Dynamics and synchronization of new hyperchaotic complex Lorenz system

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1. Introduction

Chaotic systems have been studied for more than 45 years. In 1963, Edward Lorenz discovered the first chaotic system, almost by accident, while searching for equations explaining the behavior of weather patterns [1]. Since then, many methods for the analysis and synthesis of chaotic systems have been proposed.

In 1982, Fowler introduced a complex Lorenz as a generalization of the real Lorenz [2,3]. Basic properties and chaotic synchronization of the complex Lorenz model are studied by Mahmoud et al. [4]:

\[
\begin{align*}
\dot{x} &= \alpha (y - x), \\
\dot{y} &= \gamma x - y - xz, \\
\dot{z} &= -\beta z + 1/2 (xy + \bar{xy}),
\end{align*}
\]

where \(\alpha, \gamma\) and \(\beta\) are positive real parameters, \(x = u_1 + iu_2, y = u_3 + iu_4\) are complex functions, \(i = \sqrt{-1}, z = u_5, u_i\) are real functions, \(i = 1, 2, \ldots, 5\). Dots represent derivatives with respect to time and an overbar denotes complex conjugate variables. The \(x, y, z\) variables of (1) are related respectively to electric field and the atomic polarization amplitudes and the population inversion in a ring laser system of two-level atoms, for more details, see [5–7]. In recent years, beside the famous complex Lorenz model, several other such examples have been proposed in the literature, notably the so-called complex...
Chen and Lü systems, which may be thought to belong to the same class as the Lorenz equation [8–12, and references therein]. These systems which involve complex variables are used to describe the physics of a detuned laser, rotating fluids, disk dynamos, electronic circuits and particle beam dynamics in high energy accelerators.

In the last three decades several researchers have focused their attention on the study of hyperchaotic systems in many important fields in applied nonlinear sciences, e.g., laser physics, secure communications, nonlinear circuits, synchronization, control, neural networks and active wave propagation [13–16]. A hyperchaotic attractor is typically defined as chaotic behavior with at least two positive Lyapunov exponents. The minimal dimension for a (continuous) hyperchaotic system is 4 [17,18].

Many ideas have been employed to generate hyperchaotic systems in real variables. Some of these are adding state feedback control and periodic forcing to chaotic systems (e.g. [19–21] and references therein).

In 2009, Yang et al. [22] introduced the hyperchaotic Lorenz system in real form by adding a linear controller as:

\[
\begin{align*}
\dot{x} &= 10(y - x), \\
\dot{y} &= 28x - y - xz + w, \\
\dot{z} &= xy - \frac{8}{3}z, \\
\dot{w} &= k_1x + k_2y,
\end{align*}
\]

where \(k_1\) and \(k_2\) are two constant parameters, determining the chaotic and hyperchaotic behaviors. This system has only a trivial fixed point.

Very recently, Mahmoud et al. constructed the new hyperchaotic complex Lorenz systems by extending the ideas of adding state feedback controls and introducing the complex periodic forcing to system (1) [23,24].

In this work, we introduce a new hyperchaotic complex Lorenz system by adding a linear controller to (1) as:

\[
\begin{align*}
\dot{x} &= \alpha(y - x), \\
\dot{y} &= \gamma x - y - xz + w, \\
\dot{z} &= 1/2(xy + x\bar{y}) - \beta z, \\
\dot{w} &= k_1x + k_2y,
\end{align*}
\]

where \(\alpha, \gamma, \beta\) are positive real parameters that change the behavior of the system, \(x = u_1 + \bar{u}_2, y = u_3 + \bar{u}_4, w = u_5 + \bar{u}_6\) are complex variables, \(z = u_5, u_6\) are real variables, \(l = 1, 2, \ldots, 7\) and \(k_1, k_2\) are control parameters (real parameters). Thus, system (3) is a 7-dimensional (7D) continuous real autonomous hyperchaotic system, while most of hyperchaotic systems in the literature are (4D) systems.

The goal of this paper is to construct a new hyperchaotic complex Lorenz system by adding a linear controller to (1) and to consider this new system as a generalization of the hyperchaotic real Lorenz system [22]. We investigate the dynamics and synchronization of this new system.

This paper is organized as follows: in the next section symmetry, invariance, fixed points and stability analysis of the trivial fixed points of (3) are discussed. The complex behavior of (3) is studied. Numerically the range of parameter values of the system at which hyperchaotic attractors exist is calculated based on the calculations of Lyapunov exponents. The signs of Lyapunov exponents provide a good classification of the dynamics of (3). The fractional Lyapunov dimension [25–27] is calculated to hyperchaotic attractors of (3). Bifurcation analysis is used to demonstrate chaotic and hyperchaotic behaviors of (3). Some figures are presented to show our investigations. In the third section the active control method based on stability analysis [24,28–31] is used to synchronize hyperchaotic attractors of (3). Analytical criteria concerning the stability of this technique are implemented and excellent agreement is found upon comparison with numerical experiments. Plotting our results in two figures, we demonstrate graphically the success of hyperchaos synchronization and the time evolution of their errors. In the last section the main conclusions of our investigations are summarized.

2. Dynamics of system (3)

In this section we study the complex behaviors of new system (3). The real version of (3) reads:

\[
\begin{align*}
\dot{u}_1 &= \alpha(u_3 - u_1), \\
\dot{u}_2 &= \alpha(u_4 - u_2), \\
\dot{u}_3 &= \gamma u_1 - u_1u_5 - u_3 + u_6, \\
\dot{u}_4 &= \gamma u_2 - u_2u_5 - u_4 + u_7, \\
\dot{u}_5 &= u_1u_3 + u_2u_4 - \beta u_5, \\
\dot{u}_6 &= k_1u_1 + k_2u_3, \\
\dot{u}_7 &= k_1u_2 + k_2u_4.
\end{align*}
\]

System (4) has the following basic dynamical properties:

2.1. Symmetry and invariance:

From (4), we note that this system is invariant under the transformation:

\[(u_1, u_2, u_3, u_4, u_5, u_6, u_7) \rightarrow (-u_1, -u_2, -u_3, -u_4, u_5, -u_6, -u_7).\]

Therefore, if \((u_1, u_2, u_3, u_4, u_5, u_6, u_7)\) is a solution of (4), then \((-u_1, -u_2, -u_3, -u_4, u_5, -u_6, -u_7)\) is also a solution of the same system.
2.2. Dissipation:

System (4) is dissipative under the condition $2\alpha + \beta + 2 > 0$ since:

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial u_2} + \cdots + \frac{\partial u_7}{\partial u_7} = -(2\alpha + \beta + 2).$$

2.3. Fixed points of (4)

2.3.1. When $k_1 + k_2 \neq 0$

The equilibria of system (4) in this case can be calculated by solving the following system of equations $\dot{u}_1 = 0, \dot{u}_2 = 0, \dot{u}_3 = 0, \dot{u}_4 = 0, \dot{u}_5 = 0, \dot{u}_6 = 0$ and $\dot{u}_7 = 0$ to get one isolated fixed point $E_0 = (0, 0, 0, 0, 0, 0, 0)$.

To study the stability of $E_0$, the Jacobian matrix of system (4) at $E_0$ is

$$L_{ij} = \begin{pmatrix} -\alpha & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & \alpha & 0 & 0 & 0 \\ \gamma & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & \gamma & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\beta & 0 & 0 \\ k_1 & 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & k_1 & 0 & k_2 & 0 & 0 & 0 \end{pmatrix}.$$  

The characteristic polynomial of $L_{ij}$ at the equilibrium $E_0$ is:

$$(\lambda + \beta)(\lambda^2 + \lambda^2(\alpha + 1) + \lambda(\alpha(1 - \gamma) - k_2) - \alpha(k_1 + k_2))^2 = 0.$$  

(5)

According to the Routh–Hurwitz theorem the necessary and sufficient conditions for all roots to have negative real parts is if and only if:

$$\beta > 0, \quad \alpha > -1, \quad \alpha(k_1 + k_2) < 0, \quad \text{and} \quad k_1 > \frac{k_2}{\alpha} + (\alpha + 1)(\gamma - 1).$$

Otherwise, this is an unstable fixed point.

2.3.2. When $k_1 + k_2 = 0$ and $k_1 \neq 0, k_2 \neq 0$

System (4) when $k_1 = -k_2$ will take this form:

$$\dot{u}_1 = \alpha (u_3 - u_1), \quad \dot{u}_2 = \alpha (u_4 - u_2),$$  

$$\dot{u}_3 = \gamma u_1 - u_1 u_5 - u_3 + u_6,$$

$$\dot{u}_4 = \gamma u_2 - u_2 u_5 - u_4 + u_7,$$

$$\dot{u}_5 = u_1 u_3 + u_2 u_4 - \beta u_5,$$

$$\dot{u}_6 = k_1 (u_1 - u_3), \quad \dot{u}_7 = k_1 (u_2 - u_4).$$

System (6) has an isolated one at $(0, 0, 0, 0, 0, 0, 0)$, as well as a whole circle of equilibria in $(u_1, u_2)$ space given by the expression:

$$u_1^2 + u_2^2 = \beta u_5.$$  

(7)

Eq. (7) is a circle with center $(0, 0)$ and radius $r = \sqrt{\beta}u_5$.

Let $u_1 = u_3 = r \cos \theta$ and $u_2 = u_4 = r \sin \theta$, where $\theta \in [0, 2\pi]$, we get the non-isolated (nontrivial) fixed points as:

$$E_\theta = \begin{pmatrix} r \cos \theta, & r \sin \theta, & r \cos \theta, & r \sin \theta, & r^2/\beta, & u_e^x, & u_e^y \end{pmatrix}, \quad \text{for} \ \theta \in [0, 2\pi],$$

(8)

where $u_e^x = r \cos \theta((1 - \gamma) + r^2/\beta), u_e^y = r \sin \theta((1 - \gamma) + r^2/\beta)$.

To study the stability of $(0, 0, 0, 0, 0, 0, 0)$, the characteristic polynomial of $L_{ij}$ at this equilibrium is:

$$\lambda^2(\lambda + \beta)(\lambda^2 + (\alpha + 1)\lambda + \alpha - \alpha \gamma + k_1)^2 = 0.$$  

(9)

So, this trivial fixed point is stable if $\beta > 0$ and $k_1 > \alpha(\gamma - 1)$. Otherwise, this equilibrium is unstable.

2.4. Lyapunov exponents and dimensions

In this subsection we calculate both Lyapunov exponents and the Lyapunov dimension of system (4).
Hyperchaotic attractors

Solutions approach fixed points

(iii) existence of chaotic attractors with two zero Lyapunov exponents (not with one zero Lyapunov exponent).

2.4.1. Fix \( \alpha = 14, \beta = 3, k_1 = -5, k_2 = -4 \) and vary \( \gamma \)

Using (12) we calculate \( \lambda_i, i = 1, 2, \ldots, 7 \) and the values of \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) versus \( \gamma \) are plotted in Fig. 1a, while the values of \( \lambda_5, \lambda_6 \) and \( \lambda_7 \) versus \( \gamma \) are shown in Fig. 1b. From Fig. 1b, it is clear that all the values of \( \lambda_6 \) and \( \lambda_7 \) are negative, while the values of \( \lambda_5 \) are negative or equal to zero. For other choices of \( k_1, k_2 \) and \( \alpha \) we found that \( \lambda_5, \lambda_6 \) and \( \lambda_7 \) are negative and we omitted their figures.

It is clear that from Fig. 1a, when \( \gamma \in (23.8, 60) \) the system (3) has hyperchaotic attractors, while it has chaotic attractors when \( \gamma \) lies in \( (20.3, 23.8) \). It has also solutions that approach trivial fixed points when the value of \( \gamma \) lies in \( (0, 0.7) \). The periodic attractors of (4) exist when \( \gamma \) lies in the interval \( (0.7, 19.2) \) and quasi-periodic attractors for \( \gamma \in (19.2, 20.3) \).

As it is shown in (Fig. 1a, b) and based on signs of \( \lambda_i (i = 1, \ldots, 7) \), system (4) has many interesting and new properties:

(i) existence of hyperchaotic attractors with three zero Lyapunov exponents.

(ii) existence of quasi-periodic attractors with three zero Lyapunov exponents.

(iii) existence of chaotic attractors with two zero Lyapunov exponents (not with one zero Lyapunov exponent).
2.4.2. Fix $\gamma = 35$, $\beta = 3$, $k_1 = -5$, $k_2 = -4$ and vary $\alpha$

The values of $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$ versus $\alpha$ are plotted in Fig. 1c. From Fig. 1c one can conclude that (4) has hyperchaotic attractors for $\alpha \in [6.1, 31]$, chaotic attractors for $\alpha \in [2.0, 3.8]$, and $[4.1, 5.2]$, quasi-periodic attractors for $\alpha$ lies in $[3.8, 4.1]$, $[5.2, 6.1]$ and $[31, 38]$. Solutions of system (4) give periodic attractors for the value of $\alpha \in (0, 0.1]$ and $[38, 60)$. For this case system (4) has no periodic solutions and fixed points which exist in 2.4.1.

2.4.3. Fix $\alpha = 14$, $\gamma = 35$, $k_1 = -5$, $k_2 = -4$ and vary $\beta$

As we did before, the values of $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$ versus $\beta$ are plotted in Fig. 1d. From Fig. 1d system (4) has hyperchaotic attractors for $\beta \in [0.7, 4.6]$, chaotic attractors for $\beta \in (0.2, 0.5]$, $[0.5, 0.7]$, $[5.4, 5.9]$ and $[6.3, 6.9]$, quasi-periodic solutions for $\beta$ in $[0.2, 0.5]$, $[4.6, 5.4]$, $[5.9, 6.3]$ and $[6.9, 30]$. System (4) for the above values of the system parameters has no periodic solutions and fixed points which exist in 2.4.1.
2.4.4. Fix $\alpha = 14$, $\beta = 3$, $\gamma = 35$, $k_1 = -5$ and vary $k_2$

The values of $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$ versus $k_2$ are plotted in Fig. 1e. As shown in Fig. 1e, hyperchaotic attractors exist for $k_2 \in (-33.2, -30.3]$ and $[-28.1, 5]$, while chaotic attractors for $k_2$ lie in the interval $(-50, -33.2]$. Quasi-periodic solutions appear when $k_2$ lies in $[-30.3, -28.1)$. System (4) for this case has only chaotic or hyperchaotic attractors, and quasi-periodic solutions. We have hyperchaotic and chaotic attractors with two zero Lyapunov exponents.

2.4.5. Fix $\alpha = 14$, $\beta = 3$, $\gamma = 35$, $k_2 = -4$ and vary $k_1$

Finally we calculate $\lambda_i$, and the values of $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$ versus $k_1$ are plotted in Fig. 1f. The hyperchaotic attractors for $k_1$ lie in the intervals $[-39.1, -32.2)$ and $[-30.1, 3.9]$. The chaotic attractors exist for $k_1 \in (-50, -39.1)$, quasi-periodic solutions for $k_1 \in [-32.2, 30.1)$.

As it is shown in Sections 2.4.1–2.4.5 system (4) has different solutions (attractors). From Fig. 1, the dynamics of (4) are studied for $\gamma \in (0, 60)$, $\alpha \in (0, 60)$, $\beta \in (0, 30)$, $k_2 \in (-50, 5)$ and $k_1 \in (-50, 4)$. We note that chaotic, hyperchaotic attractors and quasi-periodic solutions exist for all the parameter values of system (4). Periodic solutions appear only in the cases 2.4.1 and 2.4.2. Trivial fixed points exist only in 2.4.1.

In Section 2.4.1 we have hyperchaotic attractors and quasi-periodic solutions with three zero Lyapunov exponents, while the chaotic attractors exist for all the parameter values of system (4) with two zero Lyapunov exponents. To test this, we have numerically solved (4) (using e.g. Mathematica 7 software) in several cases and excellent agreements are found with the results of 2.4.1. For example, choosing $\alpha = 14$, $\beta = 3$, $k_1 = -5$ and $k_2 = -4$, with the initial conditions $t_0 = 0$, $u_1(0) = 1$, $u_2(0) = 2$, $u_3(0) = 3$, $u_4(0) = 4$, $u_5(0) = 5$, $u_6(0) = 6$, $u_7(0) = 7$ and when $\gamma = 0.5$ the solution of system (4) has a trivial fixed point (see Fig. 2a), a periodic solution when $\gamma = 10$ as in Fig. 2b. In Fig. 2c, $\gamma = 20$, the solution is quasi-periodic (with three zero Lyapunov exponents), a chaotic attractor (with two zero Lyapunov exponents) in Fig. 2d and hyperchaotic attractor (with three zero Lyapunov exponents) in Fig. 2e for $\gamma = 22$ and 50, respectively.

2.5. Bifurcation diagrams of system (4)

A bifurcation diagram provides a nice summary for the transition between different types of motion that can occur as one parameter of the system is varied. A bifurcation diagram plots a system parameter on the horizontal axis and a representation of an attractor on the vertical axis. So, bifurcation diagrams provide a useful method to show how a system's behavior changes according to the value of a parameter [27].

In this subsection we calculate bifurcation diagrams of system (4) which are other signs to demonstrate its dynamics. These diagrams are plotted in Fig. 3. Fig. 3a shows the $(\gamma, u_2)$ bifurcation diagram for $\gamma \in (0, 60)$. It can be observed that when $\gamma \in (0, 0.7)$, system (4) has solutions that approach fixed points and when $\gamma \in (0.7, 19.2)$ has periodic behavior and quasi-periodic attractors for $\gamma \in (19.2, 20.3)$. Chaotic and hyperchaotic attractors are evident for $\gamma \in (20.3, 23.8)$ and $\gamma \in (23.8, 60)$, respectively. Fig. 3b depicts $(\alpha, u_3)$ bifurcation diagram for $\alpha \in (0, 60)$. The results of Fig. 3b agree with the regions which are given in 2.4.2. Fig. 3c, d and e contain $(\beta, u_3)$, $(k_3, u_3)$ and $(k_1, u_3)$ bifurcation diagrams, respectively. The dynamical behavior of these figures is similar to those which are given in 2.4.3–2.4.5, respectively.

3. Synchronization of hyperchaotic attractors of (3)

In this section we study the synchronization of two identical hyperchaotic attractors of complex Lorenz system (4) using an active control technique based on Lyapunov stability analysis [24,28–31].

The drive and response systems for hyperchaotic complex Lorenz system (3) can be described, respectively as:

\[
\begin{align*}
\dot{x}_d &= \alpha (y_d - x_d), \quad \dot{y}_d = \gamma x_d - y_d - x_d y_d + w_d, \\
\dot{z}_d &= 1/2(\dot{x}_d y_d + x_d \dot{y}_d) - \beta z_d, \quad \dot{w}_d = k_1 x_d + k_2 y_d, \\
\dot{x}_r &= \alpha (y_r - x_r) + (v_1 + iv_2), \quad \dot{y}_r = \gamma x_r - x_r z_r - y_r + w_r + (v_3 + iv_4), \\
\dot{z}_r &= 1/2(\dot{x}_r y_r + x_r \dot{y}_r) - \beta z_r + v_5, \quad \dot{w}_r = k_1 x_r + k_2 y_r + (v_6 + iv_7),
\end{align*}
\]

where $x_d = u_{1d} + iu_{2d}$, $y_d = u_{3d} + iu_{4d}$ and $w_d = u_{5d} + iu_{6d}$ are complex state variables for the drive system (14), $z_d = u_{7d}$ is the real state variable, $x_r = u_{1r} + iu_{2r}$, $y_r = u_{3r} + iu_{4r}$ and $w_r = u_{5r} + iu_{6r}$ are complex state variables for the response system (15), $z_r = u_{7r}$ is real and $u_{1} + iu_{2}$, $v_3 + iv_4$, $v_5 + iv_6$ and $v_7$ are control functions to determine periodic behavior.

The complex system (14) can be rewritten as seven real first order ODEs of the form:

\[
\begin{align*}
\dot{u}_{1d} &= \alpha (u_{3d} - u_{1d}), \quad \dot{u}_{2d} = \alpha (u_{4d} - u_{2d}), \\
\dot{u}_{3d} &= \gamma u_{1d} - u_{1d}u_{3d} - u_{3d} + u_{6d}, \\
\dot{u}_{4d} &= \gamma u_{2d} - u_{2d}u_{4d} - u_{4d} + u_{7d}, \\
\dot{u}_{5d} &= u_{1d}u_{3d} + u_{2d}u_{4d} - \beta u_{5d}, \\
\dot{u}_{6d} &= k_1 u_{1d} + k_2 u_{3d}, \quad \dot{u}_{7d} = k_1 u_{2d} + k_2 u_{4d}.
\end{align*}
\]
Fig. 2. For $\beta = 3, \alpha = 14, k_1 = -5, k_2 = -4$ and vary $\gamma$ at the same initial conditions as in Fig. 1 (a) Fixed point of (3) when $\gamma = 0.5$. (b) Periodic solution of (3) when $\gamma = 10$. (c) Quasi-periodic solution of (3) when $\gamma = 20$. (d) Chaotic attractor of (3) when $\gamma = 22$. (e) Hyperchaotic attractor of (3) when $\gamma = 50$.

The complex system (15) in the real form becomes:

$$
\dot{u}_1 = \alpha (u_3 - u_1) + v_1, \quad \dot{u}_2 = \alpha (u_6 - u_2) + v_2,
\dot{u}_3 = \gamma u_1 - u_1 u_5 - u_3 + u_6 + v_3,
\dot{u}_4 = \gamma u_2 - u_2 u_5 - u_4 + u_7 + v_4,
\dot{u}_5 = u_1 u_3 + u_2 u_4 - \beta u_5 + v_5,
\dot{u}_6 = k_1 u_1 + k_2 u_3 + v_6, \quad \dot{u}_7 = k_1 u_2 + k_2 u_4 + v_7.
$$

(17)
In order to obtain the control signals, we define the errors between the drive and the response systems as:

\[
\begin{align*}
\dot{e}_{u_1} + i\dot{e}_{u_2} &= x_r - x_d = (u_{1r} - u_{1d}) + i(u_{2r} - u_{2d}), \\
\dot{e}_{u_3} + i\dot{e}_{u_4} &= y_r - y_d = (u_{3r} - u_{3d}) + i(u_{4r} - u_{4d}), \\
\dot{e}_{u_5} &= z_r - z_d = u_{5r} - u_{5d}, \\
\dot{e}_{u_6} + i\dot{e}_{u_7} &= w_r - w_d = (u_{6r} - u_{6d}) + i(u_{7r} - u_{7d}).
\end{align*}
\]

Subtracting (16) from (17) we get:

\[
\begin{align*}
\dot{e}_{u_3} + i\dot{e}_{u_4} &= \gamma (e_{u_1} + i e_{u_2}) - (e_{u_3} + i e_{u_4}) - u_{5r}(u_{1r} + i u_{2r}) + u_{5d}(u_{1d} + i u_{2d}) + (e_{u_6} + i e_{u_7}) + (v_3 + i v_4), \\
\dot{e}_{u_5} &= -\beta e_{u_5} - (u_{1d} u_{3d} + u_{2d} u_{4d}) + (u_{1r} u_{3r} + u_{2r} u_{4r}) + v_5, \\
\dot{e}_{u_6} + i\dot{e}_{u_7} &= k_1(e_{u_3} + i e_{u_4}) + k_2(e_{u_3} + i e_{u_4}) + (v_6 + i v_7).
\end{align*}
\]

Fig. 3. Bifurcation diagrams of system (3) (a) \((\gamma, u_5)\) plane, (b) \((\alpha, u_5)\) plane, (c) \((\beta, u_5)\) plane, (d) \((k_2, u_5)\) plane, (e) \((k_1, u_5)\) plane.
Eq. (19) describes a dynamical system via which the “errors” evolve in time and finally the ODEs of this system in real form become:

\[
\begin{align*}
\dot{e}_{u_1} &= \alpha (e_{u_1} - e_{u_1}) + v_1, \\
\dot{e}_{u_2} &= \alpha (e_{u_2} - e_{u_2}) + v_2, \\
\dot{e}_{u_3} &= \gamma e_{u_1} - e_{u_3} + u_{3d}u_{1d} - u_{3r}u_{1r} + e_{u_6} + v_3, \\
\dot{e}_{u_4} &= \gamma e_{u_2} - e_{u_4} + u_{4d}u_{2d} - u_{4r}u_{2r} + e_{u_7} + v_4, \\
\dot{e}_{u_5} &= -\beta e_{u_5} - (u_{1d}u_{3d} + u_{2d}u_{4d}) + (u_{1r}u_{3r} + u_{2r}u_{4r}) + v_5, \\
\dot{e}_{u_6} &= k_1 e_{u_1} + k_2 e_{u_3} + v_6, \\
\dot{e}_{u_7} &= k_1 e_{u_2} + k_2 e_{u_4} + v_7.
\end{align*}
\]

(20)

According to Lyapunov stability analysis [27] we define:

\[
\begin{align*}
s_1 &= v_1, \\
s_2 &= v_2, \\
s_3 &= u_{3d}u_{1d} - u_{3r}u_{1r} + v_3, \\
s_4 &= u_{4d}u_{2d} - u_{4r}u_{2r} + v_4, \\
s_5 &= -(u_{1d}u_{3d} + u_{2d}u_{4d}) + (u_{1r}u_{3r} + u_{2r}u_{4r}) + v_5, \\
s_6 &= v_6, \\
s_7 &= v_7.
\end{align*}
\]

So, system (20) will take the form:

\[
\begin{align*}
\dot{e}_{u_1} &= \alpha (e_{u_1} - e_{u_1}) + s_1, \\
\dot{e}_{u_2} &= \alpha (e_{u_2} - e_{u_2}) + s_2, \\
\dot{e}_{u_3} &= \gamma e_{u_1} - e_{u_3} + e_{u_6} + s_3, \\
\dot{e}_{u_4} &= \gamma e_{u_2} - e_{u_4} + e_{u_7} + s_4, \\
\dot{e}_{u_5} &= -\beta e_{u_5} + s_5, \\
\dot{e}_{u_6} &= k_1 e_{u_1} + k_2 e_{u_3} + s_6, \\
\dot{e}_{u_7} &= k_1 e_{u_2} + k_2 e_{u_4} + s_7.
\end{align*}
\]

(21)

We can determine $s_j, j = 1, \ldots, 7$ as:

\[
\begin{bmatrix}
s_1 \\
 s_2 \\
 s_3 \\
 s_4 \\
 s_5 \\
 s_6 \\
 s_7
\end{bmatrix}
= 
\begin{bmatrix}
\alpha + \xi_1 & 0 & -\alpha & 0 & 0 & 0 & 0 \\
0 & \alpha + \xi_2 & 0 & -\alpha & 0 & 0 & 0 \\
\gamma & 0 & 1 + \xi_3 & 0 & 0 & -1 & 0 \\
0 & \gamma & 0 & 1 + \xi_4 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & \beta + \xi_5 & 0 & 0 \\
-k_1 & 0 & -k_2 & 0 & 0 & \xi_6 & 0 \\
0 & -k_1 & 0 & -k_2 & 0 & 0 & \xi_7
\end{bmatrix}
\begin{bmatrix}
e_{u_1} \\
e_{u_2} \\
e_{u_3} \\
e_{u_4} \\
e_{u_5} \\
e_{u_6} \\
e_{u_7}
\end{bmatrix}.
\]

Finally, we can find $v_j$ as:

\[
\begin{align*}
v_1 &= (\xi_1 + \alpha) e_{u_1} - \alpha e_{u_3}, \\
v_2 &= (\xi_2 + \alpha) e_{u_2} - \alpha e_{u_4}, \\
v_3 &= -\gamma e_{u_1} - (\xi_3 + 1) e_{u_3} - e_{u_6} - u_{3d}u_{1d} + u_{3r}u_{1r}, \\
v_4 &= -\gamma e_{u_2} - (\xi_4 + 1) e_{u_4} - e_{u_7} - u_{4d}u_{2d} + u_{4r}u_{2r}, \\
v_5 &= (\xi_5 + \beta) e_{u_5} + (u_{1d}u_{3d} + u_{2d}u_{4d}) - (u_{1r}u_{3r} + u_{2r}u_{4r}), \\
v_6 &= \xi_6 e_{u_6} - k_1 e_{u_1} - k_2 e_{u_3}, \\
v_7 &= \xi_7 e_{u_7} - k_1 e_{u_2} - k_2 e_{u_4}.
\end{align*}
\]

(22)

When anyone chooses $\xi_1, \ldots, \xi_7 < 0$ and based on the Lyapunov stability theory, the error states $e_{u_i} = 0, j = 1, \ldots, 7$ are asymptotically stable, which means that:

\[
\lim_{t \to \infty} \| e_{u_i}(t) \| = 0.
\]

Therefore, the states of controlled response and drive systems are globally synchronized asymptotically. Systems (16) and (17) with (22) are solved numerically (using e.g. Mathematica 7 software) for $\xi_1 = \ldots = \xi_7 = -1, \alpha = 14, \gamma = 35, k_1 = -5, k_2 = -4$ and $\beta = 3$ and the initial conditions of the drive and the response systems at $t_0 = 0$ are $u_{1d}(0) = 1, u_{2d}(0) = 2, u_{3d}(0) = 3, u_{4d}(0) = 4, u_{3r}(0) = 5, u_{4r}(0) = 6, u_{1r}(0) = 7$ and $u_{1r}(0) = -1, u_{2r}(0) = -2, u_{3r}(0) = -3, u_{4r}(0) = -4, u_{5r}(0) = -5, u_{6r}(0) = -6, u_{7r}(0) = -7$, respectively. The synchronization of this hyperchaotic attractor is shown in Fig. 4, where the oscillations of the drive and response systems rapidly become totally indistinguishable. The synchronization errors are plotted in Fig. 5 and demonstrate that synchronization is achieved very fast. They are seen to converge to zero after very small values of $t$ (time/10). Hence, the active control technique of two identical hyperchaotic systems is achieved.

4. Conclusions

This paper deals with new nonlinear system where the main variables participating in the dynamics are complex. Thus, when real and imaginary parts are separated anyone obtains a higher dimensional real system. In communications doubling the number of variables may be used to increase the content and security of the transmitted information.
In this paper a linear controller is used to construct hyperchaotic complex Lorenz systems. The new system exhibits both chaotic and hyperchaotic behaviors as well as periodic, quasi-periodic and solutions that approach fixed points as shown in Figs. 1 and 2. System (3) is (7D) and contains the new parameters $k_1$ and $k_2$. Bifurcation analyses are used to confirm these behaviors as shown in Fig. 3. There are new and interesting results of new system (3), we have hyperchaotic attractors and quasi-periodic solutions with three zero Lyapunov exponents, while chaotic attractors exist for all the parameter values of the new system with two zero Lyapunov exponents. However, the hyperchaotic complex or real Lorenz systems [22,23] do not have these properties. The new hyperchaotic system has trivial fixed points when $k_1 + k_2 \neq 0$, but when $k_1 + k_2 = 0$
system (4) has both isolated (trivial) and non-isolated fixed points. The projection of non-isolated fixed points in \((u_1, u_2)\) space is a circle with center \((0, 0)\) which does not occur in the other hyperchaotic complex or real Lorenz systems [22,23]. The stability analyses of the trivial fixed points of (3) are carried out. The stability analysis of nontrivial fixed points of (3) can be similarly studied as we did for \(E_0\). The fractional Lyapunov dimension of hyperchaotic attractors of (4) is \(\approx 5.082\), and this value is greater than the others which are obtained in [23]. The new hyperchaotic complex system (3) is considered as a generalization of the hyperchaotic real Lorenz [22].

The synchronization of hyperchaotic attractors are achieved using the active control technique based on Lyapunov stability analysis and yields excellent results as shown in Figs. 4 and 5.

We hope that the results of these investigations shed some light on higher dimensions and the dynamics of complex dynamical systems, which is still far from what has been achieved to date for real dynamical systems.
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References