \cdots y_m.(\lambda x.M_0)M_1 \cdots M_l\) with \(l > 0\), then there are \(E_1, \ldots, E_r, N_0, \ldots, N_l\) such that \(N_i \overset{\beta}{\rightarrow} M_i, E_i \overset{\beta}{\rightarrow} u_i\) and \(M \overset{\beta}{\rightarrow} \lambda y_1 \cdots y_m.u_i \cdots u_r.(\lambda x.N_0)N_1 \cdots N_lE_1 \cdots E_r\).

\textit{Proof.} With the standardization for \(\beta\), the postponement of \(\eta\) and the preceding lemma. \(\square\)

So, head-reductions are important for the structure of terms in \(\lambda\) and \(\lambda\eta\). Wadworth’s theorem states, that a term is solvable in \(\lambda\) if and only if its head-reduction path is finite and terminates in a hnf.

We wish to distinguish hnf. This is done by looking at the head variable.
2.6 Definition. The head-arity of a term \( \text{har}(M) \) is defined as follows.

1. Let \( M \equiv \lambda y_1 \ldots y_m. x M_1 \ldots M_l \)
\[
\text{har}(M) := \begin{cases} 
    i & \text{if } x \equiv y_i \\
    0 & \text{else}
\end{cases}
\]

2. Let \( M \) be arbitrary.
\[
\text{har}(M) := \begin{cases} 
    \text{har}(N) & \text{if } \lambda \vdash M = N \\
    0 & \text{else}
\end{cases}
\]

3. Following [CF], we call \( M \) regular, if \( \text{har}(M) = 1 \).

Variables and unsolvable terms have \( \text{har}(M) = 0 \). Their functionality is merely virtual. If \( \text{har}(M) > 1 \), then the functionality of \( M \) is restricted. Only regular terms have the ability to be surjective.

\[ \Omega \equiv (\lambda x. xx)(\lambda x. xx) \] is the simplest unsolvable term. Substituting an unsolvable term does not change the unsolvability and it does not essentially change the head-reduction path. Substitution and head-reduction commute for unsolvable terms.

2.7 Lemma. Let \( M \) be unsolvable and \( * \) be a substitution. If \( M * \xrightarrow{\alpha} K \), then there exists a \( N \) such that \( M \xrightarrow{\alpha} N \) and \( N * \equiv K \).

3 Unsolvability

A combinator \( M \) is solvable if and only if for every \( \lambda \)-term \( N \) there is a regular \( F \) such that \( \lambda \vdash FM = N \). For unsolvable \( M \) there is a \( N \) such that \( \lambda \vdash FM = N \) never holds. This is the reason, why Barendregt proposes that unsolvable terms represent the notion of 'undefined'. Unsolvable terms seem to be objects with no functional content. As a result, every unsolvable combinator is interpreted as \( \bot \) in the 'functional' epO models \( P_\omega \) and \( D_\infty \). In these models all unsolvable terms are identified as 'the undefined'. But pure \( \lambda \)-calculus and the extensional calculus \( \lambda \eta \) do not identify all unsolvable terms. These objects have properties besides their missing (or weak) functionality. These properties are trivially represented by the equality-class \( [M]_T := \{ N | T \vdash M = N \} \). We shall need a less granular and more informative equivalence relation on the unsolvable terms. Behind this query for an equivalence relation on the unsolvable terms is the question, if, for a given unsolvable \( M \), we can find another unsolvable \( V \), which is totally different from \( M \) and may therefore serve as 'new variable' in the closed term model \( M^0(\lambda \eta) \). 'Totally different' can be understood as 'cannot be brought into congruence' by means of regular operators. In equation \( \ast \) of proposition 4.10.2 there are regular \( G,H \) such that \( (\ast) \lambda \eta \vdash G(\lambda x. F_i) = HV \) and for given unsolvable \( (\lambda x. F_j), \ldots, (\lambda x. F_k) \) we need to find a \( V \) such that \( (\ast) \) holds for no regular \( G,H \). If \( \lambda \eta \vdash GU = HV \) holds for unsolvable \( U,V \) and regular \( G,H \), then we might say that \( U \) is solvable relative to \( V \) and vice versa. We shall call such \( U \) and \( V \) equisolvable. In order to make equisolvability an equivalence relation we have to consider a transitive closure.

3.1 Definition. Let \( T \) be a \( \lambda \)-theory and \( U \) and \( V \) be unsolvable in \( T \).

\( U \) and \( V \) are equisolvable if there are unsolvable \( U_0, \ldots, U_n+1 \) and regular \( G_0, H_0, \ldots, G_n, H_n \) such that

1. \( T \vdash U = U_0 \)
2. \( T \vdash V = U_{n+1} \)
3. \( T \vdash G_iU_i = H_iU_{i+1} \).

We write \( U \simeq_T V \) for equisolvable \( U,V \) and \( U \simeq V \) for \( U \simeq_A V \).

\( \simeq_T \) is called congruence in [Fo].

The equivalence relation \( \simeq_T \) is generated by the set \( \{(M,N) | \exists PT \vdash MP = N \text{ for unsolvable } M \} \). For unsolvable \( M \) we get \( \lambda x. M \simeq_T M[x := A] \). This reflects, that if \( M \) is unsolvable, then so are \( MA, \lambda x. M \) and \( M[x := A] \) and \( M \) is the essential part of \( MA, \lambda x. M \) and \( M[x := A] \). This argument can also be found in the proof of Wadsworth's theorem as given in [Ba8.3.11-8.3.14]. A spirit of equisolvability can also be found in [St]. We understand equisolvability not only as tool to prove invertibility results, but
also as a topic of its own interest. It provides a new structure for the set of unsolvable terms. Functional properties of unsolvable terms do not play a big role in this structure. Consider \( M \) with \( \text{ord}(M) < \infty \). Then there are \( N_n \) such that \( M \simeq N_n \) and \( \text{ord}(N_n) = n \). But there is no \( N \) with \( \text{ord}(N) = \infty \) and \( M \simeq N \). This gives the first consistency result for \( \simeq \). Unsolvable terms of finite order and of infinite order cannot be equisolvable.

**Examples**

\[ \Omega \simeq \Omega \kappa \simeq \Omega \]

\[ \lambda z.((\lambda x.xz(yz))((\lambda y.xz(zy)))) \not\in \Omega \]

\[ \lambda z.((\lambda x.xz(yz))((\lambda y.xz(zy))) \not\simeq \lambda y \Omega \]

For \( \omega_n := \lambda x.x x \ldots x \) let \( \Omega_n := \omega_n \omega_n \)

\( \Omega_n \simeq \lambda y \Omega_n \) implies \( n = m \).

One could use a notion of reduction to fully analyze equisolvability in \( \lambda \) and \( \lambda y \). But equisolvability and unsolvability are not the main topics of this paper. For our needs, it suffices to introduce a function \( \text{deg} : \text{USOL} \rightarrow \mathbb{N}_0^\ast \)

- \( M \simeq \lambda y \ N \) implies \( \text{deg}(M) = \text{deg}(N) \)
- There are combinators \( \Sigma_n \) of order zero with \( \text{deg}(\Sigma_n) = n + 1 \).

The function \( \text{deg} \) will only give a rough picture of equisolvability. \( \text{deg}(M) \) measures the number of applications needed to create the unsolvability of \( M \in \text{USOL} \). \( \text{app}(M) \) counts the number of (external) applications of \( M \). We start with a function \( \text{app} : \Lambda \rightarrow \mathbb{N} \), which is not yet closed under \( \lambda - \)equality.

### Definition

\( \text{app}(x) = 0 \)

\( \text{app}(\lambda x. M) = \text{app}(M) \)

\( \text{app}(M \Pi) = \begin{cases} \text{app}(M) & M \in \text{USOL} \\ \text{app}(M) + 1 & \text{else} \end{cases} \)

\( \text{app} \) is not closed under \( \lambda - \)equality. For instance \( \text{app}(\Pi \omega) = 2 \) and \( \text{app}(\underbrace{\Pi \cdots \Pi}_{n} \Omega) = n + 1 \). That means, in order to get a measure closed under \( \lambda - \)equality, we cannot take a trivial closure of \( \text{app} \) like \( \text{max} \{ \text{app}(N) \mid \lambda \vdash M = N \} \) or \( \text{min} \{ \text{app}(N) \mid \lambda \vdash M = N \} \). Let us consider an example.

### Definition

1. \( \sigma_n \equiv \lambda y_1 \ldots y_n \lambda x. x y_1 \ldots y_n \)
2. \( \Sigma_n \equiv \underbrace{\sigma_n \ldots \sigma_n}_{n+2} \)

We have \( \Sigma_0 \equiv \Omega \) and \( \Sigma_2 \rightarrow_\beta (\lambda y_2 \lambda x x y_2 \sigma_2) \rightarrow_\lambda \rightarrow_\sigma \rightarrow_\Sigma_3 \)

All \( \Sigma_n \) are unsolvable and of order zero. They have a cyclic \( \beta - \)reduction path of order \( n + 1 \). In this path there are terms \( \{ \Sigma_n \}_{1 \leq i \leq n} \) with \( \text{app}(\Sigma_n) = i + 1 \); in particular \( \text{app}(\Sigma_n) = n + 1 \). A characteristic number for \( \{ \Sigma_n \mid 1 \leq i \leq n \} \) is the number \( n + 1 \), which is the supremum of all \( \text{app}(M) \) with \( \Sigma_n \rightarrow_\sigma M \). We are not yet finished with the construction of \( \text{deg} \). In order to delete effects from terms like \( \underbrace{\text{I} \ldots \text{I} \Sigma_n}^{n+1} \), we have to take a kind of \( \text{lim sup} \) instead of the simple supremum.

### Definition

Let \( f \) be a function \( f : \Lambda \rightarrow \mathbb{N}^\infty \).

1. \( (\text{sup } f)(M) := \sup \{ f(N) \mid M \rightarrow_\sigma N \} \)
2. \( (\text{inf } f)(M) := \inf \{ f(N) \mid M \rightarrow_\sigma N \} \)
3. \( (\text{lim sup } f) := (\text{sup } (\text{lim sup } f)) \)

### Definition

For unsolvable \( M \) let \( \text{deg}(M) := (\text{lim sup } \text{app}(M)) \).

It is easy to see that \( \text{deg}(\Sigma_n) = n + 1 \). By CR \( \text{deg} \) is invariant under \( \lambda \)-equality. This holds also for \( \lambda y \), but we have to show, why \( \eta \)-reductions are not in the way.

### Lemma

Let \( M \) be unsolvable.

1. \( M \rightarrow_\eta N \) implies \( \text{app}(M) \leq \text{app}(N) \)
2. \( M \rightarrow_\eta N \) implies \( \text{app}(M) \leq \text{app}(N) \) \( \square \).

This lemma underlines the importance of uncritical \( \eta \)-reduction. We conclude:

### Corollary

\( \text{deg} \) is invariant under \( \lambda \eta \)-equality.

**Proof.** By CR of \( \rightarrow_\eta \), it suffices to show that \( \text{deg}(M) = \text{deg}(N) \) holds for unsolvable \( M, N \) with \( M \rightarrow_\eta N \).

First we show: \( \text{deg}(M) \geq n \) implies \( \text{deg}(N) \geq n \)

Let \( N \rightarrow_\eta L \). By the postponement of \( \eta \) there is a \( K \) with \( M \rightarrow_\sigma K \) and \( K \rightarrow_\sigma L \). As \( \text{deg}(M) \geq n \), there is a \( P \) with \( K \rightarrow_\sigma P \) and \( \text{app}(P) \geq n \). By the commutativity (see [Ba], 3.3.8) of \( \beta \) and \( \eta \) there is a \( Q \) with \( P \rightarrow_\beta Q \) and \( L \rightarrow_\sigma Q \). Lemma 3.6.2 implies \( \text{app}(Q) \geq \text{app}(P) \geq n \).
Therefore $\deg(N) \geq n$.  

\[
\begin{align*}
M \xrightarrow{\beta} K & \xrightarrow{\beta} P \\
\downarrow \eta & \downarrow \eta & \downarrow \eta \\
N \xrightarrow{\beta} L & \xrightarrow{\beta} Q
\end{align*}
\]

Now we show: $\deg(N) \geq n$ implies $\deg(M) \geq n$

Let $M \xrightarrow{\beta} K$. By the commutativity of $\beta$ and $\eta$ there is a $L$ with $N \xrightarrow{\beta} L$ and $K \xrightarrow{\eta} L$. As $\deg(N) \geq n$, there is a $Q$ with $L \xrightarrow{\beta} Q$ and $\text{uapp}(Q) \geq n$. Lemma 2.4 gives a $P$ with $K \xrightarrow{\beta} P$ and $P \xrightarrow{\eta} Q$. Lemma 3.6.1 implies $\text{uapp}(P) = \text{uapp}(Q) \geq n$. Therefore $\deg(M) \geq n$.

\[
\begin{align*}
M \xrightarrow{\beta} K & \xrightarrow{\beta} P \\
\downarrow \eta & \downarrow \eta & \downarrow \eta \\
N \xrightarrow{\beta} L & \xrightarrow{\beta} Q
\end{align*}
\]

For sake of completeness, let us note, that there are unsolvable terms $M$ with $\deg(M) = \infty$. The details can be found in [Fo].

We wish to show that $\deg$ is invariant under $\lambda\eta$-equisolvability. For unsolvable $M$ we must show $\deg(MA) = \deg(M)$. It is easy to see that $\deg(\lambda x.M) = \deg(M)$. This implies $\deg(Mx) = \deg(\lambda x.Mx) = \deg(M)$. So, it suffices to prove $\deg(MA) = \deg(MB)$ for unsolvable $M$ and arbitrary $A, B$. $MA$ and $MB$ differ only at a subterm position, which is not essential for their unsolvability. We have to give a more general definition of this. If $M$ and $N$ differ only at - for their unsolvability - unessential subterm positions, we shall substitute all these positions with terms equisolvable to a given unsolvable $L$ and obtain a term $K$. This is captured in the next definition. This definition and the following lemmata are all of a very technical nature.

3.8 Definition. Let $L$ be unsolvable. The notion $L \vdash (M, K, N)$ is defined by recursion:

1. $L \vdash (M, K, N)$ for $L \simeq K$ and arbitrary $M, N$.  
2. $L \vdash (x, x, x)$ for variables $x$.  
3. If $L \vdash (M_i, K_i, N_i)$ then $L \vdash (M_1 M_2, K_1 K_2, N_1 N_2)$.  
4. If $L \vdash (M, K, K)$ then $L \vdash (\lambda x. M, \lambda x. K, \lambda x. N)$.  

Because $L \vdash (M, K, N)$ is so simple, it commutes with $\beta$.

3.9 Lemma. If $L \vdash (M, K, N)$ and $M \xrightarrow{\beta} M'$ then there are $N', K'$ such that $L \vdash (M', K', N')$ and $N \xrightarrow{\beta} N'$ and $K \xrightarrow{\beta} K'$. \qed

In $L \vdash (M, K, N)$ the terms $L$ and $K$ express something about the relation of $M$ and $N$. But $L \vdash (M, K, N)$ itself is too simple for our needs - we have $L \vdash (M, L, N)$ for arbitrary $M$ and $N$. We need something more intelligent.

3.10 Definition. We call $M$ and $N$ consistent over $L$ if there is an unsolvable $K$ such that $K \not\simeq L$ and $L \vdash (M, K, N)$. We write $L \vdash (M, N)$ for this notion.

3.11 Proposition.  

1. $L \vdash (M, N)$ implies $\text{uapp}(M) = \text{uapp}(N)$.  
2. $L \vdash (M, N)$ implies $\deg(M) = \deg(N)$.  
3. $M \simeq_{\lambda\eta} N$ implies $\deg(M) = \deg(N)$.  

Proof.  
1) is proved with the help of 3.9.  
2) is proved with 1) and two applications of 3.9.  
3) We have already reduced this to showing $\deg(MA) = \deg(MB)$ for unsolvable $M$ and arbitrary $A, B$. For $L \not\simeq M$ we get $L \vdash (MA, ML, MB)$ and $L \vdash (MA, MB)$. b) shows $\deg(MA) = \deg(MB)$. \qed

3.12 Corollary. There are infinitely many combinators of order zero, which are pairwise not equisolvable.

Proof. Take $\Sigma_n$ with $\deg(\Sigma_n) = n + 1$. \qed

We have mainly discussed equisolvability in this section. We still have to explain why and how (unsolvable) combinators of order zero can play the role of variables. This will be done in the following lemma,
which presents a kind of irreducibility property of unsolvable terms of order zero.

3.13 Lemma. Let $U_1, M$ be unsolvable and $U_1$ be of order zero. In addition, let $*$ be a substitution and $U_2, \ldots, U_{n+1} \in \Lambda$ with $\lambda \eta \vdash M^* = U_1 \ldots U_{n+1}$. Then there are $M_1, \ldots, M_{n+1}$ such that

1. $\lambda \eta \vdash M = M_1 \ldots M_{n+1}$
2. $M_1 \in \text{USOL}$
3. $\lambda \eta \vdash M_1^* = U_1$.

Proof. It is sufficient to consider the case $n = 1$. By CR $M^*$ and $U_1U_2$ have a common $\beta\eta$-reduct $N$. As $U_1$ is of order zero, $N$ has a form $(\lambda \eta N_0)N_1 \ldots N_{r+1}$ with $\lambda \eta \vdash U_1 = (\lambda \eta N_0)N_1 \ldots N_q$, $\lambda \eta \vdash U_2 = N_{q+1}$ and $q > 0$.

We come to the second application of uncritical $\eta$-reductions. By proposition 2.5.2 there is a term $K$ such that $M^* \xrightarrow{\beta\eta} K$ and $K$ has the form $K \equiv \lambda \varepsilon_1 \ldots \varepsilon_r (\lambda \eta K_0)K_1 \ldots K_{q+1}E_1 \ldots E_r$ with $\lambda \eta \vdash K_i = N_i$ and $\lambda \eta \vdash E_j = z_j$. By lemma 2.7 there is a term $L$ with $M \xrightarrow{\beta\eta} L$ and $L^* \equiv K$.

As $L$ is unsolvable, $L$ has the form

\[
\lambda \varepsilon_1 \ldots \varepsilon_r (\lambda \eta L_0)L_1 \ldots L_{r+1}F_1 \ldots F_r
\]

with $\lambda \eta \vdash L_i = K_i = N_i$ and $\lambda \eta \vdash F_j = z_j$.

Choose $M_1 \equiv \lambda \varepsilon \lambda \varepsilon_1 \ldots \varepsilon_r (\lambda \eta L_0)L_1 \ldots L_{r+1}F_1 \ldots F_r$ and $M_2 \equiv L_{q+2}$. \qed

A more detailed presentation of equisolvability and of unsolvable terms serving as variables will be given in a separate paper.

4 Invertibility

This paper was initiated by the question, if the invertibility correspondence holds for $\mathcal{M}^0(\lambda \eta)$. The proof of this correspondence differs considerably from the proof of the invertibility correspondence of $\mathcal{M}(\lambda \eta)$. Still, it is useful to have a glance at the proof of the invertibility correspondence of $\mathcal{M}(\lambda \eta)$. We shall obey which parts carry over to $\mathcal{M}^0(\lambda \eta)$ and which parts have to be amended.

Let $F$ be a $\lambda$-term which is bijective in $\mathcal{M}(\lambda \eta)$. $F$ is surjective and injective in $\mathcal{M}(\lambda \eta)$.

**sur** Let $x$ be a variable not occurring in $F$. As $F$ is surjective, there is a term $M$ such that $\lambda \eta \vdash FM = x$.

For $G \equiv \lambda x. M$ we get $\lambda \eta \vdash F \circ G = I$. $F$ is right-invertible.

**ini** Now consider $H \equiv G \circ F$. We are done, if we can prove $\lambda \eta \vdash H = I$. We get $\lambda \eta \vdash F(Hx) = (F \circ H) x = ((F \circ G) \circ F)x = Fx$. By the injectivity of $F$ we obtain $\lambda \eta \vdash Hx = x$. Therefore $\lambda \eta \vdash H = I$.

The existence of the (right-)inverse of $F$ is proved using the surjectivity of $F$. Indeed, the right-invertibility correspondence of $\mathcal{M}(\lambda \eta)$ is a part of the proof above. As the right-invertibility correspondence does not hold for $\mathcal{M}^0(\lambda \eta)$ (see theorem 4.15) this existence part will be the hard part for $\mathcal{M}^0(\lambda \eta)$. We shall have to find a proper replacement for the variable $x$ in the proof above. Certain unsolvable terms can play the role of variables. But, under what conditions can we find an unsolvable term which is 'new' for $F$ and what does that mean?

In $\mathcal{M}(\lambda \eta)$ the injectivity of $F$ proves that the right-inverse is also a left-inverse. Again a variable $x$ is used. But it is not difficult to mimic this proof in $\mathcal{M}^0(\lambda \eta)$. $\Omega$ can play the role of $x$ in (**ini**). The following lemma gives the details.

4.1 Lemma. In $\mathcal{M}^0(\lambda \eta)$ the following are equivalent.

1. $F$ is invertible.
2. $F$ is right-invertible and injective.

Proof.

1. $\rightarrow 2.$: is trivial.

2. $\rightarrow 1.$: Let $F$ be right-invertible and injective in $\mathcal{M}^0(\lambda \eta)$. Then there is a $R$ with $\lambda \eta \vdash F \circ R = I$. Now consider $H \equiv R \circ F$. We have to show $\lambda \eta \vdash H = I$ again. Using the injectivity of $F$ we obtain $\lambda \eta \vdash H \Omega = \Omega$. As $F \circ H$ is solvable, so is $H$. $H$ must be regular, it therefore possesses a lfnf $\lambda x. \lambda x_1 \ldots x_n, x R_1 \ldots R_i$. We conclude $\lambda \eta \vdash \Omega R_1[x := \Omega] \ldots R_i[x := \Omega] = \Omega x_1 \ldots x_n$. This implies $l = n$ and $\lambda \eta \vdash H_i[x := \Omega] = x_i$ and by the injectivity lemma $\lambda \eta \vdash H_i = x_i$. \qed

[BV] were the first to prove the lemma above. They use the partial validity of the $\omega$-rule in $\lambda \eta$ ([Ba], 17.3.24). The proof given is just a copy of (**ini**) in the invertibility correspondence of $\mathcal{M}(\lambda \eta)$. The role of $x$ can be taken by $\Omega$. We shall fix our attention to conditions when surjective (in $\mathcal{M}^0(\lambda \eta)$) terms are right-invertible. This will be the main topic for the rest of the paper!

From now on we will only be interested for invertibility and one sided invertibility in the extensional calculus $\lambda \eta$. We shall therefore say invertible in $\lambda \eta$ etc.

Consider a right-invertible $F$ and an right-inverse $G$ such that $\lambda \eta \vdash F \circ G = I$. $F$ must be regular, but $G$
is not necessarily regular. Take $F : = \lambda xy.x(xly)$ and $G : = \lambda xy.y(\lambda uv.xu)$. We wish to prove that bijective $F$ are invertible. In this case the anticipated inverse $G$ will be invertible and regular. We call such $F$ regular-right-invertible.

4.2 Definition. A $\lambda$-term is regular-right-invertible if it possesses a regular right-inverse.

The set of regular-right-invertible terms becomes important, because we can characterize it with the help of unsolvable terms. In addition, we obtain some information about their hnf.

4.3 Lemma. For regular $F$ and $G$ the following are equivalent.

1. $\lambda \eta \vdash F \circ G = I$

2. There are numbers $l, m$ and $\lambda$-terms $F_j, G_i$ such that

   (a) $\lambda \eta \vdash F = \lambda x.\lambda y_1 \ldots y_m.xF_1 \ldots F_l$

   (b) $\lambda \eta \vdash G = \lambda x.\lambda z_1 \ldots z_i.xG_1 \ldots G_m$

   (c) $\lambda \eta \vdash (\lambda z_1 \ldots z_i.G_i)F_1 \ldots F_l = y_i$

   for $1 \leq i \leq m$.

Proof. By evaluating $(F \circ G)x_{y_1} \ldots y_m$. □

If we know the structure of $F$, we have some information on the structure of $G$ and vice versa. But this applies only to the first level of $BT(F)$ and $BT(G)$. We cannot continue into deeper levels with this structure analysis. That is only possible if $G$ is a proper inverse of $F$. In that case the proof of [DC] and [Kl] gives information on all levels of the Böhm trees.

4.4 Corollary. $F : = \lambda x.\lambda y_1 \ldots y_m.xF_1 \ldots F_l$ is regular-right-invertible if and only if $\forall i \exists j_i \lambda \eta_i.F_{j_i}$ is left-invertible.

□

4.5 Corollary. $G : = \lambda x.\lambda z_1 \ldots z_i.xG_1 \ldots G_m$ is left-invertible if and only if there are new variables $y_1, \ldots, y_m$ and $F_j$ with $FV(F_j) \subseteq \{y_1, \ldots, y_m\}$ such that for all $i$ $\lambda \eta \vdash (\lambda z_1 \ldots z_i.G_i)F_1 \ldots F_l = y_i$. □

If we wish to prove that $F : = \lambda x.\lambda y_1 \ldots y_m.xF_1 \ldots F_l$ is regular-right-invertible, we must show that each variable $y_i$ is extractable from some $F_j$. For $\lambda \eta \vdash FN = \Omega$ we get $\lambda \eta \vdash NG_1 \ldots G_l = \Omega y_1 \ldots y_m$ (setting $G_j : = F_j[x := N]$). By CR there must be a reduction $NG_1 \ldots G_l \leadsto^{\eta} \Omega y_1 \ldots y_m$. Each $y_i$ on the rhs of this reduction must be inherited from an occurrence of $y_i$ in one of the $G_j$ - say $G_{j_i}$. Indeed, $\lambda y_1.G_{j_i} \equiv \lambda y_1.F_{j_i}[x := N]$ is left-invertible. If $N$ is unsolvable, the genericity lemma proves that $\lambda y_1.F_{j_i}$ is also left-invertible. The (left-)invertibility does not depend on unsolvable subterms. Unsolvable terms do not have any real functionality. For solvable $N$ we cannot deduce the left-invertibility of $\lambda y_1.F_{j_i}$ - solvable subterms may change all functional properties of a term. This application of the genericity lemma is one of the central ideas of this paper. This leads to the characterization of regular-right-invertibility via unsolvability. The next lemma says, that unsolvable terms of order zero can - under certain conditions - play the role of the variable $x$ in (sur) above.

4.6 Lemma. Let $F$ be regular. $F$ is regular right invertible if and only if there is an unsolvable $V$ of order zero and $N \simeq \lambda \eta F$ such that $\lambda \eta \vdash FN = V$.

Proof. If $G$ is regular with $\lambda \eta \vdash F \circ G = I$, choose $V \equiv \Omega$ and $N \equiv GV$; this direction is easy.

For the other direction let $\lambda \eta \vdash F = \lambda x.\lambda y_1 \ldots y_m.xF_1 \ldots F_l$ and $G_i : = F_i[x := N]$, so that $\lambda \eta \vdash NG_1 \ldots G_l = V y_1 \ldots y_m$. For fresh $z_1, \ldots, z_i$ and $M : = N y_1 \ldots y_m$ and $* := \text{sub}(z_1, \ldots, z_i; G_1, \ldots, G_l)$ we get $\lambda \eta \vdash M^* = V y_1 \ldots y_m$. By 3.13 there are $M_0, \ldots, M_m$ such that $\lambda \eta \vdash M_i^* = y_i$ for $1 \leq i \leq m$. By the genericity lemma ($N$ is unsolvable) we get $\lambda \eta \vdash (\lambda z_1 \ldots z_i.M_i)F_1 \ldots F_l = (\lambda z_1 \ldots z_i.M_i)G_1 \ldots G_l = y_i$.

Now choose $R : = \lambda x.\lambda z_1 \ldots z_i.xM_1 \ldots M_l$ and by lemma 4.3 we have $\lambda \eta \vdash F \circ R = I$. □

4.7 Corollary. A $\lambda$-term $F$ is right-invertible if and only if there is a $G$, an unsolvable $V$ of order zero and a $N$ such that

1. $\lambda \eta \vdash F(GN) = V$

2. $F \circ G$ is solvable. □
So far, we have only given a characterization of (regular)-right-invertibility. We wish to prove, that certain surjective terms are regular-right-invertible. We shall have to find an unsolvable \( V \) of order zero and a \( N \simeq_{\lambda \eta} V \) such that \( \lambda \eta \vdash FN = V \). In (sur) we were using an \( x \) not occurring in \( F \). For \( N \) with \( \lambda \eta \vdash FN = x \) and \( FN \not\simeq_{\lambda \eta} x \) the choice of \( x \) implies, that \( x \) occurs in \( N \) and \( \lambda \eta \vdash F \circ (\lambda x . N) = I \). For \( N \) with \( \lambda \eta \vdash FN = V \) we distinguish three cases.

**(arg)** \( V \) is derived from some subterm of the argument \( \Lambda \).

Example: \( \lambda \eta \vdash (\lambda x . x I) \Omega = \Omega \)

**(fun)** \( V \) is derived from some subterm of the function \( F \).

Example: \( \lambda \eta \vdash (\lambda x . x \Omega) I = \Omega \)

**(gen)** \( V \) is generated by the interplay of \( F \) and \( V \).

Examples: \( \lambda \eta \vdash (\lambda x . x r)(\lambda x . x r) = \Omega \), \( \lambda \eta \vdash Y_T I = (Y_T I) \)

The first case (arg) is the case which leads via corollary 4.7 to a right-inverse of \( F \). The second case is difficult to handle, because there may be infinitely many unsolvables \( U \) occurring in \( F \) (occurrence of unsolvables would have to be defined modulo \( \lambda \eta \)-equality). In the third case we totally loose track of \( V \). This possibility will lead to a term \( \lambda \), which is surjective in \( M^{0}(\lambda \eta) \) and not right-invertible.

We shall now present a set of regular terms, which has a simple syntactic definition - it only depends on the first level of the Böhm-tree of a term - and whose surjective members are regular-right-invertible. We shall call these terms strongly faithfull. The case (gen) will not be possible for strongly faithfull \( F \). The case (fun) will only be possible for \( V \) equisolvable to an unsolvable term occurring at the first level of a Böhm-tree of a strongly faithfull \( F \). Moreover, every right-inverse of a strongly faithfull \( F \) will be regular.

4.8 **Definition.** A regular \( F \) is strongly faithfull if \( \text{har}(\lambda x . F_j) = 0 \) holds for \( \lambda \eta \vdash F = \lambda x \lambda y_1 \ldots y_m . x F_1 \ldots F_l \) and \( 1 \leq j \leq l \).

\( F \equiv \lambda x \lambda y_1 \ldots y_m . x F_1 \ldots F_l \) is strongly faithfull if and only if \( \text{har}(M) > 0 \) implies \( \text{har}(\lambda x . F_j) M = 0 \) This leads to the definition of faithfullness, which we only give for sake of completeness.

4.9 **Definition.** A regular \( F \) is faithfull if \( \text{har}(M) = j \) implies \( \text{har}(\lambda x . F_j) M = 0 \) for \( \lambda \eta \vdash F = \lambda x \lambda y_1 \ldots y_m . x F_1 \ldots F_l \) and \( 1 \leq j \leq l \).

**Example** \( Y_T \) is faithfull, but not strongly faithfull.

4.10 **Proposition.** Let \( F \) be surjective in \( M^{0}(\lambda \eta) \).

1. If \( F \) is strongly faithfull, it is regular-right-invertible.

2. If \( F \) is faithfull, it is right-invertible.

**Proof.** Let \( F \) possess a hnf \( \lambda x \lambda y_1 \ldots y_m . x F_1 \ldots F_l \).

1. Let \( U \) be unsolvable and \( j > 0 \).

First we prove \( \square_j \):

\[ \text{har}(N) = j \text{ and } \lambda \eta \vdash FN = U \text{ implies } F_j \not\simeq_{\lambda \eta} U. \]

By assumption we have \( \text{har}(\lambda x . F_j) = 0 \). As \( F \) is closed, this means that either \( 1 \leq \text{har}(\lambda y_1 \ldots y_m . F_j) \leq m \) or \( F_j \) is unsolvable.

Let \( G_j : = (\lambda x . F_j) N \). For \( N \) with \( \text{har}(N) = j \) there exists a hnf \( \lambda u_1 \ldots u_p . u_j N_1 \ldots N_q \). Now

\[ (\ast) \lambda \eta \vdash U y_1 \ldots y_m = FN y_1 \ldots y_m = (\lambda u_1 \ldots u_p . u_j N_1 \ldots N_q) G_1 \ldots G_l = \lambda u_1 \ldots u_p . (\lambda x . F_j) N M_1 \ldots M_s \]

for \( 1 \leq r \leq p \) and appropriate \( M_i \).

If \( 1 \leq \text{har}(\lambda y_1 \ldots y_m . F_j) \leq m \) this implies the solvability of \( \lambda u_1 \ldots u_p . (\lambda x . F_j) N M_1 \ldots M_s \) and \( U \). This is not possible. So \( \lambda x . F_j \) is unsolvable and consequently \( U \not\simeq_{\lambda \eta} F_j \)

Therefore \( \square_j \) holds. We take advantage of it.

There are infinitely many pairwise not equisolvable combinators of order zero. Therefore it is possible to find a combinator \( V \) of order zero not equisolvable to any of the finite \( F_1 \), \ldots, \( F_l \). As \( F \) is surjective, there will be a combinator \( N \) with \( \lambda \eta \vdash FN = V \). If \( N \) were solvable, we would have \( 1 \leq \text{har}(N) \leq l \). By \( \square_{\text{har}(N)} \) we would receive \( F_{\text{har}(N)} \not\simeq_{\lambda \eta} V \). A contradiction - \( N \) cannot be solvable. Lemma 4.6 now provides a regular-right-inverse of \( F \).

2. We cannot give a complete proof of 2) in this paper. Like the strongly faithfull terms the faithfull terms have a syntactic characterization depending on the Böhm-tree. This now depends on all levels of the Böhm-tree. Furthermore, a more detailed analysis of \( \simeq_{\lambda \eta} \) is needed. The idea to find an unsolvable of order zero which has to be extracted from an argument of \( F \) carries over from 1). We can control the cases (gen) and (fun) for faithfull \( F \). As there are many different unsolvable combinators of order zero, we can find one.
which suits the case (arg). Corollary 4.7 provides a right-inverse.

We present just one application of strong faithfulness, but this provides the link between injectivity and invertibility.

4.11 Lemma. Every regular and in $M^0(\lambda \eta)$ injective $F$ is strongly faithful.

Proof. Let $F$ possess a hnf $\lambda x_1 \ldots y_m x F_1 \ldots F_l$. We have to show $\text{har}(\lambda x F_j) = 0$ for $1 \leq j \leq l$.
Assume $\text{har}(\lambda x F_j) = 1$ for some $j$.
In this case $F_j$ possesses a hnf $\lambda u_1 \ldots u_r x H_1 \ldots H_k$ with $k \geq 1$. Set
\[
M := \lambda u_1 \ldots u_r [I, I]
\]
\[
N := \lambda w_1 \ldots w_k [w_j I \ldots IF, I]
\]
and for arbitrary $P$
\[
P^+ := P[x := M]
\]
\[
P^* := P[x := N]
\]
\[
P^0 := \text{sub}(P^*; u_1, \ldots, u_r; I, \ldots, I).
\]
We obtain
\[
\lambda \eta \vdash FN y_1 \ldots y_n w_{I+1} \ldots w_k =
\]
\[
NF^{I}_1 \ldots F^{I}_r w_{I+1} \ldots w_k =
\]
\[
[F^{I}_j I \ldots IF, I] =
\]
\[
[(\lambda u_1 \ldots u_r, NH^{I}_1 \ldots H^{I}_k) I \ldots IF, I] =
\]
\[
[NH^{I}_1 \ldots H^{I}_k F, I] =
\]
\[
[H^{I}_j I \ldots IF, I] =
\]
\[
[I, I] =
\]
\[
FM y_1 \ldots y_n w_{I+1} \ldots w_k
\]
$F$ would not be injective. This case cannot occur.
Assume $\text{har}(\lambda x F_j) = i + 1 > 0$ for some $j$.
In this case $F_j$ possesses a hnf $\lambda u_1 \ldots u_r x u_1 H_1 \ldots H_k$.
This implies $\lambda \eta \vdash F(\lambda x_1 \ldots x_r x_j K_1 \ldots K_n) = K^n F(K^n)$.
$\text{har}(\lambda x F_j) > 0$ is impossible. Therefore $\text{har}(\lambda x F_j) = 0$.

4.12 Theorem. The invertibility correspondence holds in $M^0(\lambda \eta)$. Every bijective term is invertible.

Proof. Let $F$ be bijective in $M^0(\lambda \eta)$. By lemma 4.11 $F$ is strongly faithful. By proposition 4.10 $F$ is regular-right-invertible. By lemma 4.1 $F$ is invertible.

The two main ideas of Theorem 4.12 are the use of the genericity lemma 2.8 and the use of infinitely many unsolvable pairwise not equisolvable terms from corollary 3.12. Unsolvable terms play a major role in the proof. The injectivity of $F$ guarantees, that the graph of its set theoretical inverse is not too 'irregular'.
For surjective $F$ which are not injective, every set theoretical right-inverse may be very 'irregular'. That is topic of the next subsection. We shall present a term , which is surjective in $M^0(\lambda \eta)$ and not right-invertible. So, the right-invertibility correspondence does not hold in $M^0(\lambda \eta)$. The surjectivity of , must be 'not uniform'. What does 'not uniform' mean? Consider an universal enumerator $E$ (see [Ba], 8.1.6) such that $\forall M \in \Lambda^0 \exists n \in \mathbb{N} \lambda \vdash E' n = M$. Any mapping $g : M \mapsto "n"$ with $\lambda \vdash E' n = M$ is not represented by a $\lambda$-term. Assume $G \in \Lambda^0$ with $\lambda \vdash GM = g(M)$. $g(\Omega)$ has a normal form, so $G\Omega$ has a normal form. By the genericity lemma we conclude $\lambda \vdash G\Omega = GI$ and $\lambda \vdash I = E(GI) = E(G\Omega) = \Omega$. That is not possible. But this only proves that $E$ is 'not uniformly' surjective on the set $\{"n" \mid n \in \mathbb{N}\}$. As $E$ is surjective in $M(\lambda \eta)$, it must possess a right-inverse $H$ - we have just proved, that the range of $H$ is not a subset of $\{"n" \mid n \in \mathbb{N}\}$. $E$ is 'uniformly surjective' on the set $\{HM \mid M \in \Lambda^0\}$.
For , we must obtain some information on all sets $\Sigma \subseteq \Lambda^0$, such that , is surjective on $\Sigma$ (in $M^0(\lambda \eta)$). From the above discussion we see, that it would be nice to have a , such that $\lambda \vdash , M = \Omega$ can only hold for $M$ possessing a normal form. We shall use a slightly modified condition.

4.13 Lemma. Let $F, G$ and $H$ be $\lambda$-terms with $F \equiv G \circ H$. In addition let $\lambda \eta \vdash FM = \Omega$ imply the existence of a normal form of $HM$.
Then $F$ cannot be right-invertible.

Proof. Assume $R$ with $\lambda \eta \vdash F \circ R = I$. This implies $\lambda \eta \vdash G(H(R\Omega)) = F(R\Omega) = \Omega$. Therefore $H(R\Omega)$ possesses a normal form. Again by the genericity lemma we get $\lambda \eta \vdash H(R\Omega) = H(R\Omega)$ and this implies $\lambda \eta \vdash I = \Omega$. Another contradiction!

\[\square\]
For some \(G, H\) we shall define \(\equiv G \circ H\). \(H\) will be a combinator encoding the graph of arbitrary \(F\) on \(\{ n^n \mid n \in \mathbb{N} \}\) - that means \(\lambda\eta \vdash H \pi_n F = F' n^n\) or even better \(\text{(ext)} \lambda\eta \vdash H \pi_n = \lambda x. x n^n\). Here \(\pi_n\) is the projection defined in [Ba],8.2.3. We call \(\text{(ext)}\) the extensional property of \(H\). It is easy to find a combinator \(D\) such that \(\lambda\eta \vdash D \pi_n = \gamma n^n\). One could set \(H \equiv \lambda \eta \lambda x. D x\) to obtain \(\text{(ext)}\), but we need an additional intensional property of \(H\). \(\text{(int)}\) For every subtree \(A\) of \(BT(Hxy)\) the following are equivalent.

1. \(y\) is head variable of \(A\).

2. \(\exists n \in \mathbb{N} \ A = BT(y n^n)\).

In short: in the Böhm-tree of \(Hxy\) the variable \(y\) only occurs in the form \(y n^n\).

The \(\text{nf} n^n\) will be needed in the proof, that \(HM\) has a \(\text{nf}\) for \(M\) with \(\lambda\eta \vdash \lambda \pi_n \ M = (G \circ H) M = \Omega\). Some \(n^n\) will be a subterm of \(HM\). In order to show, that the 'other parts' of \(HM\) possess a \(\text{nf}\), we shall show that these 'other parts' constitute an invertible term - invertible terms always possess a \(\text{nf}\). This invertibility will be deduced from solving two equations using lemma 4.6. There will be one equation for left- and another for right-invertibility. These two equations correspond to two occurrences of \(H\) in \(\lambda\eta\). There will be a third occurrence of \(H\) in \(\lambda\eta\), which guarantees the surjectivity of \(\equiv G \circ H\).

4.14 \textbf{Definition.}

1. \(H \equiv \lambda xy. x[y n^n] n \in \mathbb{N}\)

2. \(G \equiv \lambda xy. xe \((H x(Ky))(\lambda u. z. H x(Ku))\)

3. \(\lambda \eta \equiv G \circ H\)

where \([y n^n] n \in \mathbb{N}\) denotes the (or rather an) universal generator of the uniform sequence \((y n^n) n \in \mathbb{N}\) (see [Ba],8.2 for details).

4.15 \textbf{Theorem.} \(\lambda \eta\) is surjective in \(M^0(\lambda \eta)\), but not right-invertible. The right-invertibility correspondence does not hold for \(M^0(\lambda \eta)\).

\textit{Proof.}

\textbf{Step One}

By [Ba]8.2.5 we have \(\lambda \eta \vdash H \pi_n y = y n^n\). \(H\) has the extensional property \(\text{(ext)}\). Therefore \(\lambda \eta \vdash \pi_n = \lambda y z \ E n^n \ y (\lambda u z u) = E n^n\), and \(\lambda \eta\) is surjective. Now consider \(M\) with \(\lambda \eta \vdash \lambda \pi_n \ M = \Omega\). Following lemma 4.13, we wish to show, that \(HM\) possesses a \(\text{nf}\).

\textbf{Step Two}

\(\lambda y. HM(Ky)\) is invertible.

For \(F' \equiv \lambda xyz. y x(HM(Ku))\) and \(M' \equiv \lambda y. (HE)(HM(Ky))\) we get \(\lambda \eta \vdash F' M' = \Omega\). \(M'\) must be unsolvable, so \(F'\) is regular-right-invertible by lemma 4.6. By corollary 4.4 \(FP \equiv \lambda z u. z(HM(Ku))\) is left-invertible. \(F'\) is regular and there is a \(L\) with \(\lambda \eta \vdash L \circ H' = \Omega\). By corollary 4.5 the left inverse \(L\) can be chosen as \(L \equiv \lambda x y z. F' z\) with \(\lambda \eta \vdash L \circ x M(Ku) F' = y\). Now \(\lambda \eta \vdash \lambda u. HM(Ku) \circ \lambda y F' \equiv \Omega\). \(\lambda u. HM(Ku)\) is right-invertible.

Consider \(F'' \equiv \lambda xy z. x (HM(Ku))(\lambda u. z (HM(Ku)))\). We get \(\lambda \eta \vdash F'' M = \Omega\). As \(\lambda y. (HM(Ky))\) is right-invertible - and therefore regular (this is obvious for \(\lambda z u. z(HM(Ku))\)) - \(M\) must be unsolvable. Again by lemma 4.6, \(F''\) is regular-right-invertible and possesses a left-inverse by corollary 4.4.

We have produced a left- and a right-inverse of \(\lambda u. HM(Ku)\). Therefore \(\lambda u. HM(Ku)\) is invertible.

\textbf{Step Three}

As \(\lambda y. HM(Ky)\) is invertible, it possesses a \(\text{nf}\) \(\lambda y \lambda v_1 \ldots v_m \ y J_1 \ldots J_m\) with \(J_j\) in \(\text{nf}\). Therefore \(HM\) possesses a \(\text{hnf}\) \(\lambda y \lambda v_1 \ldots v_m \ y J_0 \ldots J_m\). We have to prove that \(J_0\) possesses a \(\text{nf}\). Now the intensional property \(\text{(int)}\) of \(H\) comes into play.
By [Ba], 10.1.5 $BT(H) =$

$$
\lambda xy.x \\
\downarrow \downarrow \\
BT(y^0) \lambda z.z \\
\downarrow \downarrow \\
BT(y^1) \lambda z.z \\
\downarrow \downarrow \\
BT(y^n) \\
$$

In $BT(H)$ the variable $y$ occurs only in the form $y^n$. This property carries over to $BT(HM)$. The application of a term $\gamma^n$ to $y$ cannot be removed by reduction or substitution into $x$. Therefore, there exists a natural number $n$, such that $\lambda y y \vdash f_n = \gamma^n$. $HM$ possesses a nf. We have proved this for arbitrary $M$ with $M = \Omega$. By 4.13, $M$ is not right-invertible.

Unsolvable terms are also used in the proof of 4.15. This happens via 4.6. But this use is not necessary. We could solve the equations for suitable variables. Our main problem in the proof of 4.15 was the following: If we construct a surjective $F$ this construction yields a set $\Sigma \subseteq \Lambda^0$ on which $F$ is surjective. (For instance: $E$ is surjective on $\{\gamma^n \mid n \in \mathbb{N}\}$.) But a possible right-inverse $G$ may have a range different from $\Sigma$. We have to consider all possible ways to prove the surjectivity of $F$ - not just the one, that comes along with its construction. The solution is by putting the universal generator term $H$ in three different places of $\Sigma$. This restricts the surjectivity-possibilities of $\Sigma$. The use of an universal generator may remind us of the proof, that the $\omega$-rule is not valid in $M^0(\lambda \eta)$. That was a result showing that $M^0(\lambda \eta)$ has some functional defects. Theorem 4.15 is another instance of those functional defects.

5 Summary

We have shown, how unsolvable terms can be used to prove the invertibility-correspondence for $M^0(\lambda \eta)$ and we have seen, that this is a valuable result by giving a counterexample to the right-invertibility correspondence in $M^0(\lambda \eta)$. Unsolvable terms possess only weak functional properties. Because they are so different from bijective and invertible terms, they are suitable tools in invertibility proofs. But we consider unsolvability as a topic of its own interest.

It is still open, if the invertibility-correspondence holds for $M^0(\lambda \eta)$ or $M^0(\lambda^\ast)$. A proof would be quite different from the proof given for $M^0(\lambda \eta)$. On the other hand, defined in 4.14, that the right-invertibility correspondence does not hold in $M^0(\lambda \eta)$ or $M^0(\lambda^\ast)$.

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