The pagenumber of toroidal graphs is at most seven

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Abstract

In this paper, we show that seven pages are sufficient for a book embedding of any toroidal graph.

1. Introduction

A book embedding of a graph is an ordering of its vertices along the spine of a book (a line) and an assignment of each edge to a single page (half-plane with the spine as its boundary) so that edges on the same page do not intersect. The pagenumber $\rho(G)$ of a graph $G$ is the minimum number of pages required for a book embedding of $G$. Our concern is to establish relations between the pagenumber and the genus of graphs. Let $\Theta_g$ denote the class of genus $g$ graphs. The pagenumber $\rho(\Theta_g)$ of genus $g$ graphs is the maximum pagenumber among all graphs in $\Theta_g$. Bernhart and Kainen [1] proved that graphs with pagenumber 3 can have arbitrarily high genus, and conjectured that, for any genus $g$, the pagenumber $\rho(\Theta_g)$ of genus $g$ graphs is unbounded. Heath and Istrail [4] disproved the conjecture by embedding graphs with genus $g(\geq 1)$ in books of $O(g)$ pages. This upper bound was improved to $O(\sqrt{g})$ by Malitz [6]. For the class $\Theta_0$ of planar graphs, Yannakakis [10] presented an algorithm which embeds any planar graph in a 4-page book. In addition, he reported that there exists a planar graph $G$ which cannot be embedded in 3 pages [9]. We may, therefore, conclude that $\rho(\Theta_0) = 4$. Heath and Istrail [4] conjectured that the pagenumber of toroidal (genus 1) graphs is seven; that is $\rho(\Theta_1) = 7$, although their algorithm only guarantees a 13-page book embedding for a toroidal graph. Our purpose is to establish the upper bound.

Theorem. Any toroidal graph can be embedded in a book of 7 pages.

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2. Preliminaries

The graphs considered in this article are simple and connected. This is clearly not a restriction for our purpose. We refer to [2] for graph-theoretical terminology and notation. We adopt some more definitions. Let \( G \) be a graph drawn on the torus \( \Sigma \) (for ease of notations, we use the same symbol for a graph and its embedding). A cycle \( C \) of \( G \) is contractible if \( C \) bounds a disc \( \Delta \) in \( \Sigma \). Otherwise \( C \) is noncontractible. If \( \Delta \) exists, then it is unique. Let \( \text{Int} \ \Delta \) denote the topological interior of \( \Delta \). If \( \text{Int} \ \Delta \cap V(G) \neq \emptyset \) and \( (\Sigma - \Delta) \cap V(G) \neq \emptyset \), then \( C \) is separating. The components of \( G - V(C) \) contained in \( \Delta \) is called the inner subgraph of \( C \). Let \( C = v_1v_2 \cdots v_nv_1 \) be a cycle of \( G \). The direction induced by the labels of \( V(C) \) is called the orientation of \( C \). We denote by \( C[v_i,v_j] \) (\( C^{-1}[v_i,v_j] \)) the section of \( C \) from \( v_i \) to \( v_j \) along the orientation (opposite orientation) of \( C \). For a path \( P = v_1v_2 \cdots v_m \) of \( G \), the orientation of \( P \), \( P[v_i,v_j] \) and \( P^{-1}[v_i,v_j] \) is defined similarly. If \( v \) is a vertex of \( G \), then the adjacent vertices of \( v \) are called the neighbours of \( v \). The subgraph of \( G \) bounding the face of \( G - v \) in which \( v \) was embedded is called the neighbourhood of \( v \). For subgraphs \( S \) and \( T \) of \( G \), the set of edges with one end in \( V(S) \) and the other in \( V(T) \) is denoted by \([S,T]\).

For our purpose, it is convenient to use an alternative representation of the book embedding. Let \( D_n \) be the union of \( n \) discs with common boundary. Obviously, a graph \( G \) can be embedded in a book of \( n \) pages if and only if the vertices of \( G \) can be placed on the boundary circle of \( D_n \) so that each edge of \( G \) can be assigned to each disc of \( D_n \) so that edges on the same disc do not intersect. (This reformulation of the book embedding can be seen in [3] as the circle graph coloring.) For example, Fig.1 illustrates a 3-page book embedding of \( K_5 \). In the proof of Theorem, we consider the book embedding by the latter style.

3. Proof of the Theorem

The important tool for the proof of the Theorem is the following result by Yannakakis.
Theorem 3.1 (Yannakakis [10]). Let $G$ be a plane graph with the outer cycle $C$. Then there exists a 4-page book embedding of $G$ such that the vertices of $C$ are arranged on the spine in the cyclic order of $C$.

Following Heath and Istrail [4], we can show the Theorem by considering only the case of graphs that triangulate the torus. The following theorem establishes the desired result.

Theorem 3.2. Let $G$ be a triangulation of the torus $\Sigma$. Then there exists a 7-page book embedding of $G$.

Proof of Theorem 3.2. Let $C = v_1v_2\cdots v_mv_1$ be a shortest noncontractible cycle of $G$. In what follows, we read subscripts of the vertices of $C$ by modulo $m$.

Claim 1. If $v_i$ and $v_j$ are nonadjacent in $C$, then they are nonadjacent in $G$. Moreover, if $v_i$ and $v_j$ are nonadjacent in $C$ and have a common neighbour in $V(G) - V(C)$, then they have a common neighbour in $V(C)$.

Proof. Suppose that $v_i$ and $v_j$ are nonadjacent in $C$ but are adjacent in $G$. Then one of two cycles $v_iv_{i+1}\cdots v_jv_b$ and $v_1v_2\cdots v_iv_{j+1}\cdots v_mv_1$ is noncontractible, which contradicts the minimality of the length of $C$. Thus $v_i$ and $v_j$ are nonadjacent in $G$.

Suppose that $v_i$ and $v_j$ with $|i - j| > 2$ have a common neighbour $v$ in $V(G) - V(C)$. Then one of two cycles $v_i\cdots v_jv_b\cdots v_{i-1}$ and $v_1v_2\cdots v_iv_j$ is noncontractible, which contradicts to the minimality of the length of $C$. Thus, they have no common neighbour. □

The orientation of $C$ defines two sides to $C$ in $\Sigma$, say a right side and a left side. In what follows, we shall consider only the right side to $C$ and find a cycle $D$ which is disjoint from $C$ and homotopic to $C$ in $\Sigma$ (we refer to [7] for the terminology of topology). First, for $i = 1,2,\ldots,m$, examine whether or not $v_i$ and $v_{i+2}$ have a common neighbour in $V(G) - V(C)$. If $v_i$ and $v_{i+2}$ have a common neighbour $u_i$, then choose $u_i$ with $|V(I_i)|$ maximum, where $I_i$ denotes the inner subgraph of the cycle $u_iv_iv_{i+1}u_i$. Next, for $i = 1,2,\ldots,m$, if $u_i$ is not defined yet, define $u_i$ as the common neighbour of $v_i$ and $v_{i+1}$ such that $|V(I_i)|$ is maximum, where $I_i$ denotes the inner subgraph of the cycle $u_iv_iv_{i+1}u_i$. For ease of explanation, we set $u_{i+1} = u_i$ in the former case for $i = 1,2,\ldots,m$. Thus, we obtain the vertices $u_1,u_2,\ldots,u_m$. From Claim 1, the vertices $u_1,u_2,\ldots,u_m$ are not contained in $V(C)$ and $u_i = u_j$ implies $|i - j| \leq 1$.

Now, for $i = 1,2,\ldots,m$, if $u_{i-1} \neq u_i$, let $D_i$ be the path of $N(v_i)$ connecting $u_{i-1}$ and $u_i$ in the right side to $C$, where $N(v_i)$ denotes the neighbourhood of $v_i$ (notice that $N(v_i)$ is a cycle since $G$ is a triangulation). If $u_{i-1} = u_i$, then let $D_i$ be a single vertex $u_{i-1}$, and set $D = \bigcup_{i=1}^m D_i$.

Claim 2. $D$ is a cycle that is disjoint from $C$ and homotopic to $C$. 

Proof. If \(|i - j| > 2\), \(D_i\) and \(D_j\) are disjoint since \(v_i\) and \(v_j\) have no common neighbour from Claim 1. From the choice of the \(u_i\), \(D_i \cap D_{i+2} = \emptyset\) if \(u_i \neq u_{i+1}\), \(D_i \cap D_{i+2} = \{u_i\}\) if \(u_i = u_{i+1}\), and \(D_i \cap D_{i+1} = \{u_i\}\). Thus \(D\) is a cycle. From Claim 1, \(D\) is disjoint from \(C\). It is obvious from the definition of \(D\) that \(C\) and \(D\) are homotopic. 

Let \(A_H\) be the annulus in \(\Sigma\) bounded by \(C\) and \(D\) in the right side to \(C\), and \(H\) the subgraph of \(G\) embedded in \(A_H\). Let \(K\) be the subgraph of \(G\) that is embedded in the annulus \(A_K = \Sigma - \text{Int} A\). Clearly, \(H\) and \(K\) are triangulations of \(A_H\) and \(A_K\), respectively (therefore both \(H\) and \(K\) are planar embeddings). It holds that \(G = H \cup K\) and \(H \cap K = C \cup D\).

First, we will construct a 5-page book embedding \(\beta_K\) of \(K\). Let \(P = w_1w_2 \cdots w_n\) be a shortest path in \(K\) from a vertex \(w_1\) of \(C\) to a vertex \(w_n\) of \(D\). It follows from Claim 1 that \(n \geq 2\). The following comes from the minimality of the length of \(P\) (the proof is similar to that of Claim 1, and is omitted).

Claim 3. If \(w_i\) and \(w_j\) are nonadjacent in \(P\), then \(w_i\) and \(w_j\) are nonadjacent in \(K\). Moreover, if \(w_i\) and \(w_j\) are nonadjacent in \(P\) and have a common neighbour in \(V(K) - V(P)\), then they have a common neighbour in \(V(P)\).

The orientation of \(P\) defines two sides to \(P\) in \(A_K\), say a right side and a left side. We shall consider only the right side to \(P\) and find a path \(Q\) which is disjoint from \(P\) and homotopic to \(P\) in \(A_K\). First, for \(i = 1, 2, \ldots, n - 2\), examine whether or not \(w_i\) and \(w_{i+2}\) have a common neighbour in \(V(K) - V(P)\). If \(w_i\) and \(w_{i+2}\) have a common neighbour \(z_i\), then choose \(z_i\) with \(|V(J_i)|\) maximum, where \(J_i\) denotes the inner subgraph of the cycle \(z_iw_iw_{i+1}w_{i+2}z_i\). Next, for \(i = 1, 2, \ldots, n - 1\), if \(z_i\) is not defined yet, define \(z_i\) as the common neighbour of \(w_i\) and \(w_{i+1}\) such that \(|V(J_i)|\) is maximum, where \(J_i\) denotes the inner subgraph of the cycle \(z_iw_iw_{i+1}z_i\). For convenience, we set \(z_{i+1} = z_i\) in the former case for \(i = 1, 2, \ldots, n - 2\). Thus, we obtain the vertices \(z_1, z_2, \ldots, z_{n-1}\). From Claim 3, the vertices \(z_1, z_2, \ldots, z_{n-1}\) are not contained in \(V(P)\) and \(z_i = z_j\) implies \(|i - j| \leq 1\), moreover, each \(z_i\) is disjoint from \(C\) if \(i \neq 1\) and is disjoint from \(D\) if \(i \neq n - 1\).

For \(i = 2, 3, \ldots, n - 1\), if \(z_{i-1} \neq z_i\), let \(Q_i\) be the path of \(N(w_i)\) connecting \(z_{i-1}\) and \(z_i\) in the right side to \(P\), where \(N(w_i)\) denotes the neighbourhood of \(w_i\). If \(z_{i-1} = z_i\), then let \(Q_i\) be a single vertex \(z_{i-1}\). It may be possibly that there exists a vertex of \(Q_2\) contained in \(V(C)\). In this case, let \(x\) be the vertex of \(Q_2\) that is nearest to \(z_2\) in \(Q_2\), and exchange by \(w_1 = x\). After repeating such exchanges, if any, we may assume that all vertices of \(Q_2\) (therefore, all vertices of \(Q_2, Q_3, \ldots, Q_{n-1}\)) are not contained in \(V(C)\). Similarly, we may assume that all vertices of \(Q_2, Q_3, \ldots, Q_{n-1}\) are not contained in \(V(D)\).

If \(z_1\) is not contained in \(V(C)\), examine neighbours of \(w_1\), starting with the vertex \(z_1\) and sweep rotationally around \(w_1\) in the opposite direction toward \(w_2\) (in Fig. 2, this direction is clockwise). Let \(z_0\) be the first vertex encountered such that \(z_0 \in V(C)\), and let \(Q_1\) be the path of \(N(w_1)\) connecting \(z_0\) and \(z_1\) in the right side to \(P\), where
$N(w_1)$ denotes the neighbourhood of $w_1$. From Claim 1, $z_0$ is adjacent to $w_1$ in $C$. If $z_1 \in V(C)$, then $z_1$ is adjacent to $w_1$ in $C$, and we set $z_0 = z_1$ and $Q_1 = z_1$.

If $z_{n-1}$ is not contained in $V(D)$, then examine neighbours of $w_n$, starting with the vertex $z_{n-1}$ and sweep rotationally around $w_n$ in the opposite direction toward $w_{n-1}$ (in Fig. 2, this direction is counterclockwise). Let $z_n$ be the first vertex encountered such that $z_n \in V(D)$, and let $Q_n$ be the path of $N(w_n)$ connecting $z_{n-1}$ and $z_n$ in the right side to $P$, where $N(w_n)$ denotes the neighbourhood of $w_n$.

Now, $z_n$ may be not adjacent to $w_n$ by an edge of $D$, since $D$ is not a shortest path. In this case, let $J$ be the subgraph of $K$ that is embedded in the disc in $A_K$ bounded by $D[w_n,z_n,] \cup \{z_n,w_n\}$. Otherwise, let $J = \emptyset$. If $z_{n-1} \in V(D)$, then we set $z_n = z_{n-1}$ and $Q_n = z_{n-1}$, and define $J$ similarly. Let $Q = \bigcup_{i=1}^{n} Q_i$. The following holds (the proof is similar to that of Claim 2 and is omitted).

**Claim 4.** $Q$ is a path that is disjoint from $P$, internally disjoint from both $C$ and $D$, and homotopic to $P$ in $A_K$.

Now, we bring $P$ close to $Q$. For $i = 1, 2, \ldots, n-1$, if the cycle $z_iw_iw_{i+1}z_i$ is separating, then exchange $w_iw_{i+1}$ of $P$ for the path of neighbourhood of $z_i$ connecting $w_i$ and $w_{i+1}$ in the disc in $A_K$ bounded by $z_iw_iw_{i+1}z_i$ (see Fig. 3). If the cycle $z_iw_iw_{i+1}w_{i+2}z_i$ is separating, then exchange $w_iw_{i+1}w_{i+2}$ of $P$ for the path of neighbourhood of $z_i$ connecting $w_i$ and $w_{i+2}$ in the disc in $A_K$ bounded by $z_iw_iw_{i+1}w_{i+2}z_i$. From now on, let $P$ denote the resulting path.

Let $\hat{K}$ be the subgraph of $K$ that is embedded in the disc bounded by the cycle $F = P \cup D^{-1}[z_n,w_n] \cup Q^{-1} \cup C[z_0,w_1]$ in $A_K$. Notice that $K = \hat{K} \cup [P,Q] \cup J$ (see Fig. 4(a)). Since $\hat{K}$ is a planar embedding with outer cycle $F$, there is a 4-page book embedding $\beta_{\hat{K}}$ of $\hat{K}$ such that the vertices of $F$ are arranged along the cyclic order of $F$ on the spine of $\beta_{\hat{K}}$ from Theorem 3.1 (the orientation of $F$ is defined to be that of $C$). Let $P_1, P_2, P_3, P_4$ be the four pages required for $\beta_{\hat{K}}$. The ordering of the vertices of $P$ and $Q$ on the spine of $\beta_{\hat{K}}$ enables us to embed the edges of $[P,Q]$ in a
single page, so we embed these edges in a new page $P_3$ (see Fig. 4(b)). Thus, we have a 5-page book embedding $\beta_{\hat{K} \cup [P, Q]}$ of $\hat{K} \cup [P, Q]$.

To complete the book embedding of $K$, we must explain how to embed $J$ if $J$ is not empty. Note that $J$ is a plane embedding bounded by the cycle $L = D[w_n, z_n] \cup \{z_n w_n\}$, and so there exists a 4-page book embedding $\beta_J$ of $J$ such that the vertices of $L$ are arranged along the cyclic order of $L$ on the spine of $\beta_J$ from Theorem 3.1 (the orientation of $L$ is defined to be that of $D$). We may assume that $w_n$ is placed first on the spine of $\beta_J$. Also note that $z_n w_n$ is already embedded in $\beta_{K \cup [P, Q]}$.

We distinguish two cases.

**Case 1:** $z_n \neq z_{n-1}$. We insert the book embedding $\beta_{J - w_n}$ of $J - w_n$ induced by $\beta_J$ immediately after $z_n$ of $\beta_{K \cup [P, Q]}$ by identifying $z_n$ of $\beta_{J - w_n}$ to $z_n$ of $\beta_{K \cup [P, Q]}$ (see Fig. 5). Then we can use $P_1, P_2, P_3, P_4$ as the four pages of $\beta_{J - w_n}$ and we can embed the edges of $J$ incident to $w_n$ in $P_3$.

**Case 2:** $z_n = z_{n-1}$. We insert the book embedding $\beta_{J - z_n}$ of $J - z_n$ induced by $\beta_J$ immediately before $w_n$ of $\beta_{K \cup [P, Q]}$ by identifying $w_n$ of $\beta_{J - z_n}$ to $w_n$ of $\beta_{K \cup [P, Q]}$ (see Fig. 6). Then we can use $P_1, P_2, P_3, P_4$ as the four pages of $\beta_{J - z_n}$ and we can embed the edges of $J$ incident to $z_n$ in $P_3$. 

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**Fig. 3.**

![Figure 3](image)

**Fig. 4.**

![Figure 4](image)
The resulting 5-page book embedding is obviously a book embedding of \( K \), and is denoted by \( \beta_K \). The following comes from what has been said above.

**Claim 5.** On the spine of \( \beta_K \), the vertices of \( C \) are arranged along the orientation of \( C \), and afterwards the vertices of \( D \) are arranged along the opposite orientation of \( D \).

In the following, we explain how to embed \( H - (E(C) \cup E(D)) \) in the above book; as a consequence, we will obtain the required book embedding of \( G \). Relabel the vertices of \( C \) and \( D \) in \( H \) by \( v_1v_2 \ldots v_m \) and \( u_1u_2 \ldots u_n \), respectively, according to the ordering on the spine of \( \beta_K \) (i.e. \( v_1 = z_0 \) and \( v_m = w_1 \) at the previous step). Let \( u_k \) be the vertex
of $D$ that is the common neighbour of $v_m$ and $v_1$ in $H$ (if the path $v_m v_1 v_2$ is a part of a separating 4-cycle, then $u_k$ is the common neighbour of $v_m$ and $v_2$ in $H$). Let $v_1$ be the vertices of $C$ that is the common neighbour of $u_0$ and $u_1$ in $H$ (see Fig. 7).

From the ordering of the vertices of $C$ and $D$ on the spine of $\beta_K$, another two pages $P_6$ and $P_7$ are sufficient to embed the edges $[C,D]$ of $H$. We embed the edges incident to $v_1, v_2, \ldots, v_l$ ($v_2, v_3, \ldots, v_l$ if the path $v_m v_1 v_2$ is a part of a separating 4-cycle) but not incident to $u_1$ into $P_6$ and embed the other edges into $P_7$. This is indicated by Fig. 8.

In Fig. 8, $P_6$ and $P_7$ are illustrated by thin lines and bold lines, respectively.

Finally, we shall show that each inner subgraph of $H$ (if any) can be embedded in this book. We distinguish two cases. Let $I$ be any inner subgraph of $H$.

Case 1: $I$ is an inner subgraph by a separating 3-cycle $u_j v_i v_{i+1} u_j$. Since $I$ is planar, from Theorem 3.1, there is a 4-page book embedding $\beta_I$ of $I$. We divide into two subcases, depending on whether $i \neq m$ or $i = m$: 
(1a) \( i \neq m \). \( \beta_i \) is placed immediately after \( v_i \) in \( \beta_k \). Then we can use \( P_1, P_2, P_3, P_4 \) as the four pages of \( \beta_i \). The edges connecting the vertices of \( I \) and \( u_i \) are placed in \( P_6 \) if \( 1 \leq i \leq l - 1 \), or in \( P_7 \) if \( l \leq i \leq m - 1 \). Since these edges are placed between \( u_i v_i \) and \( u_{i+1} \), they do not conflict. The edges connecting the vertices of \( I \) and \( v_i \) are placed in \( P_5 \). The edges connecting the vertices of \( I \) and \( v_{i+1} \) are placed in \( P_7 \) if \( 1 \leq i \leq l - 1 \), or in \( P_6 \) if \( l \leq i \leq m - 1 \).

(1b) \( i = m \). \( \beta_i \) is placed immediately after \( u_k \) in \( \beta_K \) if \( u_k \) is adjacent to \( v_2 \) in \( H \), or immediately before \( u_k \) in \( \beta_K \) if \( u_k \) is adjacent to \( v_{m-1} \) (otherwise, the choice is arbitrary). Then we can use \( P_1, P_2, P_3, P_4 \) as the four pages of \( \beta_i \), and we can embed the edges connecting the vertices of \( I \) and \( u_k, v_m, v_1 \) are placed in \( P_5, P_7, P_6 \), respectively.

Case 2: \( I \) is an inner subgraph by a separating 4-cycle \( u_jv_i-1v_iv_{i+1}u_j \). Notice that \( i \neq l \). In this case, let \( \hat{I} \) be the subgraph of \( H \) embedded in the disc in \( A_H \) bounded by the cycle \( u_jv_i-1v_iv_{i+1}u_j \), and let \( \beta_j \) be a 4-page book embedding of \( \hat{I} \). We may assume, without loss of generality, that \( v_i \) is placed first on the spine of \( \beta_j \). We insert \( \beta_j \) immediately after \( v_i \) of \( \beta_k \) by identifying \( v_i \) of \( \beta_j \) to \( v_i \) of \( \beta_K \). Then we can use \( P_1, P_2, P_3, P_4 \) as the four pages of \( \beta_j \), and we can embed the edges connecting the vertices of \( I \) and \( v_i-1 \) in \( P_5 \), the edges connecting the vertices of \( I \) and \( v_{i+1} \) (resp. \( u_j \)) in \( P_7 \) (resp. \( P_6 \)) if \( 1 \leq i \leq l - 1 \), or in \( P_6 \) (resp. \( P_7 \)) if \( l + 1 \leq i \leq m - 1 \). Now the proof is complete.

4. Concluding remarks

From the theorem, we can conclude that \( \rho(\Theta_1) \leq 7 \). It remains an unsettled question whether there exists a toroidal graph \( G \) such that 7 pages are necessary for any book embedding of \( G \). We believe that such a graph exists and conjecture that \( \rho(\Theta_1) = 7 \).

Here, we shall mention an interesting relation between the pagenumber and the chromatic number of graphs. Let \( \chi(G) \) denote the usual chromatic number of a graph \( G \). The chromatic number \( \chi(\Theta_g) \) of genus \( g \) graphs is defined as the maximum chromatic number \( \chi(G) \) among all graphs \( G \) in \( \Theta_g \). The following is known (see [5], for example):

**Theorem 1.** For any \( g \geq 0 \),

\[
\chi(\Theta_g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor,
\]

where \( \lfloor x \rfloor \) denotes the largest integer not greater than \( x \).

Combining Theorem 1 with Malitz's theorem (showing that \( \rho(\Theta_g) \) is \( O(\sqrt{g}) \) [6]), we can conclude that \( \rho(\Theta_g) \) and \( \chi(\Theta_g) \) are equal asymptotically. Moreover, \( \rho(\Theta_0) = \chi(\Theta_0) = 4 \) and (possibly) \( \rho(\Theta_1) = \chi(\Theta_1) = 7 \). Is it the case that for any genus \( g \) the pagenumber \( \rho(\Theta_g) \) and the chromatic number \( \chi(\Theta_g) \) are equal?
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