Optimal orientations of products of paths and cycles

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Abstract

For a graph $G$, let $\mathcal{D}(G)$ be the family of strong orientations of $G$, $d(G) = \min \{d(D) | D \in \mathcal{D}(G)\}$ and $\rho(G) = d(G) - d(D)$, where $d(G)$ and $d(D)$ are the diameters of $G$ and $D$ respectively. In this paper we show that $\rho(G) = 0$ if $G$ is a cartesian product of (1) paths, and (2) paths and cycles, which satisfy some mild conditions.

Keywords: Path; Cycle; Bipartite graph; Diameter; Strong orientation

1. Introduction

Let $G$ (resp., $D$) be a graph (resp., digraph) with vertex set $V(G)$ (resp., $V(D)$) and edge set $E(G)$ (resp., $E(D)$). For $v \in V(G)$, the eccentricity $e(v)$ of $v$ is defined as $e(v) = \max \{d(v,x) | x \in V(G)\}$, where $d(v,x)$ denotes the distance from $v$ to $x$. The notion $e(v)$ in $D$ is similarly defined. The diameter of $G$ (resp., $D$), denoted by $d(G)$ (resp., $d(D)$), is defined as $d(G) = \max \{e(v) | v \in V(G)\}$ (resp., $d(D) = \max \{e(v) | v \in V(D)\}$).

An orientation of a graph $G$ is a digraph obtained from $G$ by assigning to each edge in $G$ a direction. An orientation $D$ of $G$ is strong if every two vertices in $D$ are mutually reachable in $D$. An edge $e$ in a connected graph $G$ is a bridge if $G - e$ is disconnected. Robbins' celebrated one-way street theorem [15] states that a connected graph $G$ has a strong orientation if and only if no edge of $G$ is a bridge. As a possible way of extending Robbins' theorem, Boesch and Tindell [1] introduced the notion $\rho(G)$ given below. For a connected graph $G$ containing no bridges, let $\mathcal{D}(G)$ be the family...
of strong orientations of $G$. Define

$$d(G) = \min\{d(D) \mid D \in \mathcal{D}(G)\} \quad \text{and} \quad \rho(G) = d(G) - d(G).$$

The problem of evaluating $\rho(G)$ for an arbitrary connected graph $G$ is very difficult. As a matter of fact, Chvátal and Thomassen [2] showed that the problem of deciding whether a graph admits an orientation of diameter two is NP-hard.

On the other hand, the parameter $\rho(G)$ has been studied in various classes of graphs including complete graphs [1, 11, 14], complete bipartite graphs [1, 3, 20], complete $k$-partite $(k \geq 3)$ graphs [4, 6, 7, 13], and $n$-cubes [12, 20]. Let $G \times H$ denote the cartesian product of two graphs $G$ and $H$ (see Section 2 for the definition), and $P_r, C_r$, and $K_r$, respectively, the path, cycle and complete graph of order $r$. Roberts and Xu [16–19], and independently Koh and Tan [5], evaluated the quantity $\rho(P_m \times P_n)$. Very recently, Koh and Tay have further determined the quantities $\rho(C_{2m} \times P_k)$ [8], $\rho(K_m \times P_k)$, $\rho(K_m \times C_{2k+1})$ and $\rho(K_m \times K_n)$ [9] and $\rho(C_{2m} \times K_n)$ [10]. In this paper, we shall evaluate $\rho(G_1 \times G_2 \times \cdots \times G_m)$, where $m \geq 2$ and $\{G_i \mid 1 \leq i \leq m\}$ is a combination of paths and cycles.

2. Cartesian product of paths

The cartesian product of a family of graphs $G_1, G_2, \ldots, G_n$, denoted by $G_1 \times G_2 \times \cdots \times G_n$ or $\prod_{i=1}^n G_i$, where $n \geq 2$, is the graph $G$ having $V(G) = V(G_1) \times V(G_2) \times \cdots \times V(G_n)$ and two vertices $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ are adjacent if and only if there exists $r \in \{1, 2, \ldots, n\}$ such that $u_r v_r \in E(G_r)$ and $u_i = v_i$ for all $i = 1, 2, \ldots, n$ with $i \neq r$. In this section, we shall evaluate $\rho(G)$, where $G$ is of the form $\prod_{i=1}^n P_{k_i}$ with $n \geq 2$ and $k_i \geq 2$ for each $i = 1, 2, \ldots, n$. For convenience, the vertices in the graph are labelled $(x_1, x_2, \ldots, x_n)$, where $1 \leq x_i \leq k_i$ for each $i = 1, 2, \ldots, n$, such that the vertices $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are adjacent iff $|a_r - b_r| = 1$ for exactly one $r \in \{1, 2, \ldots, n\}$, and $a_i = b_i$ for all $i$ with $i \neq r$.

Let $D$ be a digraph. A dipath (resp., dicycle) in $D$ is simply called a path (resp., cycle) in $D$. A path from $u$ to $v$ in $D$ is simply called a $u$-$v$ path. For $X \subseteq V(D)$, the subdigraph of $D$ induced by $X$ is denoted by $D[X]$. For $x, y \in V(D)$ and $A \subseteq V(D)$, we write $x \rightarrow y$ if $x$ is adjacent to $y$ in $D$, and write $x \rightarrow A$ (resp., $A \rightarrow y$) if $x \rightarrow y$ for each $y \in A$ (resp., for each $x \in A$).

Our first main result is as follows:

**Theorem 1.** $\rho(\prod_{i=1}^n P_{k_i}) = 0$, where $n \geq 2$, $k_1 \geq 3$, $k_2 \geq 6$ with $(k_1, k_2) \neq (3, 6)$.

Let

$$G_n = \prod_{i=1}^n P_2$$

(i.e., the $n$-cube). In proving that $\rho(G_n) = 0$ for $n \geq 4$, McCanna [12] made use of the following subtle observation due to C. Thomassen.
Lemma 1. If a bipartite graph $G$ admits an orientation of diameter at most $k$, where $k \geq 3$, such that every vertex is in a cycle of length at most $k$, then the graph $G \times P_2$ admits an orientation of diameter at most $k + 1$ such that every vertex is in a cycle of length at most $k$.

We shall now extend Thomassen's observation from $P_2$ to $\prod_{i=1}^{n} P_k$, and shall make use of the extension to prove some of our main results in this paper.

Lemma 2. If a bipartite graph $G$ admits an orientation of diameter at most $k$, where $k \geq 3$, such that every vertex is in a cycle of length at most $k$, then the graph $G \times \prod_{i=1}^{n} P_k$, where $n \geq 1$, admits an orientation of diameter at most $k - n + \sum_{i=1}^{n} k_i$ such that every vertex is in a cycle of length at most $k$.

Proof. Let $V_1$ and $V_2$ be the partite sets of $G$. Let $F \in \mathcal{O}(G)$ with $d(F) \leq k$ such that every vertex is in a cycle of length at most $k$ in $F$. We shall now orient $G \times \prod_{i=1}^{n} P_k$ inductively as follows:

(i) In $G \times P_k$, for $1 \leq i \leq k - 1$, orient $(x, i) \to (x, i+1)$ iff $x \in V_1$; and for $1 \leq i \leq k_1$, orient $(x, i) \to (y, i)$ iff $xy \in E(F)$.

(ii) Suppose $G \times \prod_{i=1}^{r} P_k$, where $1 \leq r \leq n - 1$, has been oriented. Orient $G \times \prod_{i=1}^{r+1} P_k$ so that the orientation of $G \times \prod_{i=1}^{r} P_k \times \{j\}$ is isomorphic to that of $G \times \prod_{i=1}^{r} P_k$ for each $j = 1, 2, \ldots, k_{r+1}$, and for $1 \leq i \leq k_{r+1} - 1$, orient $(x, a_1, a_2, \ldots, a_{r}, i) \to (x, a_1, a_2, \ldots, a_{r}, i+1)$ iff $x \in V_1$.

Let $F^*$ be the resulting orientation of $G \times \prod_{i=1}^{n} P_k$.

Claim. $e(u) \leq k - n + \sum_{i=1}^{n} k_i$ for each vertex $u$ in $F^*$.

Let $u = (x, b_1, b_2, \ldots, b_m, a_{m+1}, \ldots, a_n)$ and assume that $x \in V_1$, say. Take an arbitrary vertex $v = (y, b_1, b_2, \ldots, b_n)$ in $F^*$. As the cartesian product is commutative, we may assume that $a_i < b_i$ for $1 \leq i \leq m$ and $a_i > b_i$ for $m + 1 \leq i \leq n$, where $m \leq n$.

(1) Let $w = (x, b_1, b_2, \ldots, b_m, a_{m+1}, \ldots, a_n)$. Observe that there is a $u-w$ path of length at most $\sum_{i=1}^{m} k_i - m$ in $F^*$.

(2) If $x \neq y$, let $w' = (x', b_1, b_2, \ldots, b_m, a_{m+1}, \ldots, a_n)$, where $x'$ is adjacent from $x$ in an $x-y$ path of length at most $k$ in $F$. Then $w \to w'$ in $F^*$. (Note that $x' \in V_2$.) If $x = y$, take a cycle of length at most $k$ containing $x$ in $F$.

(3) There is a path of length at most $\sum_{i=m+1}^{n} k_i - (n-m)$ from $w'$ to $(x', b_1, b_2, \ldots, b_n)$ in $F^*$.

(4) There is a path of length at most $k - 1$ from $(x', b_1, b_2, \ldots, b_n)$ to $v$ in $F^*$. Combining (1)--(4), $d(u, v) \leq \sum_{i=1}^{m} k_i - m + 1 + \sum_{i=m+1}^{n} k_i - (n-m) + k - 1 = k - n + \sum_{i=1}^{n} k_i$. This proves the claim.

Thus $d(F^*) \leq k - n + \sum_{i=1}^{n} k_i$. The second part of Lemma 2 is obvious as each vertex in $F^*$ is contained in a cycle of length at most $k$ in $F$. $\square$

We need also the following result.
Lemma 3. For \( m \geq 3, \ n \geq 6 \) with \((m, n) \neq (3, 6)\), there exists \( F \in \mathcal{D}(P_m \times P_n) \) such that

(i) \( d(F) = d(P_m \times P_n) = m + n - 2 \) and

(ii) every vertex in \( P_m \times P_n \) is in a cycle of length at most \( m + n - 2 \) in \( F \).

Note. (1) \( d(P_m \times P_n) = m + n - 2 \) for all \( m, n \geq 1 \).

(2) It was shown in [5] that \( d(P_3 \times P_6) = 8 (= m + n - 1) \).

Proof of Lemma 3. Part (i) (except some isolated cases) was first obtained by Roberts and Xu [16-19]. Here, we shall use the orientations of \( P_m \times P_n \) introduced by Koh and Tan [5] to prove part (ii). Following [5], we have seven cases to consider.

Case A: \( m = 3 \) and \( n \equiv 0 \pmod{2} \) with \( n \geq 8 \). Define \( F \in \mathcal{D}(P_m \times P_n) \) as follows (see Fig. 1):

1. For \( i = 1, 3 \) and \( j = 1, 2, \ldots, n - 1 \), orient \((i, j + 1) \rightarrow (i, j)\);
2. For \( j = 1, 2, \ldots, n - 1 \), orient \((2, j) \rightarrow (2, j + 1)\);
3. For \( j = 1, 2, 3 \), orient \((1, j), (3, j) \rightarrow (2, j)\);
4. Orient \((2, 4) \rightarrow \{(1, 4), (3, 4)\}\);
5. Orient \((2, n) \rightarrow \{(1, n), (3, n)\}\) and \((2, n - 1) \rightarrow \{(1, n - 1), (3, n - 1)\}\);
6. For \( j = 5, 6, \ldots, n - 2 \), orient \((3, j) \rightarrow (2, j) \rightarrow (1, j)\) if \( j \equiv 0 \pmod{2} \); and \((1, j) \rightarrow (2, j) \rightarrow (3, j)\) if \( j \equiv 1 \pmod{2} \).

Note that \( d(F) = m + n - 2 \). Now, consider the following cycles (see also Fig. 1):

\[
\begin{align*}
(A_1) & (1, 1)(2, 1)(2, 2)(2, 3)(2, 4)(1, 4)(1, 3)(1, 2)(1, 1), \\
(A_2) & (3, 1)(2, 1)(2, 2)(2, 3)(2, 4)(3, 4)(3, 3)(3, 2)(3, 1), \\
(A_3) & (3, 5)(3, 4)(3, 3)(2, 3)(2, 4)(2, 5)(3, 5), \\
(A_4) & (1, n)(1, n - 1)(1, n - 2)(1, n - 3)(2, n - 3)(2, n - 2)(2, n - 1)(2, n)(1, n), \\
\end{align*}
\]

It can be checked that each of the above cycles is of length at most \( m + n - 2 \), and that the cycles cover vertices \((3, 5), (1, n - 3), (2, n - 3)\) and \((i, j)\), where \( i = 1, 2, 3 \) and \( j = 1, 2, 3, 4, n - 2, n - 1, n \). On the other hand, each of the remaining vertices lies in a cycle of length 4 in \( F \).

Case B: \( m = 3 \) and \( n = 7 \). Define \( F \in \mathcal{D}(P_3 \times P_7) \) as shown in Fig. 2. It can be checked that \( d(F) = 8 = m + n - 2 \) and that (ii) is satisfied as shown in Fig. 2.

Case C: \( m = 3 \) and \( n \equiv 1 \pmod{2} \) with \( n \geq 9 \). Define \( F \in \mathcal{D}(P_m \times P_n) \) as follows (see Fig. 3):
(1) $F[P_m \times P_{n-1}]$ is identical with the orientation in Case A;
(2) Orient $(2, n) \rightarrow \{(1, n), (3, n)\}$;
(3) Orient $(1, n) \rightarrow (1, n - 1), (2, n - 1) \rightarrow (2, n)$, and $(3, n) \rightarrow (3, n - 1)$.

Note that $d(F) = m + n - 2$. Now, consider the following cycles (see also Fig. 3):

$$(C_1) (1, 1)(2, 1)(2, 2)(2, 3)(2, 4)(1, 4)(1, 3)(1, 2)(1, 1),$$

$$(C_2) (3, 1)(2, 1)(2, 2)(2, 3)(2, 4)(2, 5)(3, 5)(3, 4)(3, 3)(3, 2)(3, 1),$$

$$(C_3) (1, n)(1, n - 1)(1, n - 2)(1, n - 3)(1, n - 4)(2, n - 4)(2, n - 3)(2, n - 2)(2, n - 1)(2, n)(1, n),$$


Each of these cycles is of length at most $m + n - 2$ and they cover vertices $(3, 5), (1, n - 4), (2, n - 4)$ and $(i, j)$, where $i = 1, 2, 3$ and $j = 1, 2, 3, 4, n - 3, n - 2, n - 1, n$. On the other hand, each of the remaining vertices lies in a cycle of length 4 in $F$.

Case D: $m \equiv n \equiv 0 \pmod{2}$ with $m \geq 4$ and $n \geq 6$. Define $F \in \mathcal{OP}(P_m \times P_n)$ as follows (see Fig. 4):
1. For $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n - 1$, orient

$(i, j) \rightarrow (i, j + 1)$ if $i \equiv 0 \pmod{2}$,
$(i, j + 1) \rightarrow (i, j)$ if $i \equiv 1 \pmod{2}$;

2. For $i = 1, 2, \ldots, m - 1$ and $j = 2, 3, \ldots, n - 1$, orient

$(i, j) \rightarrow (i + 1, j)$ if $j \equiv 0 \pmod{2}$,
$(i + 1, j) \rightarrow (i, j)$ if $j \equiv 1 \pmod{2}$;

3. Orient $(1, 1) \rightarrow (2, 1)$ and $(i, 1) \rightarrow \{(i - 1, 1), (i + 1, 1)\}$ for each $i = 3, 5, \ldots, m - 1$;

4. Orient $(i, n) \rightarrow \{(i - 1, n), (i + 1, n)\}$ for each $i = 2, 4, \ldots, m - 2$; and

5. Orient $(m, n) \rightarrow (m - 1, n)$.

Note that $d(F) = m + n - 2$. Now, consider the following cycles:

$(D_1)$ For $i = 1, 3, \ldots, m - 1$,

$(i, 1)(i + 1, 1)(i + 1, 2)(i + 1, 3)(i, 3)(i, 2)(i, 1)$;

$(D_2)$ For $i = 1, 3, \ldots, m - 1$,

$(i, n)(i, n - 1)(i, n - 2)(i + 1, n - 2)(i + 1, n - 1)(i + 1, n)(i, n)$.

Each of these cycles is of length not exceeding $m + n - 2$, and the cycles cover vertices $(i, j)$, where $i = 1, 2, \ldots, m$ and $j = 1, 2, 3, n - 2, n - 1, n$. On the other hand, each of the remaining vertices lies in a cycle of length 4 in $F$.

Case E: $m \equiv 0 \pmod{2}$ and $n \equiv 1 \pmod{2}$ with $m \geq 4$ and $n \geq 7$. Define $F \in \mathcal{D}(P_m \times P_n)$ as follows (see Fig. 5):

1. $F[P_m \times P_{n-1}]$ is identical with the orientation in Case D;

2. For $i = 1, 3, \ldots, m - 1$, orient $(i, n - 1) \rightarrow (i, n)$ and $(i + 1, n) \rightarrow (i + 1, n - 1)$;

3. For $i = 3, 5, \ldots, m - 1$, orient $(i, n) \rightarrow \{(i - 1, n), (i + 1, n)\}$;

4. Orient $(1, n) \rightarrow (2, n)$.
Note that $d(F) = m + n - 2$. Also, it can be shown (see Fig. 5 as an illustration) that each vertex is in a cycle of length not exceeding $m + n - 2$ in $F$.

**Case F:** $m \equiv n \equiv 1 \pmod{2}$ with $m \geq 5$ and $n \geq 7$. Define $F \in \mathcal{D}(P_m \times P_n)$ as follows (see Fig. 6):

1. $F[P_{m-1} \times P_n]$ is identical with the orientation in Case E;
2. For each $j = 2, 4, \ldots, n - 1$, orient $(m, j) \rightarrow (m, j - 1), (m, j + 1)$;
3. Orient $(m, j - 1) \rightarrow (m - 1, j - 1)$; $(m, j) \rightarrow (m, j)$ and
   
   Note that $d(F) = m + n - 2$. Also, it can be checked (see Fig. 6 as an illustration) that each vertex is in a cycle of length not exceeding $m + n - 2$ in $F$.

Finally, we consider the case when $m \equiv 1 \pmod{2}$ and $n \equiv 0 \pmod{2}$ with $m \geq 5$ and $n \geq 6$. By symmetry and the result in Case E, we need only consider the following:

**Case G:** $m = 5$ and $n \equiv 0 \pmod{2}$ with $n \geq 6$. Let $n = 2k$ and define $F \in \mathcal{D}(P_5 \times P_n)$ as follows (see Fig. 7):

1. For $i = 1, 2, 4, 5$ and $j = 1, 2, \ldots, k - 1$, orient $(i, j) \rightarrow (i, j + 1)$ and $(3, j + 1) \rightarrow (3, j)$;
2. For $i = 1, 2, 4, 5$ and $j = k + 1, k + 2, \ldots, 2k - 1$, orient $(i, j + 1) \rightarrow (i, j)$ and $(3, j) \rightarrow (3, j + 1)$;
3. For $j \neq k, k + 1$, orient $(1, j) \leftarrow (2, j) \leftarrow (3, j) \rightarrow (4, j) \rightarrow (5, j)$;
4. For $j = k, k + 1$, orient $(1, j) \rightarrow (2, j) \rightarrow (3, j) \leftarrow (4, j) \leftarrow (5, j)$;
5. Orient $(2, k) \rightarrow (2, k + 1), (3, k) \leftarrow (3, k + 1)$ and $(4, k) \rightarrow (4, k + 1)$. The edges $(1, k)(1, k + 1)$ and $(5, k)(5, k + 1)$ may be arbitrarily oriented.

Note that $d(F) = n + 3$ and each vertex is in a cycle of length not exceeding $n + 3$ (see Fig. 7 as an illustration).
The proof of Lemma 3 is now complete. □

**Proof of Theorem 1.** Let $G = P_{k_1} \times P_{k_2}$. By Lemma 3, the bipartite graph $G$ admits an orientation $F$ with $d(F) = k_1 + k_2 - 2$ such that each vertex in $G$ is in a cycle of length at most $k_1 + k_2 - 2$ in $F$. By Lemma 2, the graph $\prod_{i=1}^{n} P_{k_i}$ admits an orientation $F^{*}$ with

$$d(F^{*}) \leq (k_1 + k_2 - 2) - (n - 2) + \sum_{i=3}^{n} k_i = \sum_{i=1}^{n} k_i - n = d\left(\prod_{i=1}^{n} P_{k_i}\right).$$

The result thus follows. □

3. Cartesian product of paths and cycles

The main aim in this section is to prove the following results.

**Theorem 2.** (i) $\rho(C_{2n} \times \prod_{i=1}^{m} P_{k_i}) = 0$ for $m \geq 1$, $n \geq 2$ and $k_1 \geq 4$.

(ii) $\rho(\prod_{i=1}^{m} P_{k_i} \times \prod_{i=1}^{r} C_{n_i}) = 0$ for $m \geq 2$, $r \geq 0$, $k_1 \geq 3$ and $k_2 \geq 6$ with $(k_1, k_2) \neq (3, 6)$.

(iii) $\rho(C_{2n} \times \prod_{i=1}^{m} P_{k_i} \times \prod_{i=1}^{r} C_{n_i}) = 0$ for $m \geq 1$, $r \geq 0$, $n \geq 2$ and $k_1 \geq 4$.

Note that results (ii) and (iii) are overlapping with (ii) requiring stronger conditions on two paths whereas (iii) requiring a cycle to be even and the length of a path at least four.

In what follows, the vertices of $\prod_{i=1}^{r} C_{n_i}$ are labelled $(x_1, x_2, \ldots, x_r)$, where $1 \leq x_i \leq n_i$, $1 \leq i \leq r$ so that $(a_1, a_2, \ldots, a_r)$ and $(b_1, b_2, \ldots, b_r)$ are adjacent iff $|a_k - b_k| = 1 (\text{mod } n_k - 2)$.
for exactly one \( k, \ 1 \leq k \leq r \), and \( a_i = b_i \) for all \( i \) with \( i \neq k \). For a real \( x \), we shall denote by \( [x] \) the greatest integer not exceeding \( x \).

To prove Theorem 2(i), we need the following result.

Lemma 4. For \( n \geq 2 \) and \( k \geq 4 \), there exists \( F \in \mathcal{D}(C_{2n} \times P_k) \) such that

(i) \( d(F) = d(C_{2n} \times P_k) = n + k - 1 \)

(ii) every vertex in \( C_{2n} \times P_k \) is in a cycle of length at most \( n + k - 1 \) in \( F \).

Proof. In [8], the following orientation \( F \) of \( C_{2n} \times P_k \) was introduced (see Fig. 8):

(i) For \( i \equiv 1 \pmod{2} \) and \( 1 \leq i \leq 2n - 1 \), orient \((i, 1) \rightarrow (i+1, 1), (i-1, 1)\);

(ii) For \( j \equiv 0 \pmod{2}, 2 \leq j \leq k - 1 \), and \( 1 \leq i \leq 2n \), orient \((i, j) \rightarrow (i+1, j)\);

(iii) For \( j \equiv 1 \pmod{2}, 3 \leq j \leq k - 1 \), and \( 1 \leq i \leq 2n \), orient \((i, j) \rightarrow (i-1, j)\);

(iv) For \( i \equiv 0 \pmod{2} \) and \( 2 \leq i \leq 2n \), orient \((i, k) \rightarrow (i+1, k), (i-1, k)\);

(v) For \( i \equiv 0 \pmod{2}, 2 \leq i \leq 2n \) and \( 1 \leq j \leq k - 1 \), orient \((i, j) \rightarrow (i, j+1)\);

(vi) For \( i \equiv 1 \pmod{2}, 1 \leq i \leq 2n - 1 \) and \( 2 \leq j \leq k \), orient \((i, j) \rightarrow (i, j-1)\).

It was shown in [8] also that \( d(F) = d(C_{2n} \times P_k) = n + k - 1 \) for \( k \geq 4 \) and \( n \geq 2 \). It remains to prove (ii).

Consider the following cycles (see also Fig. 8):

\( (A_1) \) For \( i \equiv 0 \pmod{2} \) with \( 2 \leq j \leq k - 3 \) for odd \( k \) or \( 2 \leq j \leq k - 2 \) for even \( k \), and \( i \equiv 1 \pmod{2} \) with \( 1 \leq i \leq 2n - 1 \), orient \((i, j+1), (i+1, j), (i, j+1), (i, j)\);

\( (A_2) \) For \( i \equiv 1 \pmod{2}, 1 \leq i \leq 2n - 1 \), orient \((i, 1), (i, 1), (i, 1), (i, 2)\), and \( (i, 1)\);

\( (A_3) \) For \( i \equiv 1 \pmod{2}, 1 \leq i \leq 2n - 1 \), orient \((i, k), (i, k), (i, k-1), (i+1, k), (i+1, k)\) if \( k \) is odd, or \((i-1, k), (i, k), (i, k-1), (i+1, k-1), (i+1, k)\) if \( k \) is even.

Clearly, all the above cycles are of length \( 4 \) (\( \leq n+k-1 \)) and they cover \( V(C_{2n} \times P_k) \).

Proof of Theorem 2(i). Let \( G = C_{2n} \times P_k \). By Lemma 4, the bipartite graph \( G \) admits an orientation \( F \) with \( d(F) = n + k - 1 \) such that each vertex in \( G \) is in a cycle of length at most \( n + k_1 - 1 \) in \( F \). By Lemma 2, the graph \( H = C_{2n} \times \prod_{i=1}^m P_k \) admits an
orientation $F^*$ with

$$d(F^*) \leq n + k_1 - 1 - (m - 1) + \sum_{i=2}^{m} k_i$$

$$= n - m + \sum_{i=1}^{m} k_i$$

$$= d(H).$$

This proves Theorem 2(i). □

To prove Theorem 2(ii), we shall extend Thomassen's observation from $P_2$ to $\prod_{i=1}^{n} C_{k_i}$.

**Lemma 5.** If a bipartite graph $G$ admits an orientation of diameter at most $k$, where $k \geq 3$, such that each vertex is in a cycle of length at most $k$, then the graph $G \times \prod_{i=1}^{n} C_{k_i}$ admits an orientation of diameter not exceeding $k + \sum_{i=1}^{n} \left[ \frac{k_i}{2} \right]$ such that each vertex is in a cycle of length at most $k$.

**Proof.** Let $V_1$ and $V_2$ be the partite sets of $G$. Let $F \in \mathcal{D}(G)$ with $d(F) \leq k$ such that every vertex is in a cycle of length at most $k$ in $F$.

Orient $G \times \prod_{i=1}^{n} C_{k_i}$ inductively as follows:

(i) In $G \times C_{k_1}$, for $1 \leq i \leq k_1$, orient

$$(x, i) \rightarrow (x, i + 1) \text{ iff } x \in V_1,$$

$$(x, i) \rightarrow (y, i) \text{ iff } xy \in E(F).$$

(Note that the second coordinate of $(x, i + 1)$ is taken modulo $k_1$.)

(ii) Suppose $G \times \prod_{i=1}^{r} C_{k_i}$, where $1 \leq r \leq n - 1$, has been oriented. Orient $G \times \prod_{i=1}^{r+1} C_{k_i}$ so that the orientation of $G \times \prod_{i=1}^{r} C_{k_i} \times \{j\}$ is isomorphic to that of $G \times \prod_{i=1}^{r} C_{k_i}$ for each $j = 1, 2, \ldots, k_{r+1}$, and for $1 \leq i \leq k_{r+1}$, orient $(x, a_1, a_2, \ldots, a_r, i) \rightarrow (x, a_1, a_2, \ldots, a_r, i + 1)$ iff $x \in V_1$ (note that the last coordinate is taken modulo $k_{r+1}$).

Let $F^*$ be the resulting orientation of $G \times \prod_{i=1}^{n} C_{k_i}$. We shall now show that there is a path of length at most $k + \sum_{i=1}^{n} \left[ \frac{k_i}{2} \right]$ from an arbitrary vertex $u$ to any other vertex $v$ in $F^*$. Let $u = (x, a_1, a_2, \ldots, a_m)$ and $v = (y, b_1, b_2, \ldots, b_n)$. We may assume that $x \in V_1$. As the cartesian product is commutative, we further assume that

$$0 \leq b_i - a_i \leq \left\lfloor \frac{k_i}{2} \right\rfloor \pmod{k_i} \text{ for } i = 1, 2, \ldots, m$$

and

$$0 \leq a_i - b_i \leq \left\lfloor \frac{k_i}{2} \right\rfloor \pmod{k_i} \text{ for } i = m + 1, \ldots, n.$$

(1) Let $w = (x, b_1, b_2, \ldots, b_m, a_{m+1}, \ldots, a_n)$. Clearly, there is a $u-w$ path in $F^*$ of length at most $\sum_{i=1}^{m} \left[ \frac{k_i}{2} \right]$. 
(2) If \( x \neq y \), let \( w' = (x', b_1, b_2, \ldots, b_m, a_{m+1}, \ldots, a_n) \), where \( x' \) is adjacent from \( x \) in an \( x \rightarrow y \) path of length at most \( k \) in \( F \). Observe that \( x' \in V_2 \). If \( x = y \), take a cycle of length at most \( k \) containing \( x \) in \( F \).

(3) Let \( w^* = (x', b_1, b_2, \ldots, b_n) \). Clearly, there is a \( w' \rightarrow w^* \) path of length at most \( \sum_{i=m+1}^{n} \left\lfloor \frac{k_i}{2} \right\rfloor \) in \( F^* \).

(4) There is a \( w^* \rightarrow v \) path of length at most \( k - 1 \) in \( F^* \).

Combining (1)-(4), we have

\[
d(u, v) \leq \sum_{i=1}^{m} \left\lfloor \frac{k_i}{2} \right\rfloor + 1 + \sum_{i=m+1}^{n} \left\lfloor \frac{k_i}{2} \right\rfloor + k - 1
\]

\[= k + \sum_{i=1}^{n} \left\lfloor \frac{k_i}{2} \right\rfloor.
\]

This shows that \( d(F^*) \leq k + \sum_{i=1}^{n} \left\lfloor \frac{k_i}{2} \right\rfloor \). The second part of Lemma 5 is obvious as each vertex in \( F^* \) is contained in a cycle of length at most \( k \) in \( F \). \( \square \)

**Proof of Theorem 2(ii).** Let \( G = \prod_{i=1}^{m} P_{k_i} \). By Theorem 1 and Lemma 2, the bipartite graph \( G \) admits an orientation \( F \) with \( d(F) = \sum_{i=1}^{m} k_i - m \), and every vertex in \( G \) lies in a cycle of length not exceeding \( k_1 + k_2 - 2 \) (\( \leq \sum_{i=1}^{m} k_i - m \)) in \( F \). Thus by Lemma 5, the graph \( H = G \times \prod_{i=1}^{r} C_{n_i} \) admits an orientation \( F^* \) with

\[
d(F^*) \leq \sum_{i=1}^{m} k_i - m + \sum_{i=1}^{r} \left\lfloor \frac{n_i}{2} \right\rfloor
\]

\[= d(H).
\]

This proves Theorem 2(ii). \( \square \)

**Proof of Theorem 2(iii).** Let \( G = C_{2n} \times \prod_{i=1}^{m} P_{k_i} \). By Theorem 2(i) and Lemma 2, the bipartite graph \( G \) admits an orientation \( F \) with \( d(F) = n + \sum_{i=1}^{m} k_i - m \), and every vertex in \( G \) lies in a cycle of length at most \( 4( \leq n + \sum_{i=1}^{m} k_i - m ) \) in \( F \) (see the proof of Lemma 4(ii)). Thus by Lemma 5, the graph \( H = G \times \prod_{i=1}^{r} C_{n_i} \) admits an orientation \( F^* \) with

\[
d(F^*) \leq n + \sum_{i=1}^{m} k_i - m + \sum_{i=1}^{r} \left\lfloor \frac{n_i}{2} \right\rfloor
\]

\[= d(H).
\]

This proves Theorem 2(iii). \( \square \)

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References