The Generalized Fibonomial Matrix

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**Abstract**

The Fibonomial coefficients are known as interesting generalization of binomial coefficients. In this paper, we derive a \((k + 1)\)th recurrence relation and generating matrix for the Fibonomial coefficients, which we call *generalized Fibonomial matrix*. We find a nice relationship between the eigenvalues of the Fibonomial matrix and the generalized right-adjusted Pascal matrix that they have the same eigenvalues. We obtain generating functions, combinatorial representations, many new interesting identities and properties of the Fibonomial coefficients. Several cases of our results are given as examples.

*Key words:* The Fibonomial coefficients, generating matrix, recurrence relation, generating function

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1 Introduction

The well known Fibonacci numbers are defined by

\[
F_n = F_{n-1} + F_{n-2}
\]

with initial conditions \(F_0 = 0\) and \(F_1 = 1\), for \(n > 1\).

The Fibonomial coefficient is defined by the relation for \(n \geq m \geq 1\)

\[
\begin{bmatrix} n \\ m \end{bmatrix}_F = \frac{F_1 F_2 \ldots F_n}{(F_1 F_2 \ldots F_{n-m}) (F_1 F_2 \ldots F_m)}
\]

with \([n \atop 0]_F = [n \atop n]_F = 1\) where \(F_n\) is the \(n\)th Fibonacci number. These coefficients satisfy the relation:

\[
\begin{bmatrix} n \\ m \end{bmatrix}_F = F_{m+1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_F + F_{n-m-1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_F.
\]
Let $p$ be a nonzero integer. Define the generalized Fibonacci and Lucas sequences by the recurrences:

\begin{align*}
u_n &= pu_{n-1} + u_{n-2} \\
v_n &= pv_{n-1} + v_{n-2}
\end{align*}

where $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = p$, respectively, for all $n \geq 2$.

When $p = 1$ and $p = 2$, then $u_n = F_n$ (n-th Fibonacci number) and $u_n = P_n$ (n-th Pell number), respectively.

The authors [14] were the first to study generalized Fibonomial coefficients formed by terms of sequence $\{u_n\}$ as follows: for $n \geq m \geq 1$

$$\binom{n}{m} = \frac{u_1u_2 \ldots u_n}{(u_1u_2 \ldots u_{n-m})(u_1u_2 \ldots u_m)}$$

with $\binom{n}{0} = \binom{n}{n} = 1$. When $p = 1$, the generalized Fibonomial coefficient $\binom{n}{m}$ is reduced to the Fibonomial coefficient $\binom{n}{m}_{F}$.

The $n \times n$ generalized Pascal matrix $V_n$ whose $(i, j)$ entry is given by

$$(V_n)_{ij} = \left(\frac{j-1}{j+i-n-1}\right)p^{j+n-1}.$$ 

For example,

$$V_4 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 3p \\
0 & 1 & 2p & 3p^2 \\
1 & p & p^2 & p^3
\end{bmatrix}.$$ 

Recently there have been increasing interest in both the Fibonomial coefficients and certain generalized matrix of binomial coefficients, which we call them as generalized Pascal matrices. Regarding left or right adjustments, and certain coefficients generalizations, several authors give various names to Pascal matrices. For example Carlitz [1] considered the right adjusted Pascal matrix and he called it as "matrix of binomial coefficients". In [5], Edelman and Gilbert considered left adjusted matrix of binomial coefficients and he called it "Pascal matrix". In [23], the author considered right adjusted and coefficient generalized matrix of binomial coefficient and he gave a name "Netted Matrix" to this matrix.

Regarding generalization of binomial coefficients, several authors have studied
the generalized Fibonomial coefficients and their properties (see for more details [7,10,14,21,26,27]). Meanwhile, some authors consider the spectral properties of the generalized Pascal matrix ([1,3,11,22]). Since some relationships between the generalized Pascal matrix and the Fibonomial coefficients have been constructed, the Fibonomial coefficients are considered by some author. In this paper, we give more powerful relationships between the Fibonomial coefficients and certain Pascal matrix.

Matrix methods and generating matrices are very useful for solving some problems stemming from number theory. In this paper, we define the generalized Fibonomial matrix and derive an \((k + 1)\)th order linear recurrence relation for the generalized Fibonacci coefficients. Also we show that the generalized Fibonomial and Pascal matrices have the same characteristic polynomials and therefore same eigenvalues. We obtain some explicit and closed formulas for the coefficients and their sums by matrix methods. We give generating functions, properties and combinatorial representations for them. Further we present some relationships between determinants of certain matrices and the generalized Fibonacci coefficients.

2 Generalized Fibonomial Coefficients

This section is mainly devoted to derive a recurrence relation and generating matrix for the generalized Fibonomial coefficients. For the sake of compactness, we shall use the following notations, for fixed \(k\) such that \(1 \leq i \leq k + 1\):

\[
a_{n,i} = (-1)^{(i-1)(i-2)/2} \left\{ \begin{array}{c} n + k \\ k - i + 1 \end{array} \right\} \left\{ \begin{array}{c} n + i - 2 \\ i - 1 \end{array} \right\}
\]

where \(\left\{ \begin{array}{c} n \\ m \end{array} \right\}\) stands for the generalized Fibonomial coefficients and it is supposed \(\left\{ \begin{array}{c} n \\ m \end{array} \right\} = 1\) when either \(m > n\) and \(n \leq 0\).

For later use, we give the following useful result.

**Lemma 1** For \(n > 0\) and \(1 \leq i \leq k\)

\[
a_{1,i}a_{n,1} + a_{n,i+1} = a_{n+1,i}
\]

where \(a_{n,i}\) be as before.

**Proof.** For the first case \(i = 1\), the proof can be found in [10]. For the other cases, that is, \(i > 1\), if we simplify the equality \(a_{1,i}a_{n,1} + a_{n,i+1} = a_{n+1,i}\),
it is reduced to the form:

\[ u_{k+1}u_{n+i} + (-1)^{i-1} u_{n}u_{k-i+1} = u_{i}u_{n+k+1}. \]

The last equality can be easily obtained from the Binet formula of \( \{u_n\} \). Thus the proof is complete.

Define the \((k + 1) \times (k + 1)\) companion matrix \( G_k \) and the matrix \( H_{n,k} \) as follows:

\[
G_k = \begin{bmatrix}
  a_{1,1} & a_{1,2} & \ldots & a_{1,k+1} \\
  1 & & & \\
  \vdots & & \ddots & \\
  0 & 1 & 0 & \\
\end{bmatrix}
\quad \text{and} \quad
H_{n,k} = \begin{bmatrix}
  a_{n,1} & a_{n,2} & \ldots & a_{n,k+1} \\
  a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-2,k+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n-k,1} & a_{n-k,2} & \ldots & a_{n-k,k+1} \\
\end{bmatrix}. \tag{1}
\]

The matrix \( G_k \) is said to be generalized Fibonomial matrix.

Now we give one of the our main results by the following Theorem.

**Theorem 2** For all \( n > 0 \),

\[ G_k^n = H_{n,k}. \]

**PROOF.** By the definitions of matrix \( H_{n,k} \) and Fibonomial coefficients, the proof can be easily seen for the case \( n = 1 \). Suppose that the equation holds for \( n \geq 1 \). Now we show that the equation holds for \( n + 1 \). Thus we write

\[ G_{k}^{n+1} = G_k G_k^n = G_k H_{n,k}. \]

From Lemma 1 and a property of matrix multiplication, we get

\[ G_{k}^{n+1} = G_k H_{n,k} = H_{n+1,k}. \]

Thus the theorem is proven.

It is valuable to note that when \( A = 1 \) and \( k = 1 \), we obtain well-known fact:

\[
G_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad H_{1,n} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.
\]

Now we give a linear recurrence relation for the generalized Fibonomials coefficient.
Corollary 3 For $n, k > 0$, the generalized Fibonomial coefficients satisfy the following order-$(k + 1)$ linear recurrence relation

$$a_{n+1,1} = \sum_{i=1}^{k+1} a_{1,i} a_{n-i+1,1}$$

or clearly

$$\binom{n+k+1}{k} = \binom{k+1}{k} \binom{n+k}{k} + \binom{k+1}{k-1} \binom{n+k-1}{k} + \ldots + (-1)^{(k-1)(k-2)/2} \binom{k+1}{1} \binom{n+1}{k} + (-1)^{k(k-1)/2} \binom{n}{k}$$

PROOF. Since $a_{n,1} = \binom{n+k}{k}$ and from matrix multiplication, by equating $(1,1)$ entries in the equation $H_{1,k} H_{n,k} = H_{n+1,k}$, the proof is easily seen.

Considering the generalized Fibonomial matrix, we obtain following identities.

Corollary 4 For $n > 0$, the following identities hold

$$a_{n-1,1} = a_{n,k+1}$$
$$a_{m+n+1-i,j} = \sum_{t=1}^{k+1} a_{n+1-i,t} a_{m+1-t,j} \text{ for all } m > 0$$
$$a_{n+t+1-i,j} = \sum_{m=1}^{k+1} a_{n+r+1-i,m} a_{t-r+1-i,j} \text{ for } t > 0 \text{ and } t > r$$
$$a_{n+1,1} = a_{1,1} a_{n,1} + a_{n,2}$$
$$a_{n+1,k+1} = a_{n,1} a_{1,k+1}$$
$$a_{n+1,i} = a_{1,i} a_{n,1} + a_{n,i+1} \text{ for } 2 \leq i \leq k$$

PROOF. The all identities above can be proved by considering a property of matrix multiplication in $H_{n+1} = H_n H_1$, $H_{n+m} = H_n H_m$ and $H_{n+t} = H_{n+r} H_{t-r}$ for $n, m > 0$ and $t > r$.

3 The eigenvalues of matrix $G_k$

In this section we determine the eigenvalues of matrix $G_k$. Since the matrix $G_k$ is a companion matrix, its characteristic polynomial can be easily derived. Let $f_{n,k}(x)$ denote the characteristic polynomial of the matrix $H_{n,k}$, then we have the following Corollary.
Corollary 5 For \( n, k > 0 \),

\[
f_{n,k}(x) = \sum_{t=0}^{k+1} (-1)^{t+1/2} \binom{n+k}{k-t+1} \binom{n+t-2}{t-1} x^{k+1-t}.
\]

Clearly the characteristic polynomial of \( G_k \) is given by

\[
f_{1,k}(x) = \sum_{i=0}^{k+1} (-1)^{i(i+1)/2} \binom{k+1}{i} x^{k-i+1}.
\]

Here we should note that in [13,9,10,4], the authors gave the characteristic equation of generalized Fibonomial coefficients as shown:

\[
C_n(x) = \sum_{h=0}^{n} (-1)^{h(h+1)/2} \binom{n}{h} x^{n-h}
\]

where \( \binom{n}{h} \) is defined as before.

Moreover in [4], the authors proved the conjecture of Horadam and Mahon, and they gave a very nice relationship between the characteristic polynomials of the generalized Fibonomial coefficients and the \( n \times n \) generalized Pascal matrix \( V_n \). Let \( R_n(x) \) be the characteristic polynomial of matrix \( V_n \). From [4], we have that

\[
C_n(x) = R_n(x)
\]

Therefore we derive a nice relationship between the characteristic polynomials of matrix \( G_k \) and the polynomial of \( V_n \) as follows:

\[
f_{1,n-1}(x) = C_n(x) = R_n(x).
\]

So we have the following Corollary.

Corollary 6 ([4]) Let \( \alpha, \beta = \left( A \pm \sqrt{A^2 + 4} \right) / 2 \). The characteristic roots of \( C_{m+1}(x) = f_{1,m}(x) \) are:

\[
\begin{align*}
\left\{ (-1)^j \alpha^{m-2j}, \ (1)^j \beta^{m-2j} \right\} & \quad j=0,1,...,k-1 & \text{if } m = 2k-1, \\
\left\{ (-1)^k, \ (1)^j \alpha^{m-2j}, \ (1)^j \beta^{m-2j} \right\} & \quad j=0,1,...,k-1 & \text{if } m = 2k.
\end{align*}
\]

Equivalently, the roots of \( f_{1,m} \) are

\[
\begin{align*}
\left\{ (-1)^{m/2}, \ (1)^j \alpha^{m-2j}, \ (1)^j \beta^{m-2j} \right\} & \quad 0 \leq j < m/2 & \text{if } m \equiv 0, 2 \pmod{4}, \\
\left\{ (-1)^j \alpha^{m-2j}, \ (1)^j \beta^{m-2j} \right\} & \quad 0 \leq j \leq m/2 & \text{if } m \equiv 1, 3 \pmod{4}.
\end{align*}
\]

As an example, when \( k = 4 \), after some simplifications, we write
\[
G_5 = \begin{bmatrix}
 a_1 & b_1 & -c_1 & -d_1 & e_1 & 1 \\
 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

and

\[
H_{n,5} = \begin{bmatrix}
 a_n & b_n & -c_n & -d_n & e_n & a_{n-1} \\
 a_{n-1} & b_{n-1} & -c_{n-1} & -d_{n-1} & e_{n-1} & a_{n-2} \\
 a_{n-2} & b_{n-2} & -c_{n-2} & -d_{n-2} & e_{n-2} & a_{n-3} \\
 a_{n-3} & b_{n-3} & -c_{n-3} & -d_{n-3} & e_{n-3} & a_{n-4} \\
 a_{n-4} & b_{n-4} & -c_{n-4} & -d_{n-4} & e_{n-4} & a_{n-5} \\
 a_{n-5} & b_{n-5} & -c_{n-5} & -d_{n-5} & e_{n-5} & a_{n-6} \\
\end{bmatrix}
\]

where

\[
a_n = \begin{cases} n + 5 \\
5 \\
\end{cases},
b_n = \begin{cases} n + 5 \\
4 \\
1 \\
\end{cases},
c_n = \begin{cases} n + 5 \\
3 \\
2 \\
\end{cases},
d_n = \begin{cases} n + 5 \\
2 \\
3 \\
\end{cases},
e_n = \begin{cases} n + 5 \\
1 \\
4 \\
\end{cases}
\]

The characteristic equation and its roots of \(G_5\) are given by

\[
f_{1,5} (x) = \sum_{i=0}^{6} (-1)^{i(i+1)/2} \binom{6}{i} x^{6-i}
\]

and \(\lambda_6 = \alpha^5, \lambda_5 = \beta^5, \lambda_4 = -\alpha^3, \lambda_3 = -\beta^3, \lambda_2 = \alpha, \lambda_1 = \beta\) where \(\alpha, \beta = \left( A \pm \sqrt{A^2 + 4} \right)/2\).

Thus we have the following result without proof.

**Corollary 7** For \(n, k > 0\),

\[
\prod_{i=1}^{k+1} (x - \lambda_i^n) = \sum_{t=0}^{k+1} (-1)^{t(t+1)/2} \binom{n+k}{k-t+1} \binom{n+t-2}{t-1} x^{k+1-t}
\]

and especially for \(n = 1\),

\[
\prod_{i=1}^{k+1} (x - \lambda_i) = \sum_{i=0}^{k+1} (-1)^{(i+1)/2} \binom{k+1}{i} x^{k+1-i}.
\]
Considering the results of Corollary 6, we derive the following facts:

\[
\begin{align*}
  f_{1,4m+4}(x) &= (x^4 - c_{4m}x^3 - d_{4m}x^2 - c_{4m}x + 1)f_{1,4m} \\
  f_{1,4m+5}(x) &= (x^4 - c_{4m+1}x^3 - d_{4m+1}x^2 + c_{4m+1}x + 1)f_{1,4m+1} \\
  f_{1,4m+6}(x) &= (x^4 - c_{4m+2}x^3 - d_{4m+2}x^2 - c_{4m+2}x + 1)f_{1,4m+2} \\
  f_{1,4m+7}(x) &= (x^4 - c_{4m+3}x^3 - d_{4m+3}x^2 + c_{4m+3}x + 1)f_{1,4m+3}.
\end{align*}
\]

In general we obtain the following identity:

\[
f_{1,t+4}(x) = (x^4 - c_{t}x^3 - d_{t}x^2 + (-1)^{t+1} c_{t}x + 1)f_{1,t}
\]

where \( c_{t} = v_{t+4} - v_{t+2} \) and \( d_{t} = v_{t+4}v_{t+2} + (-1)^{t+1} 2. \)

After arranging the right hand side of (2), equating the corresponding coefficients of \( x^n \) in the both sides gives us the following new result:

**Corollary 8** For all \( t \geq j, \)

\[
\begin{align*}
  \begin{pmatrix} t + 5 \\ i \end{pmatrix} &= \begin{pmatrix} t + 1 \\ i \end{pmatrix} + (-1)^{i+1} c_{t} \begin{pmatrix} t + 1 \\ i - 1 \end{pmatrix} + d_{t} \begin{pmatrix} t + 1 \\ i - 2 \end{pmatrix} + (-1)^{i+t} c_{t} \begin{pmatrix} t + 1 \\ i - 3 \end{pmatrix} + \begin{pmatrix} t + 1 \\ i - 4 \end{pmatrix}
\end{align*}
\]

where \( c_{t} \) is defined as before.

**PROOF.** From (2), we write
\[ f_{1,t+4}(x) = (x^4 - c_t x^3 - d_t x^2 + (-1)^{t+4} c_t x + 1)f_{1,t} \]
\[ = x^{t+5} - x^{t+4} \left( \begin{array}{c} t+1 \\ 1 \end{array} \right) + c_t \left( \begin{array}{c} t+1 \\ 0 \end{array} \right) \]
\[ - x^{t+3} \left( \begin{array}{c} t+1 \\ 2 \end{array} \right) - c_t \left( \begin{array}{c} t+1 \\ 1 \end{array} \right) + d_t \left( \begin{array}{c} t+1 \\ 0 \end{array} \right) \]
\[ + x^{t+2} \left( \begin{array}{c} t+1 \\ 3 \end{array} \right) + c_t \left( \begin{array}{c} t+1 \\ 2 \end{array} \right) + d_t \left( \begin{array}{c} t+1 \\ 1 \end{array} \right) - c_t \left( \begin{array}{c} t+1 \\ 0 \end{array} \right) \]
\[ + x^{t+1} \left( \begin{array}{c} t+1 \\ 4 \end{array} \right) - c_t \left( \begin{array}{c} t+1 \\ 3 \end{array} \right) + d_t \left( \begin{array}{c} t+1 \\ 2 \end{array} \right) + c_t \left( \begin{array}{c} t+1 \\ 1 \end{array} \right) \]
\[ + x^t \left( \begin{array}{c} t+1 \\ 5 \end{array} \right) + c_t \left( \begin{array}{c} t+1 \\ 4 \end{array} \right) + d_t \left( \begin{array}{c} t+1 \\ 3 \end{array} \right) - c_t \left( \begin{array}{c} t+1 \\ 2 \end{array} \right) + c_t \left( \begin{array}{c} t+1 \\ 1 \end{array} \right) \]
\[ \ldots \]
\[ + (-1)^{i+1/2} x^{t+5-i} \left( \begin{array}{c} t+1 \\ i \end{array} \right) + (-1)^{i+1} c_t \left( \begin{array}{c} t+1 \\ i-1 \end{array} \right) + d_t \left( \begin{array}{c} t+1 \\ i-2 \end{array} \right) \]
\[ + (-1)^{i+t} c_t \left( \begin{array}{c} t+1 \\ i-3 \end{array} \right) + \left( \begin{array}{c} t+1 \\ i-4 \end{array} \right) \]
\[ \ldots \]
\[ +1 \]

Comparing the coefficient of \( x^i \) for \( 1 \leq i \leq n \) in the above equation and the polynomial \( f_{1,t+4} \), the proof is complete.

In [3], the authors show that
\[ \text{tr}(V_n) = \frac{u_{(k+1)n}}{u_k} \]
where \( V_n \) is the generalized Pascal matrix.

Since the matrices \( H_{n,k} \) and \( V_n \) have the same eigenvalues, alternatively we have also that
\[ \text{tr}(H_{n,k}) = \frac{u_{(k+1)n}}{u_n}. \]

By Corollary 6, we can give the following result for both the generalized Fibonacci and Pascal matrices.

**Theorem 9** For \( n > 0 \),
\[ \text{tr}(H_{n,k}) = \begin{cases} \sum_{i=0}^{[m/2]} (-1)^i v_{(m-2i)n} & \text{if } m \equiv 1, 3 \pmod{4}, \\ \sum_{i=0}^{(m-2)/2} (-1)^i v_{(m-2i)n} & \text{if } m \equiv 0, 2 \pmod{4}. \end{cases} \]
4 Diagonalization of the matrix $G_k$ and the Generalized Binet formula

In this section we consider diagonalization of the matrix $G_k$ and then give the generalized Binet formula for the generalized Fibonomial coefficients. From Corollary 6, we know that if $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ be the eigenvalues of matrix $G_k$, then they are all distinct. Thus we can diagonalize the matrix $G_k$.

Define the $(k + 1) \times (k + 1)$ Vandermonde matrix $V$ and diagonal matrix $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{k+1})$ as shown:

$$V = \begin{bmatrix} 
\lambda_1^k & \lambda_2^k & \cdots & \lambda_{k+1}^k \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k+1}^2 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_{k+1} \\
1 & 1 & 1 & 1
\end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_{k+1}
\end{bmatrix}$$

Since $\lambda_i \neq \lambda_j$ all $i$ and $j$ for $1 \leq i, j \leq k + 1$, $\det V \neq 0$.

Let $V_j^{(i)}$ is the $(k + 1) \times (k + 1)$ matrix obtained from $V^T$ by replacing the $j$th column of $V$ by $w_i$ where

$$w_i = \begin{bmatrix} 
\lambda_1^{n-i+k+1} & \lambda_2^{n-i+k+1} & \cdots & \lambda_{k+1}^{n-i+k+1}
\end{bmatrix}^T.$$ 

Recalling $a_{n,i} = (-1)^{(i-1)(i-2)/2} \binom{n+k}{k-i+1} \binom{n+i-2}{i-1}$, we give the generalized Binet formulas for the generalized Fibonomial coefficients by the following Theorem.

**Theorem 10** For $n,k > 0$,

$$a_{n-i+1,j} = \frac{\det (V_j^{(i)})}{\det (V)}.$$

**PROOF.** One can check that $G_k V = V D$. Since $V$ is the invertible matrix and $G_k^n = H_{n,k}$, we write $G_k^n V = H_{n,k} V = V D^n$. Clearly we get the following linear equation system:

$$h_{i1} \lambda_1^k + h_{i2} \lambda_1^{k-1} + \ldots + h_{i,k-2} \lambda_1^2 + h_{i,k} \lambda_1 + h_{i,k+1} = \lambda_1^{n-i+k+1},$$

$$h_{i1} \lambda_2^k + h_{i2} \lambda_2^{k-1} + \ldots + h_{i,k-2} \lambda_2^2 + h_{i,k} \lambda_2 + h_{i,k+1} = \lambda_2^{n-i+k+1},$$

$$\vdots$$

$$h_{i1} \lambda_{k+1}^k + h_{i2} \lambda_{k+1}^{k-1} + \ldots + h_{i,k-2} \lambda_{k+1}^2 + h_{i,k} \lambda_{k+1} + h_{i,k+1} = \lambda_{k+1}^{n-i+k+1}. $$
Thus by the Cramer solution of the above system, we have the conclusion.

After some calculations, we present some identities as examples of Theorem 10.

**Case I** When \( k = 3 \), \( \det(V) = -u_2^2u_3D^3 \) and for \( n > 0 \) we have that

\[
\begin{align*}
\begin{pmatrix} n+2 \\ 3 \end{pmatrix} u_1u_2u_3(u_{3n+3} + (-1)^{n+1}u_{n+1}) / D,
\begin{pmatrix} n+3 \\ 2 \end{pmatrix} \begin{pmatrix} n \\ 1 \end{pmatrix} = (u_{3n+5} + (-1)^{n+1}(u_2u_n + u_{n+5}) / u_2D,
\begin{pmatrix} n+3 \\ 1 \end{pmatrix} \begin{pmatrix} n+1 \\ 2 \end{pmatrix} = (u_3u_{3n+4} + (-1)^{n+1}[u_2^2(u_{n+4} + u_n) - u_{n-4}]) / u_2u_3D
\end{align*}
\]

where \( D = A^2 + 4 \).

Especially when \( A = 1 \), \( u_n = F_n \) (nth Fibonacci number) and so

\[
\begin{align*}
F_nF_{n+1}F_{n+2} &= \left[ 4 (F_{3n+3} + 2(-1)^{n+1}F_{n+1}) \right] / 5 ,
F_nF_{n+2}F_{n+3} &= \left[ F_{3n+5} + (-1)^{n+1}(F_{n+5} + F_{n+1}) \right] / 5,
F_nF_{n+1}F_{n+3} &= \left[ F_{3n+4} + (-1)^n(F_{n+1} + L_n) \right] / 5.
\end{align*}
\]

**Case II** When \( k = 4 \), \( \det(V) = u_2^4v_2u_3^2D^5 \) and for \( n \geq 0 \)

\[
\begin{align*}
\begin{pmatrix} n+3 \\ 4 \end{pmatrix} v_{4n+6} - v_4 + v_1^2 + (-1)^{n+1}v_1v_2v_{2n+3} / u_2^2v_2u_3D^2,
\begin{pmatrix} n+4 \\ 3 \end{pmatrix} \begin{pmatrix} n \\ 1 \end{pmatrix} = v_{4n+9} + v_5 - v_3 + v_1 + (-1)^n(v_1v_{2n+2} - v_{2n+5} - v_{2n+9}),
\begin{pmatrix} n+4 \\ 2 \end{pmatrix} \begin{pmatrix} n+1 \\ 2 \end{pmatrix} = v_{4n+8} - v_6 - v_2 + v_0 + (-1)^n(v_1v_{2n+1} + v_{2n+6} - v_{2n+8}),
\begin{pmatrix} n+4 \\ 1 \end{pmatrix} \begin{pmatrix} n+2 \\ 3 \end{pmatrix} = v_{4n+7} - v_5 + v_3 - v_1 + (-1)^n(v_1v_{2n+4} - v_{2n+7} - v_{2n-1}).
\end{align*}
\]

From these results, we get

\[
\begin{align*}
F_nF_{n+1}F_{n+2}F_{n+3} &= (L_{4n+6} - 6 + 3(-1)^{n+1}L_{2n+3}) / 25, \\
F_nF_{n+2}F_{n+3}F_{n+4} &= (L_{4n+9} + 8 + (-1)^n(L_{2n+2} - 3L_{2n+7})) / 25, \\
F_nF_{n+1}F_{n+3}F_{n+4} &= (L_{4n+8} - 19 + (-1)^n(L_{2n+1} - L_{2n+7})) / 25, \\
F_nF_{n+1}F_{n+2}F_{n+4} &= (L_{4n+7} - 8 + (-1)^{n+1}(L_{2n-1} + L_{2n+3} + L_{2n+6})) / 25.
\end{align*}
\]

**Case III** When \( k = 5 \), \( \det(V) = u_2^4u_3^4u_4^2u_5D^{15} \) and for \( n \geq 0 \)

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As applications to the Fibonacci sequence, we obtain

\[
\begin{align*}
\{n+4\} \{5\} &= \frac{u_{5n+10}+(-1)^{n+1}(u_{2u3n+9}+u_3u_{3n+4})-\left(u_3(u_{n+6}+u_{n-2})+u_2^2u_{n+2}\right)}{u_2u_3u_4D^2}, \\
\{n+5\} \{n\} &= \frac{\left(u_{5n+14}+(-1)^{n+1}(u_{3u3n+14}+u_3u_{3n+10}-u_2u_3u_{3n+6})+u_4u_{n+7}-u_3(u_{n+2}+u_{n-2})\right)}{u_2u_3u_4D^2}, \\
\{n+5\} \{n+1\} &= \frac{\left(u_{5n+13}+(-1)^{n+1}(u_{2u3n+12}-u_2u_3u_{3n+5})-u_{n+11}-u_2u_3u_{n+4}-u_3u_{n-3}\right)}{u_2u_3u_4D^2}, \\
\{n+5\} \{n+2\} &= \frac{\left(u_{5n+12}+(-1)^{n+1}(u_{3u3n+10}+u_2u_3u_{3n+3})-u_4u_{n+7}-2u_2u_{n+1}+u_{n-2}-u_{n-6}\right)}{u_2u_3u_4D^2}, \\
\{n+5\} \{n+3\} &= \frac{\left(u_{5n+11}+(-1)^{n+1}(u_{2u3n+10}+u_3u_{3n+6})-u_{3n+11}-u_3u_{n+7}-u_3u_{n+3}-u_4u_{n-2}\right)}{u_2u_3u_4D^2}.
\end{align*}
\]

As applications to the Fibonacci sequence, we obtain

\[
\begin{align*}
F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4} &= (F_{5n+10}+(-1)^{n+1}(F_{3n+9}+2F_{3n+4}) \\
&\quad -2(F_{n+6}+F_{n-2})+F_{n+2})/25, \\
F_nF_{n+2}F_{n+3}F_{n+4}F_{n+5} &= (F_{5n+14}+(-1)^{n+1}(F_{3n+14}+F_{3n+9}+F_{3n+5}) \\
&\quad +3(F_{n+7}-2F_{n})))/25, \\
F_nF_{n+1}F_{n+3}F_{n+4}F_{n+5} &= (F_{5n+13}+(-1)^{n+1}(F_{3n+12}-2F_{3n+5}) \\
&\quad -(F_{n+11}+2F_{n+4}+2F_{n-3})))/25, \\
F_nF_{n+1}F_{n+2}F_{n+4}F_{n+5} &= (F_{5n+12}+(-1)^{n+1}(2F_{3n+10}+F_{3n+3}) \\
&\quad -(3F_{n+7}+2F_{n+1}+3F_{n-4})))/25, \\
F_nF_{n+1}F_{n+2}F_{n+3}F_{n+5} &= (F_{5n+11}+(-1)^{n+1}(F_{3n+10}+F_{3n+6}-F_{3n+1}) \\
&\quad -(2F_{n+7}+2F_{n+3}+3F_{n-2})))/25.
\end{align*}
\]

Let \(V_j^{(e_i)}\) be a \(k \times k\) matrix obtained from the Vandermonde matrix \(V\) by replacing the \(j\)th column of \(V\) by \(e_i\), where \(V\) is defined as before and \(e_i\) is the \(i\)th element of the natural basis for \(\mathbb{R}^n\), that is,

\[
e_i = (0,\ldots,0,\overbrace{1}^{\text{ith}},0,\ldots,0)^T
\]
Let $q_j^{(i)} = \frac{|v_j^{(e_i)}|}{|V|}$ where the $(k \times k)$ matrices $V_j^{(e_i)}$ and $V$ are defined as before.

Then we give the following Theorem.

**Theorem 11** Let $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ be the distinct roots of $x^{k+1} - a_{1,1}x^k - a_{1,2}x^{k-1} - \ldots - a_{1,k-1}x - a_{1,k+1} = 0$. For any integer $n$ and $1 \leq i \leq k+1$,

$$a_{n,i} = \sum_{j=1}^{k+1} q_j^{(i)} \lambda_j^{n+k}.$$ 

**PROOF.** We consider the following system of $k$ linear equations with $k$ unknowns $x_1, x_2, \ldots, x_k$: for $1 \leq i \leq k$

$$\begin{align*}
\lambda_1^k x_1 + \lambda_2^k x_2 + \ldots + \lambda_{k+1}^k x_{k+1} &= 0 \\
\vdots & \\
\lambda_1^{k-i+1} x_1 + \lambda_2^{k-i+1} x_2 + \ldots + \lambda_{k+1}^{k-i+1} x_{k+1} &= 0 \\
\lambda_1^{k-i} x_1 + \lambda_2^{k-i} x_2 + \ldots + \lambda_{k+1}^{k-i} x_{k+1} &= 1 \\
\lambda_1^{k-i-1} x_1 + \lambda_2^{k-i-1} x_2 + \ldots + \lambda_{k+1}^{k-i-1} x_{k+1} &= 0 \\
\vdots & \\
\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_{k+1} x_{k+1} &= 0 \\
x_1 + x_2 + \ldots + x_{k+1} &= 0.
\end{align*}$$
The system above is equivalent to

\[
\begin{bmatrix}
\lambda_1^k & \lambda_2^k & \ldots & \lambda_{k+1}^k \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{k-i+1} & \lambda_2^{k-i+1} & \ldots & \lambda_{k+1}^{k-i+1} \\
\lambda_1^{k-i} & \lambda_2^{k-i} & \ldots & \lambda_{k+1}^{k-i} \\
\lambda_1^{k-i-1} & \lambda_2^{k-i-1} & \ldots & \lambda_{k+1}^{k-i-1} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1 & \lambda_2 & \ldots & \lambda_{k+1} \\
1 & 1 & \ldots & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_j \\
\vdots \\
x_k \\
x_{k+1}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

by the solution of Vandermonde’s determinants and Cramer rule, we get

\[q_j^{(i)} = \frac{|V_j^{(e_i)}|}{|V|} (i = 1, 2, \ldots, k + 1).\]

Thus for \(n, k > 0 \) and \(1 \leq i \leq k + 1\),

\[f_n^i = \sum_{j=1}^{k+1} q_j^{(i)} \lambda_j^{n+k},\]

which completes the proof.

As an example of the above result, when \(k = 2, \gamma_1 = \alpha^2, \gamma_2 = \beta^2, \gamma_3 = -1\) are the roots of \(x^3 - a_{1,1}x^2 - a_{1,2}x - a_{1,3} = 0\). After some computations, we get

\[
\begin{align*}
q_1^{(1)} &= \frac{1}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)}, \\
q_2^{(1)} &= \frac{1}{(\gamma_2 - \gamma_3)(\gamma_2 - \gamma_1)}, \\
q_3^{(1)} &= \frac{1}{(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)}, \\
q_1^{(2)} &= -\frac{\gamma_2 + \gamma_3}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)}, \\
q_2^{(2)} &= \frac{\gamma_1 + \gamma_3}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_2)}, \\
q_3^{(2)} &= -\frac{\gamma_1 + \gamma_2}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)}, \\
q_1^{(3)} &= \frac{\gamma_2 \gamma_3}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)}, \\
q_2^{(3)} &= -\frac{\gamma_1 \gamma_3}{(\gamma_1 - \gamma_2)(\gamma_2 - \gamma_3)}, \\
q_3^{(3)} &= \frac{\gamma_1 \gamma_2}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)}.
\end{align*}
\]

Therefore, by Theorem 11, we get
\[ a_{n,1} = \binom{n+2}{2} \]
\[ = \frac{\gamma_1^{n+2}}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)} + \frac{\gamma_2^{n+2}}{(\gamma_2 - \gamma_3)(\gamma_2 - \gamma_1)} + \frac{\gamma_3^{n+2}}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)}, \]
\[ a_{n,2} = \binom{n+2}{1} \binom{n}{1} \]
\[ = -\frac{(\gamma_2 + \gamma_3)\gamma_1^{n+2}}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)} + \frac{(\gamma_1 + \gamma_3)\gamma_2^{n+2}}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_2)} - \frac{(\gamma_1 + \gamma_2)\gamma_3^{n+2}}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)} \]
and since \( \gamma_1\gamma_2\gamma_3 = -1, \)
\[ a_{n,3} = -\binom{n+1}{2} \]
\[ = \frac{\gamma_2\gamma_3\gamma_1^{n+2}}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)} + \frac{\gamma_1\gamma_3\gamma_2^{n+2}}{(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3)} + \frac{\gamma_1\gamma_2\gamma_3^{n+2}}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)} \]
\[ = -\frac{\gamma_1^{n+1}}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)} + \frac{\gamma_2^{n+1}}{(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3)} + \frac{\gamma_3^{n+1}}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)} \]
\[ = -a_{n-1,1}. \]

Since the definition of \( \{a_{n,i}\} \) for \( k = 2 \), the fact \( a_{n,3} = -a_{n-1,1} \) can also be seen.

5 On sums of generalized Fibonomial coefficients

In this section, we consider the sum of the generalized Fibonomial coefficients. To compute sum, we define a new generating matrix by extending the matrix \( G_k \) given in (1).

Define the \((k + 2) \times (k + 2)\) matrices \( T_k \) and \( W_n \) as follows:

\[
T_k = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
1 & & & \\
0 & G_k & & \\
& \vdots & & \\
0 & & &
\end{bmatrix}
\]
and
\[
W_n = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
S_n & & & \\
& \vdots & & H_{n,k} \\
S_{n-k} & & &
\end{bmatrix}
\]

where the matrices \( G_k \) and \( H_{n,k} \) be as before and also \( S_n \) is given by

\[
S_n = \sum_{i=0}^{n-1} a_{i,1} = \sum_{i=0}^{n-1} \binom{k+i}{k}.
\]
Then we have the following result.

**Theorem 12** For \( n,k > 0 \),
\[
T_k^n = W_{n,k}.
\]

**Proof.** Since \( S_{n+1} = a_{n,1} + S_n \) and by Theorem 2, we write the matrix recurrence relation \( W_{n,k} = W_{n-1,k} T_k \). By the induction method, we write \( W_{n,k} = W_{1,k} T_k^{n-1} \). From the definition of \( W_{n,k} \), we obtain \( W_{1,k} = T_k^1 \) and so \( W_{n,k} = T_k^n \). Thus we have the conclusion.

Here we should note that from Corollary 6, we know that the polynomial \( f_{1,m} \) has the root 1 for \( m \equiv 0 \pmod{4} \). Expanding the \( \det (\lambda I_{k+2} - T_k) \) with respect to the first row, it is easily seen that the matrix \( T \) has also the eigenvalue 1. Thus we see that the matrix \( T \) has double eigenvalue for \( m \equiv 0 \pmod{4} \). For \( m \not\equiv 0 \pmod{4} \), we can diagonalize the matrix \( T_k \) and so we derive an explicit formula for this sum.

Define the \( (k+2) \times (k+2) \) matrix \( M \) as shown:
\[
M = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
\delta & \ddots & \vdots & V \\
\delta & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]
where \( \delta = \left( 1 - \sum_{i=1}^{k+1} a_{1,i} \right)^{-1} \) and the Vandermonde matrix \( V \) is defined as before.

We can check that \( T_k M = M D_1 \) where \( T_k \) is as before and \( D_1 \) is a diagonal matrix such that \( D_1 = \text{diag} (1, \lambda_1, \lambda_2, \ldots, \lambda_{k+1}) \). Considering the matrix \( V \), we compute \( \det M \) with respect to the first row and then we find \( \det M = \det V \).

Then we give the following Theorem.

**Theorem 13** For \( n,k > 0 \) and \( k \not\equiv 0 \pmod{4} \),
\[
S_n = \frac{a_{n,1} + a_{n,2} + \ldots + a_{n,k+1} - 1}{\sum_{i=1}^{k+1} a_{1,i} - 1}.
\]

**Proof.** Since the matrix \( M \) is invertible, we write \( M^{-1} T_k M = D_1 \). Thus the matrix \( T_k \) is similar to the matrix \( D_1 \). Then we write \( T_k^n M = M D_1^n \). By
Theorem 12, \( W_{n,k}M = MD^n \). Equating the \((2,1)\)th elements of \( W_{n,k}M = MD^n \) and from a matrix multiplication, we obtain
\[
S_n + \delta \left( a_{n,1} + a_{n,2} + \ldots + a_{n,k+1} \right) = \delta.
\]
Thus
\[
S_n = \delta - \delta \left( a_{n,1} + a_{n,2} + \ldots + a_{n,k+1} \right) = \frac{(a_{n,1} + a_{n,2} + \ldots + a_{n,k+1} - 1)}{\sum_{i=1}^{k+1} a_{1,i} - 1}.
\]
The proof is complete.

As examples of Theorem 13, we give the following cases without required computations.

**Case I** When \( k = 3 \), after some simplifications, we obtain that for \( n \geq 0 \)
\[
\sum_{i=0}^{n} \binom{3+i}{3} = u_2u_3(u_nu_{n+2}u_{n+4} + u_{n+1}u_{n+3}u_{n+5} - u_3) / \nu_3.
\]

**Case II** When \( k = 4 \), the polynomial \( f_{1,4} \) has the roots \( \lambda_1 = \alpha^4, \lambda_2 = \beta^4, \lambda_3 = -\alpha^2, \lambda_4 = -\beta^2, \lambda_5 = 1 \). Since \( f_{1,4} \) has the root 1 and so the matrix \( T_4 \) has double eigenvalue 1 and linear dependent eigenvector, we can not diagonalize it.

**Case III** When \( k = 5 \), we get
\[
\sum_{i=0}^{n} \binom{i+5}{5} = \left[ \binom{n+5}{5} + \binom{n+5}{1} - \binom{n+5}{3} \right] / \nu_1\nu_3\nu_5.
\]

6 Generating functions

In this section, we give generating functions of the generalized Fibonomial coefficients. In [17], the author considers \( k \) sequences of general higher order linear recurrence relation with arbitrary initial conditions. Then the author give the generating functions for each sequence. Now we use the special results of [17] to obtain generating functions of generalized Fibonomial coefficients.

Define \( k \) sequences of \( k \)-th order linear recurrence relation \( \{f_n^i\} \) as shown, for \( n > 0 \) and \( 1 \leq i \leq k \)
\[
f_n^i = c_1f_{n-1}^i + c_2f_{n-2}^i + \ldots + c_kf_{n-k}^i.
\]
with initial conditions

\[ f^n_i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise,} \end{cases} \text{ for } 1 - k \leq n \leq 0 \]

where \( c_j, 1 \leq j \leq k, \) are constant coefficients, and \( f^n_i \) is the \( n \)th term of the \( i \)th sequence.

Using the approach of Kalman [15], Er [6] showed that

\[ G_n = A^n \]

where \( k \times k \) companion matrix \( A \) and the matrix \( G_n \) are as follows:

\[
A = \begin{bmatrix}
c_1 & c_2 & \ldots & c_{k-1} & c_k \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{bmatrix}
\]

and

\[
G_n = \begin{bmatrix}
f^1_n & f^2_n & \ldots & f^k_n \\
f^1_{n-1} & f^2_{n-1} & \ldots & f^k_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
f^1_{n-k+1} & f^2_{n-k+1} & \ldots & f^k_{n-k+1}
\end{bmatrix}
\]

Let \( G(i, x) = f^0_0 x^0 + f^1_1 x^1 + f^i_2 x^2 + \ldots + f^i_n x^n + \ldots \)

For the reader convenience, we give the following result with proof.

**Theorem 14** For \( n > 0 \) and \( 1 \leq i \leq k, \)

\[
G(i, x) = \frac{f^i_0 - \left( \sum_{v=m+1}^{k} c_v f^i_{m-v} \right) x^m}{1 - c_1 x - c_2 x^2 - \ldots - c_k x^k} \text{ for } m = 0, 1, 2, \ldots, k
\]

**PROOF.** Let \( G(i, x) = f^i_0 x^0 + f^i_1 x^1 + f^i_2 x^2 + \ldots + f^i_n x^n + \ldots \) Consider

\[
\left( 1 - c_1 x - c_2 x^2 - \ldots - c_k x^k \right) G(i, x)
\]

\[ = f^i_0 + f^i_1 x + f^i_2 x^2 + \ldots + f^i_k x^k + \ldots + f^i_n x^n + \ldots \\
- c_1 f^i_0 x - c_1 f^i_1 x^2 - c_1 f^i_2 x^3 - \ldots - c_1 f^i_{k-1} x^k - \ldots - c_1 f^i_{n-1} x^n - \ldots \\
- c_k f^i_0 x^k - c_k f^i_1 x^{k+1} - c_k f^i_2 x^{k+2} - \ldots - c_k f^i_{n-k} x^n - \ldots \\
\]

After some arrangements, we write
\[(1 - c_1 x - c_2 x^2 - \ldots - c_k x^k) G(i, x)\] 
\[= f_0^i + (f_1^i - c_1 f_0^i) x + (f_2^i - c_1 f_1^i - c_2 f_0^i) x^2 + \ldots +
\]
\[(f_{k-1}^i - c_1 f_{k-2}^i - \ldots - c_{k-1} f_0^i) x^{k-1} +
\]
\[+ (f_k^i - c_1 f_{k-1}^i - \ldots - c_{k-1} f_0^i - c_k f_1^i) x^k + \ldots +
\]
\[(f_n^i - c_1 f_{n-1}^i - \ldots - c_k f_{n-k}^i) x^n + \ldots.\]

Now we compute the coefficients of \(x^n\) of the equation above. Before this, from the definition of \(\{f_n^i\}\), we get

\[f_1^i = c_1 f_0^i + c_2 f_{i-1}^i + \ldots + c_k f_{i-k}^i\]
\[f_2^i = c_1 f_1^i + c_2 f_0^i + \ldots + c_k f_{2-k}^i\]
\[\ldots\]
\[f_{k-1}^i = c_1 f_{k-2}^i + c_2 f_{k-3}^i + \ldots + c_k f_0^i + c_k f_{-1}^i\]
\[f_k^i = c_1 f_{k-1}^i + c_2 f_{k-2}^i + \ldots + c_{k-1} f_1^i + c_k f_0^i\]
\[\ldots\]
\[f_n^i = c_1 f_{n-1}^i + c_2 f_{n-2}^i + \ldots + c_k f_{n-k}^i.\]

and so

\[f_1^i - c_1 f_0^i = -(c_2 f_{i-1}^i + \ldots + c_k f_{i-k}^i)\]
\[f_2^i - c_1 f_1^i - c_2 f_0^i = -(c_3 f_{i-1}^i + \ldots + c_k f_{2-k}^i)\]
\[\vdots\]
\[f_{k-1}^i - c_1 f_{k-2}^i - \ldots - c_{k-1} f_0^i = -c_k f_{-1}^i.\]

For \(n \geq k\) and from the definition of \(\{f_n^i\}\), the coefficients of \(x^n\) are all 0. Thus the proof is complete.

Let \(g(i, x) = a_{0,i} + a_{1,i} x + a_{2,i} x^2 + a_{3,i} x^3 + \ldots + a_{n,i} x^i + \ldots.\)

Then we have the following Corollary.

**Corollary 15** For \(1 \leq i \leq k + 1,\)

\[g(i, x) = \frac{a_{0,i} - (\sum_{m=m+1}^{k+2} a_{1,v} a_{m-v, i}) x^m}{1 - a_{1,1} x - a_{1,2} x^2 - \ldots - a_{1,k+1} x^{k+1}} \text{ for } m = 0, 1, 2, \ldots, k\]

where \(a_{n,i}\) is defined as before.
Especially when \( i = 1 \) in Corollary 15, we get
\[
\sum_{n=0}^{\infty} \binom{n + k}{k} = \frac{1}{1 - \binom{k+1}{k} x - \binom{k+1}{k-1} x^2 + \binom{k+1}{k-2} x^3 + \ldots + (-1)^{k+1} x^{k+1}}.
\]

Now we consider first three cases:

**Case I** When \( k = 3 \), we obtain
\[
\sum_{n=1}^{\infty} \binom{n + 3}{3} x^n = \frac{1}{1 - \frac{4}{3} x - \frac{4}{2} x^2 + \frac{4}{1} x^3 + x^4},
\]
\[
\sum_{n=1}^{\infty} \binom{n}{1} \binom{n + 3}{2} x^n = \frac{u_3 u_4 x - v_1 v_3 x^2 - v_1 x^3}{1 - \frac{4}{3} x - \frac{4}{2} x^2 + \frac{4}{1} x^3 + x^4}
\]
and
\[
\sum_{n=1}^{\infty} \binom{n + 1}{2} \binom{n + 3}{1} x^n = \frac{u_2 u_4 x + v_1 x^2}{1 - \frac{4}{3} x - \frac{4}{2} x^2 + \frac{4}{1} x^3 + x^4}.
\]

For example, the following generating function for the triple product of consecutive Fibonacci numbers can be found [18]:
\[
\sum_{n=0}^{\infty} F_n F_{n+1} F_{n+2} x^n = \frac{2x}{1 - 3x - 6x^2 + 3x^3 + x^4}.
\]

Indeed the above generating function can also be rewritten via the Fibonomial coefficients \( \binom{n}{3}^F \) as shown:
\[
\sum_{n=0}^{\infty} \binom{n + 2}{3}^F x^n = \frac{x}{1 - 3x - 6x^2 + 3x^3 + x^4}.
\]

For the generating function of the powers of Fibonacci numbers, we can refer to [27] and [12].

**Case II** When \( k = 4 \), we get
\[
\sum_{n=0}^{\infty} \binom{n + 4}{4} x^n = \frac{1}{1 - \frac{5}{4} x - \frac{5}{3} x^2 + \frac{5}{2} x^3 + \frac{5}{1} x^4 - x^5},
\]
\[
\sum_{n=0}^{\infty} \binom{n}{1} \binom{n + 4}{3} x^n = \frac{v_2 u_5 x - v_2 u_5 x^2 - u_5 x^3 + x^4}{1 - \frac{5}{4} x - \frac{5}{3} x^2 + \frac{5}{2} x^3 + \frac{5}{1} x^4 - x^5},
\]
\[
\sum_{n=0}^{\infty} \binom{n + 1}{2} \binom{n + 4}{2} x^n = \frac{v_2 u_5 x + u_5 x^2 - x^3}{1 - \frac{5}{4} x - \frac{5}{3} x^2 + \frac{5}{2} x^3 + \frac{5}{1} x^4 - x^5},
\]
\[
\sum_{n=0}^{\infty} \binom{n + 2}{2} \binom{n + 4}{1} x^n = \frac{u_5 x - x^2}{1 - \frac{5}{4} x - \frac{5}{3} x^2 + \frac{5}{2} x^3 + \frac{5}{1} x^4 - x^5},
\]
\[
\sum_{n=0}^{\infty} \binom{n + 2}{3} \binom{n + 4}{1} x^n = \frac{u_5 x - x^2}{1 - \frac{5}{4} x - \frac{5}{3} x^2 + \frac{5}{2} x^3 + \frac{5}{1} x^4 - x^5},
\]
Case III When $k = 5$, we get

\[
\sum_{n=0}^{\infty} \left\{ \frac{n+5}{5} \right\} x^n = \frac{1}{1 - \frac{6}{5} x - \frac{6}{4} x^2 + \frac{6}{3} x^3 + \frac{6}{2} x^4 - \frac{6}{1} x^5 - x^6},
\]

\[
\sum_{n=0}^{\infty} \left\{ \frac{n+5}{4} \right\} x^n = \frac{u_5u_6u_2x - u_5u_6u_3x^2 - u_5u_6x^4 + x^5}{1 - \frac{6}{5} x - \frac{6}{4} x^2 + \frac{6}{3} x^3 + \frac{6}{2} x^4 - \frac{6}{1} x^5 - x^6},
\]

\[
\sum_{n=0}^{\infty} \left\{ \frac{n+5}{3} \right\} x^n = \frac{v_3u_5x + (u_5u_6/u_2)x^2 - u_6x^3 - x^4}{1 - \frac{6}{5} x - \frac{6}{4} x^2 + \frac{6}{3} x^3 + \frac{6}{2} x^4 - \frac{6}{1} x^5 - x^6},
\]

\[
\sum_{n=0}^{\infty} \left\{ \frac{n+5}{2} \right\} x^n = \frac{u_5u_6x - u_6x^2 - x^3}{1 - \frac{6}{5} x - \frac{6}{4} x^2 + \frac{6}{3} x^3 + \frac{6}{2} x^4 - \frac{6}{1} x^5 - x^6},
\]

\[
\sum_{n=0}^{\infty} \left\{ \frac{n+5}{1} \right\} x^n = \frac{(u_5u_6/u_4)x + u_2x^2}{1 - \frac{6}{5} x - \frac{6}{4} x^2 + \frac{6}{3} x^3 + \frac{6}{2} x^4 - \frac{6}{1} x^5 - x^6}.
\]

7 Combinatorial Representations of the generalized Fibonomial Coefficients

In this section, we give combinatorial representations for the generalized Fibonomial coefficients. In [2], by considering a $k \times k$ companion matrix $A$ given in (4) and its $n$th power, the authors derive an explicit formula for the elements in the $n$th power of the matrix $A$. Let us recall their result by the following Theorem.

**Theorem 16** ([2]) Let the matrix $A = (a_{ij})$ be as in (4). The $(i, j)$ entry $a_{ij}^{(n)}$ in the matrix $A_k^n$ is given by the following formula:

\[
a_{ij}^{(n)}(c_1, c_2, \ldots, c_k) = \sum_{(t_1, t_2, \ldots, t_k)} \frac{t_1 + t_2 + \ldots + t_k}{t_i + t_2 + \ldots + t_k} \times \binom{t_1 + t_2 + \ldots + t_k}{t_i, t_2, \ldots, t_k} c_1^{t_1} \cdots c_k^{t_k} \tag{5}
\]

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \ldots + kt_k = n - i + j$, and the coefficients in (5) is defined to be 1 if $n = i - j$.

Thus we give the following results.

**Corollary 17** Let $a_{n,i}$ denote the generalized Fibonomial coefficient. Then

\[
a_{n-i+1,j} = \sum_{(t_1, t_2, \ldots, t_k)} \frac{t_1 + t_2 + \ldots + t_k}{t_1 + t_2 + \ldots + t_k} \times \binom{t_1 + t_2 + \ldots + t_k + 1}{t_1, t_2, \ldots, t_k} a_{1,1}^{t_1} \cdots a_{1,k+1}^{t_k+1}
\]

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \ldots + (k+1)t_{k+1} = n - i + j$.
PROOF. Considering matrices $G_k$ and $A$, the proof is seen from the result of Theorem 16.

**Corollary 18** For $n \geq 0$,

$$\binom{n+k}{k} = \sum_{(t_1, t_2, \ldots, t_{k+1})} \binom{t_1 + t_2 + \ldots + t_{k+1}}{t_1, t_2, \ldots, t_{k+1}} a_{t_1,1} \cdots a_{t_{k+1},k+1}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \ldots + (k+1)t_{k+1} = n$.

**PROOF.** In Corollary 17, if we take $i = j = 1$, then $a_{n,1} = \binom{n+k}{k}$ and we have the conclusion from (1).

For example, when $k = 3$, we obtain the following representations from Corollary 17:

$$\binom{n+3}{3} = \sum_{(r_1, r_2, r_3, r_4)} \binom{r_1+r_2+r_3+r_4}{r_1, r_2, r_3, r_4} (-1)^{r_3+r_4} \binom{4}{3} \binom{r_1+r_3}{2} \binom{4}{r_2}$$

where the summation is over nonnegative integers satisfying $r_1 + 2r_2 + 3r_3 + 4r_4 = n - 1$.

$$\binom{n+1}{1} \binom{n+3}{2} = \sum_{(r_1, r_2, r_3, r_4)} \frac{r_2+r_3+r_4}{r_1+r_2+r_3+r_4} \binom{r_1+r_2+r_3+r_4}{r_1, r_2, r_3, r_4} (-1)^{r_3+r_4} \binom{4}{3} \binom{r_1+r_3}{2} \binom{4}{r_2}$$

where the summation is over nonnegative integers satisfying $r_1 + 2r_2 + 3r_3 + 4r_4 = n + 1$.

$$\binom{n+1}{2} \binom{n+3}{1} = \sum_{(r_1, r_2, r_3, r_4)} \frac{r_3+r_4}{r_1+r_2+r_3+r_4} \binom{r_1+r_2+r_3+r_4}{r_1, r_2, r_3, r_4} (-1)^{r_3+r_4} \binom{4}{3} \binom{r_1+r_3}{2} \binom{4}{r_2}$$

where the summation is over nonnegative integers satisfying $r_1 + 2r_2 + 3r_3 + 4r_4 = n + 2$.

**8 Determinantal Representations for the coefficients**

In this section, we determine some relationships between determinants of certain matrices and the generalized Fibonomial coefficients. Some similar relationships have been derived by some authors (see for more detail [19,20,24]).
Especially Lind [19] gave the first result for the relationship between determinant of certain Hessenberg matrices and the generalized Fibonomial coefficients. For convenience, we give the result of Lind [19]: Let $D_{n,k}$ denote the recurrent $n \times n$ determinant $|a_{rs}|$, where $a_{rs} = (-1)^{(s+r+1)(s-r+2)/2} \binom{k+1}{s-r+1}$, $r, s = 1, 2, \ldots, n$. Then the author showed that $D_{n,k} = n_{n+k}^k F$, where $n_{n+i}^i F$ is the Fibonomial coefficient. The analogous result holds when the Fibonacci sequence is replaced by an ordinary second-order recurring sequence.

Now, by constructing super-diagonal matrices, we give some new general results as given by Lind above.

**Definition 19** For $n > k > 0$, let $M_n = [m_{ij}]$ denote the $k$-superdiagonal matrix of order $n$ with $m_{ii} = a_{i,1}$ for $1 \leq i \leq n$, $m_{i,i+1} = a_{i,2}$ for $1 \leq i \leq n-1$, $\ldots$, $m_{i,i+k} = a_{i,k+1}$ for $1 \leq i \leq n-k$.

Clearly the matrix $M_n$ takes the form

$$M_n = \begin{bmatrix}
  a_{1,1} & a_{1,2} & \ldots & a_{1,k+1} \\
  -1 & a_{1,1} & a_{1,2} & \ldots & a_{1,k+1} \\
  \vdots & \vdots & \ddots & \ddots & \ddots \\
  a_{1,1} & a_{1,2} & \ldots & a_{1,k+1} \\
  -1 & a_{1,1} & \ldots & \vdots \\
  -1 & a_{1,1} & a_{1,2} \\
  0 & -1 & a_{1,1} \\
\end{bmatrix}.$$

(6)

**Theorem 20** Then for $n > 0$,

$$|M_n| = a_{n,1}.$$

where $|M_1| = a_1$.

**Proof.** (Induction on $n$) If $n = 2$, then we have

$$|M_2| = \begin{vmatrix}
  a_{1,1} & a_{1,2} \\
  -1 & a_{1,1} \\
\end{vmatrix} = a_{1,1}a_{1,1} + a_{1,2} = a_{2,1}.$$

Suppose that the equation holds for $n$. Then we show that the equation is true for $n + 1$. Expanding $|M_{n+1}|$ by the Laplace expansion of determinant according to the last column and by the definition of $M_n$, we get

$$|M_{n+1}| = a_{1,1} |M_n| + a_{1,2} |M_{n-1}| + a_{1,3} |M_{n-2}| + \ldots + a_{1,k+1} |M_{n-k}|.$$

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By our assumption and the recurrence relation of \( \{a_{n,1}\} \), we write

\[
|M_{n+1}| = a_{1,1}a_{n,1} + a_{1,2}a_{n-1,1} + a_{1,3}a_{n-2,1} + \ldots + a_{1,k+1}a_{n-k,1} = a_{n+1,1}.
\]

Thus the theorem is proven.

For example, we take \( A = 1 \), then \( u_n = F_n \) (\( n \)th Fibonacci number) and by Theorem 20, we have

\[
\begin{vmatrix}
3 & 6 & -3 & -1 & 0 \\
-1 & 3 & 6 & -3 & \cdot \cdot \\
-1 & 3 & 6 & \cdot \cdot & -1 \\
\cdot \cdot & \cdot \cdot & \cdot \cdot & -3 \\
0 & -1 & 3 & 6 & \cdot \cdot \\
\end{vmatrix}_{n \times n}
= \begin{bmatrix} n + 3 \\ 3 \end{bmatrix}_F.
\]

Let \( M_n (k) \) denote the matrix obtained from the matrix \( M_n = [m_{ij}] \) by taking \( m_{1,j} = 0 \) from \( 1 \leq j \leq k \). For example \( M_3 (1,2) \) has the form:

\[
M_4 (2) = \begin{bmatrix}
0 & 0 & a_{1,3} & a_{1,4} \\
-1 & a_{1,1} & a_{1,2} & a_{1,3} \\
0 & -1 & a_{1,1} & a_{1,2} \\
0 & 0 & -1 & a_{1,1} \\
\end{bmatrix}.
\]

Now we determine some relationships between the sequences \( \{a_{n,i}\} \) for \( 1 < i \leq k \) and the consecutive determinants of matrix \( M_n (1, k) \) for some certain \( k \).

**Theorem 21** For \( n > k \geq i \geq 1 \),

\[
|M_n (i)| = a_{n-i,i+1}.
\]

**PROOF.** Expanding \( |M_n (i)| \) according to the first row by the definitions of \( M_n (i) \) and \( M_n \),

\[
|M_n (i)| = a_{1,i+1} |M_{n-i-1}| + a_{1,i+2} |M_{n-i-2}| + \ldots + a_{1,k+1} |M_{n-k-1}|.
\]
Expanding the value of $\det M_n$ with respect to the first row by considering the definition of $M_n$ and the value of $\det M_n(i)$, we may write after some simplifications

$$|M_n(i)| = |M_n| - a_{1,1} |M_{n-1}| - a_{1,2} |M_{n-2}| - a_{1,3} |M_{n-3}| - \ldots - a_{1,i} |M_{n-i}|$$

which, by our assumption and Lemma 1, satisfies

$$|M_n(i)| = a_{n,1} - a_{1,1}a_{n-1,1} - a_{1,2}a_{n-2,1} - a_{1,3}a_{n-3,1} - \ldots - a_{1,i}a_{n-i,1}$$

$$= a_{n-1,2} - a_{1,2}a_{n-2,1} - a_{1,3}a_{n-3,1} - \ldots - a_{1,i}a_{n-i,1}$$

$$= a_{n-2,3} - a_{1,3}a_{n-3,1} - \ldots - a_{1,i}a_{n-i,1}$$

$$\vdots$$

$$= a_{n-i+1,i} - a_{1,i}a_{n-i,1}$$

$$= a_{n-i,i+1}.$$

Thus the proof is complete.

We now define an $n \times n$ upper Hessenberg matrix $D_n$ as in the following compact form:

$$D_n = \begin{bmatrix}
1 & 1 & \ldots & 1 \\
-1 & 0 & M_{n-1} \\
0 & \end{bmatrix}$$

(7)

where $M_n$ is as before.

Then we give the following result without proof.

**Theorem 22** Then for $n > 1$

$$|D_n| = S_n$$

where $S_n$ is as before.

**PROOF.** (Induction on $n$) If $n = 2$, then

$$|D_2| = \begin{vmatrix}
1 & 1 \\
-1 & a_1 \\
\end{vmatrix} = a_{1,1} + 1 = a_{1,1} + a_{0,1} = S_2.$$

Suppose that the equation holds for $n$ and $n \geq 2$. Then we show that the equation holds for $n + 1$. Expanding $|D_{n+1}|$ according to the first column and
by our assumption, we obtain

\[ |D_{n+1}| = |M_n| + |D_n| \]

which, by Theorem 20, satisfies

\[ |D_{n+1}| = a_{n,1} + S_n = S_{n+1}. \]

Thus the proof is complete.

To derive other similar some relationships between determinants of certain matrices and the sums of the other products, we define \( n \times n \) matrix \( T_{n,i} \) for \( 1 \leq i \leq k \) as follows:

\[
T_{n,i} = \begin{bmatrix}
0 & & & & 0 \\
& \cdots & & & \\
& 0 & \cdots & & M_n(i) & 0 \\
& & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & & -1 & 1
\end{bmatrix}
\]

where the matrix \( M_n(i) \) is as before.

Expanding \( |T_{n,i}| \) according to the last row, we have the following Theorems without proof.

**Theorem 23** For \( n \geq k \geq i \geq 1 \),

\[ |T_{n,i}| = \sum_{i=0}^{n-i-1} a_{i,1}. \]

9 Conclusion

Throughout this paper we consider the recurrence \( \{u_n\} \) and its generalized Fibonacci coefficients. Using given results in this paper, one can obtain many applications to the recurrence \( \{u_n\} \) or its special cases, Fibonacci or Pell sequences. We give some special cases of our results by taking \( k = 3, 4 \) and \( 5 \). For the case \( k = 2 \), the results can be found in [16]. Moreover one can obtain many analogous results for the recurrence \( \{U_n\} \) defined by \( U_n = AU_{n-1} - BU_{n-2} \) with \( U_0 = 0 \) and \( U_1 = 1 \). However one should be aware of that in case of recurrence \( \{U_n\} \), a generator matrix can not be found by itself.
References


