A generalization of Tardiff’s fixed point theorem in probabilistic metric spaces and applications to random equations

Olga Hadžić, Endre Pap*, Mirko Budinčević

Department of Mathematics and Informatics, Faculty of Sciences and Mathematics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia and Montenegro

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Abstract

Using the infinitely countable extension of triangular norms, a generalization of Tardiff’s fixed point theorem in probabilistic metric spaces is proved. As a consequence, an application to random equations is obtained. © 2005 Elsevier B.V. All rights reserved.

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1. Introduction and preliminaries

The theory of probabilistic metric spaces [17] was developed by many authors. The study of contraction mappings for probabilistic metric spaces was initiated by Sehgal, Sherwood and Bharucha-Reid [18–20]. Some further results on the existence of the fixed point of a $q$-probabilistic contraction can be found in [1,5–7,14–16].

We investigated in [5,7] the countable extension of t-norms and we introduced a new notion: the geometrically convergent (briefly g-convergent) t-norm, which is closely related to the fixed point theory. We proved that t-norms of $H$-type and some subclasses of Dombi, Aczél–Alsina, and Sugeno–Weber families of t-norms are geometrically convergent, see [7]. A new approach to the fixed point theory in probabilistic metric spaces is given in Tardiff’s paper [21], where some additional growth conditions for the mapping $F : S \times S \to D^+$ are assumed under the condition $T \geq T_L$. V. Radu [13] introduced a

* Corresponding author. Tel./fax: +381 21 6350458.
E-mail addresses: pap@im.ns.ac.yu, pape@eunet.yu (E. Pap).

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stronger growth condition for $F$ than in Tardiff’s paper (under the condition $T \geq T_L$, which enables him to define a metric) and by metric approach an estimation of the convergence with respect to the solution can be obtained, see [5].

Using the countable extension of triangular norms we prove in this paper, a generalization of Tardiff’s fixed point theorem in probabilistic metric spaces. An application to random equations is given.

Let $D^+$ be the set of all distribution functions $F$ such that $F(0) = 0$ ($F$ is a nondecreasing, left continuous mapping from $\mathbb{R}$ into $[0, 1]$ such that $\sup_{x \in \mathbb{R}} F(x) = 1$).

The ordered pair $(S, F)$ is said to be a probabilistic metric space if $S$ is a nonempty set and $F : S \times S \to D^+$ ($F_F(p, q)$ is denoted by $F_{p,q}$ for every $(p, q) \in S \times S$) satisfies the following conditions:

1. $F_{u,v}(x) = 1$ for every $x > 0 \iff u = v$ ($u, v \in S$).
2. $F_{u,v} = F_{v,u}$ for every $u, v \in S$.
3. $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1 \Rightarrow F_{u,w}(x + y) = 1$ for $u, v, w \in S$ and $x, y \in \mathbb{R}^+$.

If only (1) and (2) from above holds, the ordered pair $(S, F)$ is said to be a probabilistic semi-metric space. A Menger space (see [17]) is a triple $(S, F, T)$, where $(S, F)$ is a probabilistic metric space, $T$ is a triangular norm (abbreviated t-norm) and the following inequality holds:

$$F_{u,v}(x + y) \geq T(F_{u,w}(x), F_{w,v}(y))$$

for every $u, v, w \in S$ and every $x > 0, y > 0$.

Recall that a mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (a t-norm), see [10], if the following conditions are satisfied:

- $T(a, 1) = a$ for every $a \in [0, 1]$, $T(a, b) = T(b, a)$ for every $a, b \in [0, 1]$,
- $a \geq b, c \geq d \Rightarrow T(a, c) \geq T(b, d)$ ($a, b, c, d \in [0, 1]$),
- $T(a, T(b, c)) = T(T(a, b), c)$ ($a, b, c \in [0, 1]$).

**Example 1.** The following are the three basic continuous t-norms:

(i) The minimum t-norm, $T_M$, is defined by

$$T_M(x, y) = \min(x, y),$$

(ii) The product t-norm, $T_P$, is defined by

$$T_P(x, y) = x \cdot y,$$

(iii) The Łukasiewicz t-norm $T_L$ is defined by

$$T_L(x, y) = \max(x + y - 1, 0).$$

As regards the pointwise ordering, we have the inequalities $T_L < T_P < T_M$. Each t-norm $T$ can be extended (by associativity) in a unique way to an $n$-ary operation taking for $(x_1, \ldots, x_n) \in [0, 1]^n$, $n \in \mathbb{N}$, the values $T(x_1, \ldots, x_n)$ which are defined by

$$\prod_{i=1}^{0} x_i = 1, \quad \prod_{i=1}^{n} x_i = T\left(\prod_{i=1}^{n-1} x_i, x_n\right) = T(x_1, \ldots, x_n).$$

We have for two important t-norms $T_L$ and $T_M$ that

$$T_L(x_1, \ldots, x_n) = \max\left(\sum_{i=1}^{n} x_i - (n - 1), 0\right)$$
and
\[ T_M(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n), \]
respectively.

A t-norm \( T \) can be extended to a countable infinitary operation taking for any sequence \((x_n)_{n \in \mathbb{N}}\) from \([0, 1]\) the values
\[ \prod_{i=1}^{\infty} x_i = \lim_{n \to \infty} \prod_{i=1}^{n} x_i. \tag{1} \]
The limit on the right-hand side of (1) exists since the sequence \((\prod_{i=1}^{n} x_i)_{n \in \mathbb{N}}\) is nonincreasing and bounded from below. Some sufficient conditions for \( T \) were given in [5,7] to ensure that \( \lim_{n \to \infty} \prod_{i=n}^{\infty} x_i = 1 \).

The \((\varepsilon, \lambda)\)-topology in \((S, \mathcal{F})\) is generated by the family of neighbourhoods
\[ U = (U_{v}(\varepsilon, \lambda))_{(v, \varepsilon, \lambda) \in S \times \mathbb{R}_+ \times (0,1)}, \text{ where } U_{v}(\varepsilon, \lambda) = \{ u \mid u \in S, F_{u,v}(\varepsilon) > 1 - \lambda \}. \]
The \((\varepsilon, \lambda)\)-uniformity in \((S, \mathcal{F})\) is given by the family \((U_{(\varepsilon, \lambda)})_{(\varepsilon, \lambda) \in \mathbb{R}_+ \times (0,1)}\), where
\[ U_{(\varepsilon, \lambda)} = \{ (u, v) \mid u, v \in S, F_{u,v}(\varepsilon) > 1 - \lambda \}, \]
and it exists (only) if \( \sup_{x<1} T(x, x) = 1 \), otherwise it is a semi-uniformity.

A sequence \((p_n)_{n \in \mathbb{N}}\) in a probabilistic metric space \((S, \mathcal{F})\) is an \(\mathcal{F}\)-Cauchy sequence if for every \( \varepsilon > 0 \) and \( \lambda \in (0, 1) \) there exists \( n_0(\varepsilon, \lambda) \in \mathbb{N} \) such that \( F_{p_n,p_m}(\varepsilon) > 1 - \lambda \), for every \( n, m \geq n_0(\varepsilon, \lambda) \). A probabilistic metric space \((S, \mathcal{F})\) is complete if every \(\mathcal{F}\)-Cauchy sequence converges in \( S \).

Let \((\Omega, \Sigma, \mathbb{P})\) be a probability measure space, and \((M, d)\) a complete separable metric space. Let \( S \) be the space of all classes \( \hat{X} \) of equivalence of measurable mappings \( X : \Omega \to M \), i.e., \( X, Y \in \hat{X} \) if and only if \( X = Y \) a.e.. Then \((S, \mathcal{F}, T_M)\) is a complete Menger space, where, for every \( \hat{X}, \hat{Y} \in S \),
\[ \mathcal{F}_{\hat{X}, \hat{Y}}(\varepsilon) = P(\{ \omega \mid \omega \in \Omega, d(X(\omega), Y(\omega)) < \varepsilon \}) \quad (\varepsilon > 0). \]
If a t-norm \( T \) is such that \( \sup_{x<1} T(x, x) = 1 \), then \((S, \mathcal{F}, T)\) is, with the \((\varepsilon, \lambda)\) topology, a metrizable topological space.

As an extension of the Banach contraction principle the first theorem on the existence of the fixed point in a Menger space \((S, \mathcal{F}, T_M)\) was proved in [18,19].

**Theorem A.** Let \((S, \mathcal{F}, T_M)\) be a complete Menger space and \( f : S \to S \). If there exists \( q \in (0, 1) \) such that for all points \( x, y \in S \) and all \( t > 0 \)
\[ F_{f^n x, f^n y}(qt) \geq F_{x,y}(t), \tag{2} \]
then there exists a unique globally attractive fixed point of \( f \).

Generally, we shall write simply \( f^p \) instead of \( f(p) \). If \( f : S \to S \) satisfies (2) then \( f \) is called \( q \)-probabilistic contraction.
Since 1972 many authors investigated the possibility of the weakening of the condition that t-norm \( T \) is equal to \( T_M \), see [5]. If \( T \geq T_L \), an interesting result is obtained by Tardiff in [21], where a growth condition for \( \mathcal{F} \) is introduced.

**Theorem B** (Tardiff [21]). Let \((S, \mathcal{F}, T)\) be a complete Menger space such that \( T \geq T_L \), \( f : S \to S \) a \( q \)-probabilistic contraction and there exists \( x_0 \in S \) such that
\[
\int_{1}^{\infty} \ln u \, dF_{x_0, f \circ x_0}(u) < \infty.
\]

Then there exists a unique fixed point of \( f \).

A generalization of the notion of the \( q \)-probabilistic contraction [18], the so-called \( \psi \)-probabilistic contraction, is investigated by many authors [2–5,9] for single-valued and multi-valued mappings.

**Definition 2.** Let \((S, \mathcal{F})\) be a probabilistic metric space, \( f : S \to S \) and \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \). The mapping \( f \) is called a \( \psi \)-probabilistic contraction iff for every \( x, y \in S \) and every \( t > 0 \)
\[
F_{f \circ x, f \circ y} (\psi(t)) \geq F_{x, y}(t).
\]

If \( \psi(t) = qt \), for every \( t > 0 \), where \( q \in (0, 1) \), then a \( \psi \)-probabilistic contraction is a \( q \)-probabilistic contraction.

If \( T \) is a t-norm let for every \( x \in [0, 1] \) and \( n \in \mathbb{N} \)
\[
x_T^{(n)} = \begin{cases} 1, & n = 0, \\ T(x_T^{(n-1)}, x), & \text{otherwise}. \end{cases}
\]

A t-norm \( T \) is of \( H \)-type if the family of functions \( \{x \to x_T^{(n)}\}_{n \in \mathbb{N}} \) is equicontinuous at the point \( x = 1 \). By \( \psi^{(n)} \) we shall denote the \( n \)th iteration of a mapping \( \psi \) \((n \in \mathbb{N})\).

In [9] a fixed point theorem for a multi-valued \( \psi \)-probabilistic contraction is proved. From Theorem 1 in [9] the next corollary follows.

**Corollary 3.** Let \((S, \mathcal{F}, T)\) be a complete Menger space and \( T \) be of \( H \)-type.
Let \( f : S \to S \) be a \( \psi \)-probabilistic contraction and \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) be nondecreasing and such that
\[
\sum_{n \in \mathbb{N}} \psi^{(n)}(t) < \infty \text{ for every } t > 0.
\]

Then there exists a fixed point of the mapping \( f \).

A similar result is obtained in [3] where \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a strictly increasing continuous function with \( \psi(0) = 0 \) and \( \psi(t) < t \), for every \( t > 0 \).

**Remark 4.** In Lemma 2.1 [3] it is necessary to suppose that \( \psi(\mathbb{R}^+) = \mathbb{R}^+ \) since in the proof the existence of \( \psi^{-1} : \mathbb{R}^+ \to \mathbb{R}^+ \) is supposed.
In [3] the continuity of \( \psi \) is used in order to prove that for every \( t > 0 \)
\[
\lim_{n \to \infty} \psi^{(n)}(t) = 0 \tag{4}
\]
\[
\lim_{n \to \infty} \psi^{-1}^{(n)}(t) = \infty. \tag{5}
\]

In [6] the following result is proved.

**Theorem C.** Let \((S, \mathcal{F}, T)\) be a complete Menger space, \( f : S \to S \) be a \( \psi \)-probabilistic contraction and \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) be a strictly increasing bijection such that (4) holds. If there exists \( x_0 \in S \) such that
\[
\lim_{n \to \infty} \bigcap_{i=n}^{\infty} F_{x_0, f x_0}(\psi^{-(i)}(t)) = 1
\]
for some \( t > 0 \), then there exists a unique fixed point \( \bar{x} = \lim_{n \to \infty} f^n x_0 \) of the mapping \( f \).

**Remark 5.** If \( T \) is of \( H \)-type the condition (6) is satisfied for every \( x_0 \in S \).

**Remark 6.** If in Theorem C, \( \psi(t) = qt \ (t > 0) \), where \( q \in (0, 1) \) and \( T \geq T_L \) condition (6) is satisfied if
\[
\int_1^{\infty} \ln u \ d F_{x_0, f x_0}(u) < \infty \tag{7}
\]
since (7) implies that by [21]
\[
\lim_{n \to \infty} \left( \bigcap_{i=n}^{\infty} T_L \right) F_{x_0, f x_0}\left( \frac{t}{q^n} \right) = 1 \quad \text{for every } t > 0
\]
and so
\[
\lim_{n \to \infty} \bigcap_{i=n}^{\infty} F_{x_0, f x_0}\left( \frac{t}{q^n} \right) \geq \lim_{n \to \infty} \left( \bigcap_{i=n}^{\infty} T_L \right) F_{x_0, f x_0}\left( \frac{t}{q^n} \right) = 1.
\]

We prove in this paper a fixed point theorem for a class of \( \psi \)-probabilistic contractions where \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a bijection such that (3) and (5) hold.

### 2. Iterative roots of a given function

In the proof of the fixed point theorem for \( \psi \)-probabilistic contraction, we shall use an iterative root of the mapping \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \), i.e., a mapping \( \alpha : \mathbb{R}^+ \to \mathbb{R}^+ \) such that
\[
(\alpha \circ \alpha)(t) = \alpha^{(2)}(t) = \psi(t) \quad \text{for every } t > 0. \tag{8}
\]

There is a large literature on iterative roots of a bijection [8,11,12]. Here, we recall a result of Lojasiewicz [12] on the existence of an iterative root of a bijection.

Let \( X \) be an arbitrary nonempty set and \( \psi : X \to X \) be a bijection. By \( L_k \ (k \in \mathbb{N} \cup \{0\}) \) the cardinality of the set of \( k \)-cycles of the mapping \( \psi \) is denoted. Let \( d_0 = n \) and \( d_k = n/n_k \ (k \in \mathbb{N}) \), where \( n_k \) is the greatest divisor of \( n \) which is relative prime to \( k \).
Theorem L (Lojasiewicz [12]). Let ψ : X → X be a bijection. The iterative functional equation

\[ x^{(n)}(t) = \psi(t) \quad \text{for every } t \in X \]  

(9)

has a solution \( x : X \to X \) iff for every \( k \in \mathbb{N} \cup \{0\} \), \( L_k \) is infinite or \( L_k \) is divisible by \( d_k \).

If in (9) \( n = 2 \) then \( d_0 = 2 \) and \( d_k = 2/n_k \) (\( k \in \mathbb{N} \)), where \( n_k \) is the greatest divisor of 2 which is relative prime to \( k \). Hence

\[ d_{2n} = 2, \quad d_{2n-1} = 1, \quad n \in \mathbb{N}. \]

From Theorem L the next corollary follows.

Corollary 7. Let ψ : X → X be a bijection. Then the iterative functional equation

\[ x^{(2)}(t) = \psi(t) \quad \text{for every } t \in X \]  

(10)

has a solution \( x : X \to X \) iff for every \( k \in \mathbb{N} \cup \{0\} \), \( L_k \) is infinite or \( L_k \) is divisible by 2 for even \( k \).

Corollary 8. Let ψ : \( \mathbb{R}^+ \to \mathbb{R}^+ \) be a bijection such that (4) holds. Then the iterative functional equation

\[ x^{(2)}(t) = \psi(t) \quad \text{for every } t > 0 \]  

(11)

has a solution \( x : \mathbb{R}^+ \to \mathbb{R}^+ \).

Proof. From (4) it follows that for every \( k \in \mathbb{N} \) the set of \( k \)-cycles is empty. Hence

\[ L_k = 0 \quad \text{for every } k \in \mathbb{N} \]

and by Corollary 7 there exists a solution \( x \) of iterative functional equation (11), i.e., \( x \) is an iterative root of the mapping \( \psi \).

Lemma 9. Let ψ : \( \mathbb{R}^+ \to \mathbb{R}^+ \) be a bijection such that (3) holds. Then every iterative root \( x : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfies the condition

\[ \sum_n x^{(n)}(t) < \infty \quad \text{for every } t > 0. \]  

(12)

Proof. From (3) it follows that (4) is satisfied and by Corollary 8 there is an iterative root \( x \) of the mapping \( \psi \). On the other hand for \( n > 1 \) and \( t > 0 \)

\[ \sum_n x^{(n)}(t) = \sum_n x^{(2n)}(t) + \sum_n x^{(2n+1)}(t) \]

\[ = \sum_n \psi^{(n)}(t) + \sum_n \psi^{(n)}(x(t)) \]

and (3) implies (12). \( \square \)
Lemma 10. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a bijection such that (5) holds.
Let $H : \mathbb{R}^+ \to [0, 1]$ be such that $\lim_{t \to \infty} H(t) = 1$ and
\[
\lim_{n \to \infty} \prod_{i=n}^{\infty} H(\psi^{-i}(t)) = 1 \quad \text{for every } t > 0. \tag{13}
\]
Then
\[
\lim_{n \to \infty} \prod_{i=n}^{\infty} H(x^{-i}(t)) = 1 \quad \text{for every } t > 0. \tag{14}
\]

Proof. From (13) it follows that $\sup_{x<1} T(x, x) = 1$ which implies that $\lim_{(x,y)\to (1,1)} T(x, y) = 1$. Since
\[
\prod_{i=n}^{\infty} H(x^{-i}(t)) \geq T \left( \prod_{i=0}^{\infty} H(x^{-i}(t)), \prod_{i=0}^{\infty} H(x^{-i}(t)) \right) \\
\geq T \left( \prod_{i=[n/2]}^{\infty} H(\psi^{-i}(t)), \prod_{i=[n-1/2]}^{\infty} H(\psi^{-i}(x^{-1}(t))) \right),
\]
(13) implies (14). \square

By $\mathcal{M}$ we shall denote the class of all bijections $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that (3) and (4) holds. It is obvious that the function $\psi(t) = qt$ $(t > 0)$, where $q \in (0, 1)$, belongs to the class $\mathcal{M}$.

3. A fixed point theorem for $\psi$-probabilistic contractions

Theorem 11. Let $(S, \mathcal{F}, T)$ be a complete Menger space, $\psi \in \mathcal{M}$ and $f : S \to S$ be a $\psi$-probabilistic contraction. If there exists $x_0 \in S$ such that for every $t > 0$
\[
\lim_{n \to \infty} \prod_{i=n}^{\infty} F_{x_0, f x_0}(\psi^{-i}(t)) = 1, \tag{15}
\]
then there exists a unique fixed point $x$ of the mapping $f$ and $x = \lim_{n \to \infty} f^n x_0$.

Proof. Since (15) implies that $\sup_{x<1} T(x, x) = 1$ the $(\varepsilon, \lambda)$-topology of $S$ is metrizable. We shall prove that $f$ is an uniformly continuous mapping. Let
\[
N(\varepsilon, \lambda) = \{(u, v) | (u, v) \in S \times S, F_{u, v}(\varepsilon) > 1 - \lambda\},
\]
where $\varepsilon > 0$, $\lambda \in (0, 1)$. The family $\mathcal{N} = \{N(\varepsilon, \lambda)\}_{\varepsilon > 0, \lambda \in (0, 1)}$ defines the $(\varepsilon, \lambda)$-uniformity of $S$. It suffices to prove that for every $\eta > 0$ and $\lambda \in (0, 1)$ there exists $\varepsilon > 0$ and $\zeta \in (0, 1)$ such that
\[
(\forall (x, y) \in S \times S) \quad (x, y) \in N(\varepsilon, \zeta) \to (f x, f y) \in N(\eta, \lambda). \tag{16}
\]
Let $\eta > 0$ and $\lambda \in (0, 1)$ be given. Since $\lim_{n \to \infty} \psi(n)(t) = 0$, for every $t > 0$, there exists $\epsilon > 0$ such that $\psi(\epsilon) < \eta$. Then $F_{f_x, f_y}(\eta) \geq F_{f_x, f_y}(\psi(\epsilon)) \geq F_{x, y}(\epsilon)$ and $F_{x, y}(\epsilon) > 1 - \lambda$ implies that $F_{f_x, f_y}(\eta) > 1 - \lambda$. Hence (16) holds for $\zeta = \lambda$.

Let $x_{n+1} = f_{x_n}$, for every $n \in \mathbb{N} \cup \{0\}$. We shall prove that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, i.e., that for every $\epsilon > 0$ and $\lambda \in (0, 1)$ there exists $n_0(\epsilon, \lambda) \in \mathbb{N}$ such that

$$F_{x_{n+m+1}, x_n}(\epsilon) > 1 - \lambda \quad \text{for every } n \geq n_0(\epsilon, \lambda) \text{ and every } m \in \mathbb{N}. \quad (17)$$

Let $\epsilon > 0$ and $\lambda \in (0, 1)$ be given. Since $\psi \in \mathcal{M}$ there exists an iterative root $x : \mathbb{R}^+ \to \mathbb{R}^+$ of the mapping $\psi$. By Lemma 9, $\sum_{n=0}^{\infty} x^{(n)}(t) < \infty$ for every $t > 0$. Let $t$ be a fixed number from $\mathbb{R}^+$. Since $\sum_{n=0}^{\infty} x^{(n)}(t) < \infty$, there exists $n_1(\epsilon) \in \mathbb{N}$ such that $\sum_{i \geq n_1(\epsilon)} x^{(i)}(t) < \epsilon$. Hence, for every $n \geq n_1(\epsilon)$ and every $m \in \mathbb{N}$

$$F_{x_{n+m+1}, x_n}(\epsilon) \geq F_{x_{n+m+1}, x_{n+1}} \left( \sum_{i \geq n_1(\epsilon)} x^{(i)}(t) \right)$$

$$\geq F_{x_{n+m+1}, x_{n+1}} \left( \sum_{i=n}^{n+m-1} x^{(i)}(t) \right)$$

$$\geq T(T(\ldots T(F_{x_n, x_{n+1}}(x^{(n)}(t)), F_{x_{n+1}, x_{n+2}}(x^{(n+1)}(t))), F_{x_{n+1}, x_{n+2}}(x^{(n+1)}(t))))$$

$$\ldots \text{ etc.}$$

$$= T(T(\ldots T(F_{x_0, x_1}(\psi^{-(n)}(x^{(n)}(t))), F_{x_0, x_1}(\psi^{-(n+1)}(x^{(n+1)}(t))))$$

$$\ldots \text{ etc.}$$

$$\geq \prod_{i=n}^{\infty} F_{x_0, x_1}(x^{-(i)}(t)).$$

Applying Lemma 10 for $H(t) = F_{x_0, x_0}(t)$, since (15) holds, we have that

$$\lim_{n \to \infty} \prod_{i=n}^{\infty} F_{x_0, x_0}(x^{-(i)}(t)) = 1.$$

Let $n_2(\lambda) \in \mathbb{N}$ be such that for every $n \geq n_2(\lambda)$

$$\prod_{i=n}^{\infty} F_{x_0, x_0}(x^{-(i)}(t)) > 1 - \lambda.$$

Then (17) holds for $n_0(\epsilon, \lambda) = \max\{n_1(\epsilon), n_2(\lambda)\}$. The space $S$ is complete and there exists $\overline{x} = \lim_{n \to \infty} x_n = \lim_{n \to \infty} f^n x_0$. Since $f$ is continuous, $\overline{x}$ is a fixed point of the mapping $f$. Suppose that
y ∈ S, y = fy. Then for every ε > 0
\[ F_{\bar{x},y}(\varepsilon) = F_{f\bar{x},fy}(\varepsilon) \geq F_{\bar{x},y}(\psi^{-1}(\varepsilon)) \geq \cdots \geq F_{\bar{x},y}(\psi^{-(n)}(\varepsilon)) \]
and so from (4) it follows that
\[ F_{\bar{x},y}(\varepsilon) = 1 \quad \text{for every } \varepsilon > 0. \]
This implies that \( \bar{x} = y. \) □

The family \((T_{SW}^{\lambda})_{\lambda \in (-1,\infty]}\) of Sugeno–Weber t-norms, see [5,10], which contains \( T_{L} \), is given by
\[
T_{SW}^{\lambda}(x, y) = \begin{cases} 
T_{P}(x, y) & \text{if } \lambda = \infty, \\
\max(0, x + y - 1 + \lambda xy) & \text{otherwise.}
\end{cases}
\] (18)

**Corollary 12.** Let \((S, F, T)\) be a complete Menger space and there exists a t-norm \( T_{1} \in \bigcup_{\lambda \in (-1,\infty)}\{T_{SW}^{\lambda}\} \) such that \( T \geq T_{1} \). Let \( \psi \in \mathcal{M} \) and \( f : S \to S \) be a \( \psi \)-probabilistic contraction. If \( \sum_{n} 1/\psi^{-(n)}(t) < \infty \), for every \( t > 0 \) and for some \( x_{0} \in S \)
\[
\sup_{s > 0} s(1 - F_{x_{0},fx_{0}}(s)) < \infty,
\]
then there exists a unique fixed point \( \bar{x} \) of the mapping \( f \) and \( \bar{x} = \lim_{n \to \infty} f^{n}x_{0}. \)

**Proof.** Let \( M > 0 \) be such that
\[
s(1 - F_{x_{0},fx_{0}}(s)) \leq M \quad \text{for every } s > 0,
\]
i.e.,
\[
F_{x_{0},fx_{0}}(s) > 1 - \frac{M}{s} \quad \text{for every } s > 0.
\]
Let \( s_{0} > 0 \) be such that \( 1 - M/s_{0} > 0 \). Then for every \( s \geq s_{0}, \ F_{x_{0},fx_{0}}(s) > 1 - M/s > 0. \) Let \( t > 0. \) If \( n_{0}(t) \in \mathbb{N} \) is such that
\[
\psi^{-(n)}(t) \geq s_{0} \quad \text{for } n \geq n_{0}(t),
\]
then for every \( n \geq n_{0}(t) \)
\[
F_{x_{0},fx_{0}}(\psi^{-(n)}(t)) > 1 - \frac{M}{\psi^{-(n)}(t)} > 0.
\]
Since \( T_{1} \in \bigcup_{\lambda \in (-1,\infty)}\{T_{SW}^{\lambda}\} \) we have by [5] the equivalence
\[
\sum_{n} \frac{M}{\psi^{-(n)}(t)} < \infty \iff \lim_{n \to \infty} \left( \lim_{i \to \infty} \left( 1 - \frac{M}{\psi^{-(i)}(t)} \right) \right) = 1.
\]
Hence
\[
\lim_{n \to \infty} \prod_{i=n}^{\infty} \left( 1 - \frac{M}{\psi^{-i}(t)} \right) \geq \lim_{n \to \infty} \prod_{i=n}^{\infty} \left( 1 - \frac{M}{\psi^{-i}(T_i)} \right) = 1
\]
and so
\[
\lim_{n \to \infty} \prod_{i=n}^{\infty} F_{x_0, fX_0}(\psi^{-i}(t)) \geq \lim_{n \to \infty} \prod_{i=n}^{\infty} \left( 1 - \frac{M}{\psi^{-i}(T_i)} \right) = 1.
\]
Hence, the relation (15) holds and we obtain the conclusion by Theorem 11. □

Now, by using Corollary 12, we can prove the following random fixed point theorem.

**Theorem 13.** Let \((\Omega, \Sigma, P)\) be a probability measure space, \((M, d)\) a complete separable metric space and \(f : \Omega \times M \to M\) a continuous random operator.

Suppose that there exists a mapping \(\psi \in \mathcal{M}\) such that \(\sum_n 1/\psi^{-n}(t) < \infty\) for every \(t > 0\) and the following conditions hold:

(a) For every \(X, Y \in S\) and every \(\varepsilon > 0\)
\[
P(\{\omega \mid d(f(\omega, X(\omega)), f(\omega, Y(\omega))) < \psi(\varepsilon)\})
\geq P(\{\omega \mid d(X(\omega), Y(\omega)) < \varepsilon\})
\]
for some \(q \in (0, 1)\).

(b) There exists \(X_0 \in S\) such that
\[
\sup_{u > 0} P(\{\omega \mid d(X_0(\omega), f(\omega, X_0(\omega))) \geq u\}) < \infty.
\]

Then there exists a measurable mapping \(X : \Omega \to M\) such that \(X(\omega) = f(\omega, X(\omega))\) a.e.

**Proof.** The mapping \(\hat{f} : S \to S\), defined by \((\hat{f}X)(\omega) = f(\omega, X(\omega))\) \((\omega \in \Omega)\) satisfies all the conditions of Corollary 12 for \(T = T_L\). □

**Remark 14.** Condition (b) in Theorem 13 is satisfied if \(E(d(X_0(\omega), f(\omega, X_0(\omega)))) < \infty\) since
\[
P(\{\omega \mid d(X_0(\omega), f(\omega, X_0(\omega))) \geq u\}) \leq \frac{E(d(X_0(\omega), f(\omega, X_0(\omega))))}{u}, \quad u > 0,
\]
holds.

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