Some extremal problems of graphs with local constraints

Mieczysław Borowiecki*, Elżbieta Sidorowicz

Institute of Mathematics, University of Zielona Góra, Podgórska 50, 65-246 Zielona Góra, Poland

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Abstract

Let \( \mathcal{P} \) be a family of graphs. A graph \( G \) is said to satisfy a property \( \mathcal{P} \) locally if \( G[N(v)] \in \mathcal{P} \) for every \( v \in V(G) \). The class of graphs that satisfies the property \( \mathcal{P} \) locally will be denoted by \( L(\mathcal{P}) \) and we shall call such a class a local property.

Let \( \mathcal{P} \) be a hereditary property. A graph is said to be maximal with respect to a hereditary property \( \mathcal{P} \) (shortly \( \mathcal{P} \)-maximal) if it belongs to \( \mathcal{P} \) and none of its proper supergraphs of the same order has the property \( \mathcal{P} \). A graph is \( \mathcal{P} \)-extremal if it has the maximum number of edges among all \( \mathcal{P} \)-maximal graphs of given order. This number is denoted by \( \text{ex}(n; \mathcal{P}) \). If the number of edges of a \( \mathcal{P} \)-maximal graph of order \( n \) is minimum, then the graph is called \( \mathcal{P} \)-saturated and its number of edges is denoted by \( \text{sat}(n; \mathcal{P}) \).

In this paper, we shall describe the numbers \( \text{ex}(n; L(O_k)) \) and \( \text{ex}(n; L(S_k)) \) for \( k \geq 1 \). Also, we give \( \text{sat}(n; L(O_k)) \) and \( \text{sat}(n; L(S_k)) \) for \( k = 1, 2 \). © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction and notation

We consider finite undirected graphs without loops or multiple edges. A graph \( G \) has a vertex set \( V(G) \) and an edge set \( E(G) \). Let \( v(G) \), \( e(G) \) denote the number of vertices and the number of edges of \( G \), respectively. We say that \( G \) contains \( H \) whenever \( G \) contains a subgraph isomorphic to \( H \). For a subset \( U \subseteq V(G) \) we denote by \( G[U] \) the subgraph of \( G \) induced by \( U \). For a vertex \( x \in V(G) \) we denote by \( N(x) \) the open neighbourhood of \( x \) and by \( N[x] \) the closed neighbourhood of \( x \) (i.e., \( N[x] = N(x) \cup \{x\} \)). The degree of \( x \) is denoted by \( d(x) \). For a vertex \( x \in V(G) \) and a set \( S \subseteq V(G) \)

* Corresponding author. Tel.: +48-68-328-2424; fax: +48-68-324-7448.
E-mail addresses: m.borowiecki@im.uz.zgora.pl (M. Borowiecki), e.sidorowicz@im.uz.zgora.pl (E. Sidorowicz).
or for the subgraph induced by \( S \), let \( N_S(x) = N(x) \cap S \) and \( d_S(x) = |N_S(x)| \). We denote by \( \Delta(G) \) and \( \delta(G) \) the maximum and the minimum degree of \( G \), respectively. Let \( F, S \) be vertex disjoint subgraphs of \( G \). Then the number of edges of \( G \) joining vertices of \( F \) with vertices of \( S \) will be denoted by \( e(F, S) \).

Let \( I \) denote the class of all graphs, with isomorphic graphs being regarded as equal. If \( \mathcal{P} \) is a proper nonempty subclass of \( I \), then \( \mathcal{P} \) will also denote the property of being in \( \mathcal{P} \). We shall use the terms class of graphs and property of graphs interchangeably.

A property \( \mathcal{P} \) is called hereditary if every subgraph of a graph with property \( \mathcal{P} \) also has property \( \mathcal{P} \).

We list some properties to introduce the necessary notation, which will be used in the paper. Let \( k \) be a nonnegative integer.

\[
\mathcal{O} = \{ G \in I : G \text{ is totally disconnected} \},
\]

\[
\mathcal{O}_k = \{ G \in I : \text{each component of } G \text{ has at most } k + 1 \text{ vertices} \},
\]

\[
\mathcal{I}_k = \{ G \in I : G \text{ contains no subgraph isomorphic to } K_{k+2} \},
\]

\[
\mathcal{S}_k = \{ G \in I : c(G) \leq k \}.
\]

It is easy to verify that \( \mathcal{O}_k \subseteq \mathcal{I}_k \subseteq \mathcal{S}_k \) and \( \mathcal{O}_0 = \mathcal{I}_0 = \mathcal{S}_0 = \emptyset \), \( \mathcal{O}_1 = \mathcal{I}_1 \).

For any hereditary property \( \mathcal{P} \), which is distinct from \( I \), there exists a number \( c(\mathcal{P}) \) (called the completeness of \( \mathcal{P} \)) defined as follows: \( c(\mathcal{P}) = \max\{ k : K_{k+1} \in \mathcal{P} \} \). Obviously, \( c(\mathcal{O}_k) = c(\mathcal{I}_k) = c(\mathcal{S}_k) \).

The following results describe the structure of additive hereditary properties of graphs.

**Theorem 1.1** (Borowiecki and Mihók [1]). Let \( L \) be the set of all hereditary properties. Then \( (L, \subseteq) \) is a complete and distributive lattice in which the join and the meet correspond to set-union and set-intersection, respectively.

**Theorem 1.2** (Borowiecki and Mihók [1]). For every nonnegative \( k \), \( L_k = \{ \mathcal{P} \in L : c(\mathcal{P}) = k \} \) is a complete and distributive sublattice of \( (L, \subseteq) \) with the least element \( \mathcal{O}_k \) and the greatest element \( \mathcal{I}_k \).

For a hereditary property \( \mathcal{P} \) we define the set of minimal forbidden subgraphs of \( \mathcal{P} \) by

\[
F(\mathcal{P}) = \{ G \in I : G \notin \mathcal{P} \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P} \}.
\]

A direct consequence of this definition is

**Lemma 1.3.** Let \( \mathcal{P} \) be a hereditary property. Then \( G \in \mathcal{P} \) if and only if no subgraph of \( G \) is in \( F(\mathcal{P}) \).

Thus any hereditary property is uniquely determined by its set of minimal forbidden subgraphs. An alternative way is to characterise \( \mathcal{P} \) by the set of graphs containing all
the graphs in \( \mathcal{P} \) as subgraphs, i.e., the set of \( \mathcal{P} \)-maximal graphs:

\[
M(\mathcal{P}) = \{ G \in \mathcal{P} : G + e \notin \mathcal{P} \text{ for each } e \in E(G) \}.
\]

The set of \( \mathcal{P} \)-maximal graphs of order \( n \) is denoted by \( M(n, \mathcal{P}) \).

The concept of maximal graphs with respect to hereditary properties is important also in connection with extremal graph theory. A problem of this type was first formulated by Turán and his original problem asked for the maximum number of edges in any graph of order \( n \) which does not contain a complete graph \( K_p \) (i.e., in any \( \mathcal{I}_{p-2} \)-maximal graph), see [14].

A general extremal problem, in our terminology, can be formulated as follows: Given a family \( \mathcal{F}(\mathcal{P}) \) of forbidden subgraphs, find the number

\[
ex(n, \mathcal{P}) = \max \{ e(G) : G \in M(n, \mathcal{P}) \}.
\]

The set of all \( \mathcal{P} \)-maximal graphs of order \( n \) with exactly \( ex(n, \mathcal{P}) \) edges is denoted by \( \text{Ex}(n, \mathcal{P}) \). The members of \( \text{Ex}(n, \mathcal{P}) \) are called \( \mathcal{P} \)-extremal graphs.

The minimum number of edges in \( \mathcal{P} \)-maximal graphs of order \( n \) is denoted by \( \text{sat}(n, \mathcal{P}) \), i.e.,

\[
\text{sat}(n, \mathcal{P}) = \min \{ e(G) : G \in M(n, \mathcal{P}) \}.
\]

By the symbol \( \text{Sat}(n, \mathcal{P}) \) we shall denote the set of all \( \mathcal{P} \)-maximal graphs on \( n \) vertices with \( \text{sat}(n, \mathcal{P}) \) edges. These graphs are called \( \mathcal{P} \)-saturated. The first result concerning saturated graphs was given by Erdős et al. [5], who found the minimum number of edges of \( K_{p+2} \)-free graphs. They showed that

\[
\text{sat}(n, \mathcal{I}_p) = pn - \frac{1}{2} (p + 1) p \quad (\text{if } n \geq p \geq 1).
\]

Let \( \mathcal{P} \) be a property of graphs. A graph \( G \) is said to satisfy a property \( \mathcal{P} \) locally if \( G[N(v)] \in \mathcal{P} \) for every \( v \in V(G) \). The class of graphs that satisfy the property \( \mathcal{P} \) locally will be denoted by \( L(\mathcal{P}) \) and we shall call such a class a local property.

The word “local” was first used in connection with infinite graphs or digraphs in concepts such as locally finite, or locally countable infinite, referring to the vertex degrees of an infinite graph. Finite graphs with a given degree sequence have also been studied [3] and local properties for finite graphs were defined using neighbourhoods, [10–13].

Early investigations dealt mostly with the case \( |\mathcal{P}| = 1 \); i.e., when all neighbourhoods are isomorphic. Summaries of results of this type can be found in the survey papers of Hell [6] and Sedláček [8]. More recently, the cases when \( \mathcal{P} \) consists of all cycles, all paths, all matchings, or all forests were investigated. Also, results concerning some extremal problems on such classes of graphs have been obtained [4,9].

The hereditary property \( \mathcal{I}_p \) is obviously a local property. Indeed let \( G \in \mathcal{I}_p \), then \( G[N(v)] \in \mathcal{I}_{p-1} \), for every \( v \in V(G) \), i.e., any subgraph induced by neighbours of any
vertex of a $K_{p+2}$-free graph is a $K_{p+1}$-free graph. Moreover, the converse of this statement also holds. So $\mathcal{I}_p = L(\mathcal{I}_{p-1})$.

The set of forbidden subgraphs and the structure of $L(\mathcal{P})$, when $\mathcal{P}$ is a hereditary property, have been described by Borowiecki and Mihók [2]. They proved that, for a hereditary property $\mathcal{P}$,

$$F(L(\mathcal{P})) = \{K_1 + H : H \in F(\mathcal{P})\},$$

where $+$ denotes the join of graphs.

Property $\mathcal{I}_k$ is the greatest element of the sublattice $L_k$. Since $L(\mathcal{I}_k) = \mathcal{I}_{k+1}$, the $L(\mathcal{I}_k)$-extremal and $L(\mathcal{I}_k)$-saturated graphs are known. In the lattice $L_k$ the least element is the property $\mathcal{O}_k$. We shall determine numbers $\text{ex}(n, L(\mathcal{O}_k))$ for $k \geq 1$ and $\text{sat}(n, L(\mathcal{O}_k))$ for $k = 1, 2$. Another important property is $\mathcal{I}_k$, the class of graphs of maximum degree $k$. We will determine the numbers $\text{ex}(n, L(\mathcal{I}_k))$ for $k \geq 1$ and $\text{sat}(n, L(\mathcal{I}_k))$ for $k = 1, 2$.

2. Extremal graphs for some local properties

The complete bipartite graph $K_{r,s}$ is in $M(r+s, L(\mathcal{O}_k))$ and $M(r+s, L(\mathcal{I}_k))$, for $r, s \geq k + 1$. The complete bipartite graph of order $n$ with the maximum possible number of edges is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ and has $\lfloor n^2/4 \rfloor$ edges. In the next theorem, we will show that if $n$ is large enough then $\text{ex}(n, L(\mathcal{I}_k)) = \lfloor n^2/4 \rfloor$.

**Theorem 2.1.** Let $n \geq 18k$ and $k \geq 1$. Then

$$\text{ex}(n, L(\mathcal{I}_k)) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

**Proof.** Since the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil} \in L(\mathcal{I}_k)$ for $n \geq 18k$, we immediately have $\text{ex}(n, L(\mathcal{I}_k)) \geq \lfloor n^2/4 \rfloor$. We shall prove that $\text{ex}(n, L(\mathcal{I}_k)) \leq \lfloor n^2/4 \rfloor$.

Assume $G \in L(\mathcal{I}_k)$ and $\text{e}(G) = \text{ex}(n, L(\mathcal{I}_k))$. Let $S$ be a subgraph of $G$ with maximum possible number of vertices such that $S \subseteq \mathcal{I}_k$. Let $F = G[V(G) - V(S)]$. We consider two cases.

**Case 1:** $e(F) \geq 4k$. First we shall prove that $e(S)e(F) - e(S,F) \leq 2e(S)$. Since $G \in L(\mathcal{I}_k)$ then for any two adjacent vertices $x, y$ of $S$

$$|N_F(x) \cap N_F(y)| \leq k. \quad (1)$$

Suppose that there is a vertex $x \in V(S)$ such that $|V(F) - N_F(x)| \leq d_S(x) - 1$. Let $T$ denote the set of all vertices in $F$ which are not adjacent to the vertex $x$. Then $|T| \leq d_S(x) - 1$. Condition (1) implies that for every $y \in N_S(x)$ we have $d_F(y) \leq |T| + |N_F(x) \cap N_F(y)| \leq |T| + k$. Hence $|V(F) - N_F(y)| \geq 4k - (|T| + k) \geq 4k - (d_S(x) - 1 + k)$. Since $S \subseteq \mathcal{I}_k$, we have $d_S(x) \leq k$, so

$$|V(F) - N_F(y)| \geq 2k + 1. \quad (2)$$
This implies that the vertices of \( S \) which have fewer nonneighbours in \( F \) than neighbours in \( S \) form an independent set. We denote this vertex set by \( V_1 \). Let \( V_2 \) be the set of all vertices in \( S \), which are adjacent to at least one vertex of \( V_1 \). From (2) it follows that each vertex of \( V_2 \) has at least \( 2k + 1 \) nonneighbours in \( F \). Let \( V_3 = S - (V_1 \cup V_2) \). Then any \( v \in V_3 \) is not adjacent to any vertex of \( V_1 \) and \( v \) has at least \( d_S(v) \) nonneighbours in \( F \). Hence
\[
v(S)v(F) - e(F,S) \geq |V_2|(2k + 1) + \sum_{v \in V_1} d_S(v) \geq 2 \sum_{v \in V_2} k + \sum_{v \in V_3} d_S(v)
\]

\[
\geq 2 \sum_{v \in V_2} d_S(v) + \sum_{v \in V_3} d_S(v).
\]

Since
\[
e(S) = \frac{1}{2} \sum_{v \in V(S)} d_S(v)
\]

\[
= \sum_{v \in V_2} d_{V_2}(v) + \frac{1}{2} \left( \sum_{v \in V_2} (d_S(v) - d_{V_1}(v)) + \sum_{v \in V_3} d_S(v) \right)
\]

\[
\leq \frac{1}{2} \sum_{v \in V_2} d_S(v) + \frac{1}{2} \sum_{v \in V_2} d_S(v) + \frac{1}{2} \sum_{v \in V_3} d_S(v)
\]

\[
= \sum_{v \in V_2} d_S(v) + \frac{1}{2} \sum_{v \in V_3} d_S(v),
\]

we see that
\[
e(S) \leq \frac{1}{2} (v(S)v(F) - e(F,S)).
\]

On the other hand, since \( G[N(v)] \in \mathcal{F}_k \) for each \( v \in V(G) \) and \( S \) is a subgraph of \( G \) of maximum order belonging to \( \mathcal{F}_k \), it follows that \( \Delta(G) \leq v(S) \). Hence \( d_S(x) + d_F(x) = d(x) \leq v(S) \) for every \( x \in V(G) \). Thus \( v(S) - d_S(x) \geq d_F(x) \). This gives
\[
\sum_{v \in V(F)} (v(S) - d_S(v)) \geq \sum_{v \in V(F)} d_F(v) = 2e(F). \]

But the left side of this inequality is equal to \( v(S)v(F) - e(F,S) \). Thus
\[
e(F) \leq \frac{1}{2} (v(S)v(F) - e(F,S)) \]

and finally
\[
e(G) = e(F) + e(S) + e(F,S) \leq v(F)v(S).
\]

The product \( v(F)v(S) \) achieves the maximum value when \( v(F) = \lfloor n/2 \rfloor \) and \( v(S) = \lfloor n/2 \rfloor \) or \( v(S) = \lfloor n/2 \rfloor \) and \( v(F) = \lfloor n/2 \rfloor \). Then
\[
e(G) \leq v(F)v(S) \leq \left\lfloor \frac{n^2}{4} \right\rfloor.
\]

**Case 2:** \( v(F) < 4k \).

For the graph \( G \) we have
\[
e(G) \leq \frac{1}{2} (v(F)\Delta(G) + v(S)\Delta(S) + v(F)).
\]
Since \( A(G) \leq v(S) \), so we have
\[
e(G) \leq \frac{1}{2} (v(F)v(S) + v(S)(A(S) + v(F))) = \frac{1}{2} v(S)(v(F) + A(S) + v(F))
\]
\[
\leq \frac{1}{2} v(S)(8k + A(S)) \leq \frac{1}{2} v(S)9k \leq \frac{9}{2} nk \leq \left\lfloor \frac{n^2}{4} \right\rfloor , \quad \text{for } n \geq 18k. \quad \square
\]

The next result follows immediately from the definitions.

**Lemma 2.2.** If \( \mathcal{P}_1 \subseteq \mathcal{P}_2 \), then \( \text{ex}(n, \mathcal{P}_1) \leq \text{ex}(n, \mathcal{P}_2) \), for every \( n \).

Since \( \mathcal{O}_k \subseteq \mathcal{F}_k \), it follows from Lemma 2.2 that \( \text{ex}(n, L(\mathcal{O}_k)) \leq \left\lfloor \frac{n^2}{4} \right\rfloor \), for \( n \geq 18k \).

But the complete bipartite graph \( K_{[n/2],[n/2]} \) has the property \( \mathcal{O}_k \), thus \( \text{ex}(n, L(\mathcal{O}_k)) = \left\lfloor \frac{n^2}{4} \right\rfloor \), for \( n \geq 18k \). However, this result can be improved.

**Theorem 2.3.** Let \( n \geq 14k \) and \( k \geq 1 \). Then
\[
\text{ex}(n, L(\mathcal{O}_k)) = \left\lfloor \frac{n^2}{4} \right\rfloor .
\]

**Proof.** Since the complete bipartite graph \( K_{[n/2],[n/2]} \subseteq L(\mathcal{O}_k) \), it follows that \( \text{ex}(n, L(\mathcal{O}_k)) \geq \left\lfloor \frac{n^2}{4} \right\rfloor \). We shall prove that \( \text{ex}(n, L(\mathcal{O}_k)) \leq \left\lfloor \frac{n^2}{4} \right\rfloor \).

Let \( G \in L(\mathcal{O}_k) \) and \( e(G) = \text{ex}(n, L(\mathcal{O}_k)) \). Let \( v \in V(G) \) be a vertex of degree \( A(G) \).

Let \( S = G[N(v)] \) and \( F = G[V(G) - N(v)] \). Obviously, the graph \( S \) belongs to \( L(\mathcal{O}_k) \). Consider two cases.

Case 1: \( v(F) \geq k + 2A(S) \). For any two adjacent vertices \( x, y \in V(S) \) the set \( N_S(x) \cup (N_F(x) \cap N_F(y)) \) induces a connected subgraph in \( G[N_G(x)] \). Thus \( |N_S(x) \cup (N_F(x) \cap N_F(y))| \leq k + 1 \). Hence
\[
|N_F(x) \cap N_F(y)| \leq k + 1 - d_S(x). \quad (3)
\]

Suppose that there is a vertex \( x \in V(S) \) such that \( |V(F) - N_F(x)| \leq d_S(x) - 1 \). Let us denote by \( T \) the set of vertices in \( F \) which are not adjacent to the vertex \( x \). Then \( |T| \leq d_S(x) - 1 \) and by (3), for every \( y \in N_S(x) \), we have \( d_F(y) \leq |T| + |N_F(x) \cap N_F(y)| \leq |T| + k + 1 - d_S(x) \).

Hence
\[
|V(F) - N_F(y)| \geq k + 2A(S) - (|T| + k + 1 - d_S(x))
\]
\[
= 2A(S) + d_S(x) - 1 - |T| \geq 2A(S).
\]

From this, it follows that the vertices of \( S \) which have fewer nonneighbours in \( F \) than neighbours in \( S \) form an independent set. Denote this vertex set by \( V_1 \). Let \( V_2 \) be the set of vertices in \( S \) which are adjacent to at least one vertex of \( V_1 \). We denote the remaining vertices of \( S \) by \( V_3 \). Every vertex \( v \in V_3 \) is nonadjacent to every vertex
of $V_1$ and $v$ has at least $d_S(v)$ nonneighbours in $F$. Thus

$$v(S)e(F) - e(F, S) \geq 2|V_2|\Delta(S) + \sum_{v \in V_3} d_S(v)$$

$$\geq 2\sum_{v \in V_2} d_S(v) + \sum_{v \in V_3} d_S(v).$$

Since $e(S) \leq \sum_{v \in V_2} d_S(v) + \frac{1}{2} \sum_{v \in V_3} d_S(v)$, it follows that

$$e(S) \leq \frac{1}{2}(v(S)e(F) - e(F, S)).$$

On the other hand, since $\Delta(G) = v(S)$, we have $|v(S) - d_S(u)| \geq d_F(u)$ for any $u \in V(G)$. Hence $v(F)v(S) - e(F, S) \geq \sum_{u \in V(F)} d_F(x) = 2e(F)$. This gives $e(F) \leq \frac{1}{2}(v(S)v(F) - e(F, S))$, and finally we have $e(G) = e(F) + e(S) + e(F, S) \leq v(F)v(S) \leq \lceil n^2/4 \rceil$.

**Case 2:** $v(F) < k + 2\Delta(S)$. For graph $G$ we have $e(G) \leq \frac{1}{2}(v(F)\Delta(G) + v(S)(\Delta(S) + v(F)))$. Since $\Delta(G) = v(S)$, we have

$$e(G) \leq \frac{1}{2}(v(F)v(S) + v(S)(\Delta(S) + v(F)))$$

$$= \frac{1}{2}v(S)(v(F) + \Delta(S) + v(F))$$

$$\leq \frac{1}{2}v(S)(2k + 5\Delta(S)).$$

But $\Delta(S) \leq k$, since $G \in L(\mathcal{C}_k)$. Thus we have

$$e(G) \leq \frac{1}{2}v(S)(7k) \leq \frac{1}{2}n(7k) \leq \left\lfloor \frac{n^2}{4} \right\rfloor,$$

for $n \geq 14k$. \(\square\)

### 3. Saturated graphs for some local properties

The following can be obtained by an easy observation.

**Proposition 3.1.** Let $G$ be a graph of order $n$, $n = \lfloor (k + 2)/2 \rfloor t + r$, $0 \leq r < \lfloor (k + 2)/2 \rfloor$ and let $r = ts + r'$, $0 \leq r' < t$. Then $G$ is $\mathcal{C}_k$-maximal with the minimum number of edges if and only if

$$G = \begin{cases} tk_p \cup K_r, & \text{for } p = \lfloor \frac{k+2}{2} \rfloor \text{ and } r = \lfloor \frac{k+2}{2} \rfloor, \\ (t - r')K_{p_1} \cup r'K_{p_2}, & \text{for } p_1 = \lfloor \frac{k+2}{2} \rfloor + s, \; p_2 = \lfloor \frac{k+2}{2} \rfloor + s + 1 \text{ and } r < \lfloor \frac{k+2}{2} \rfloor. \end{cases}$$

If $n$ is even then $r < \lfloor (k + 2)/2 \rfloor$. If $G \in M(L(\mathcal{C}_k))$, then for any edge $e = uv \in E(\tilde{G})$ the graph $G + e$ contains a forbidden subgraph “located” in the neighbourhood of $u$ or $w$, or in the neighbourhood of a vertex, which is adjacent to both $u$ and $w$. Hence we have the following
Proposition 3.2. Let $G \in M(L(C_k))$. Two vertices $u, w \in V(G)$ are not adjacent in $G$ if and only if one of the following assertions holds:

(i) There exists a vertex $v \in V(G)$ that is adjacent to both $u$ and $w$, such that $N(v)$ has two different components, $H_1$ and $H_2$, with $u$ in $H_1$ and $w$ in $H_2$ and $v(H_1) + v(H_2) \geq k + 2$.

(ii) The graph $G[N(u) \cup \{w\}]$ or the graph $G[N(w) \cup \{u\}]$ has a connected component with at least $k + 2$ vertices.

By Proposition 3.2 we have immediately.

Lemma 3.3. Let $G \in M(L(C_k))$. Then $\text{diam}(G) = 2$ for $v(G) \geq k + 3$.

To establish the minimum size of a graph with the property $L(C_2)$, we need the following lemma.

Lemma 3.4. Let $G \in M(L(C_k))$ and $G'$ be a connected subgraph of $G$ on 3 vertices such that $G' \subseteq G[N(v)]$ for some vertex $v \in V(G)$. Then $d_G(x) = 3$ for each $x \in V(G')$.

Proof. Let $V(G') = \{x, y, z\}$. Suppose $d_G(x) = 2$. Then $x$ is adjacent to only one of the vertices $y, z$. Without loss of generality, let $xz \in E(G)$. Therefore $yz \in E(G)$, because $G'$ is connected in $G[N(v)]$. Since $xy$ is not an edge in $G$, then one of the assertions of Proposition 3.2 holds.

Case 1: Assume that (i) of Proposition 3.2 applies. Then there is a vertex $w \in V(G)$ such that $xw, yw \in E(G)$ and $x$ and $y$ lie in different components of $N(w)$. Clearly, $w \neq z$ and $w \neq v$; hence $d_G(x) > 2$, a contradiction.

Case 2: Assume that (ii) of Proposition 3.2 applies. Then the subgraph $G[N(y) \cup \{x\}]$ has a component with at least 4 vertices. Then in $G[N(v)]$ or in $G[N(z)]$ there is a connected subgraph on 4 vertices, which contradicts that $G \in L(C_2)$. □

Theorem 3.5. Let $k = 1, 2$ and $n \geq 2$. Then

$$\text{sat}(n, L(C_k)) = n - 1 + \text{sat}(n - 1, C_k).$$

Proof. Let $G = K_1 + H$, where $H \in \text{Sat}(n-1, C_k)$. Clearly, $G \in M(n, L(C_k))$ and $e(G) = n - 1 + \text{sat}(n - 1, C_k)$. Thus $\text{sat}(n, L(C_k)) \leq n - 1 + \text{sat}(n - 1, C_k)$.

If $2 \leq n \leq 3$ then all $L(n, C_1)$-maximal and $L(n, C_2)$-maximal graphs are complete. Thus $\text{sat}(n, L(C_k)) = n - 1 + \text{sat}(n - 1, C_k)$ for $k = 1, 2$ and $n \leq 3$.

We shall show that $\text{sat}(n, L(C_k)) \geq n - 1 + \text{sat}(n - 1, C_k)$ for $k = 1, 2$ and $n \geq 4$. Suppose that $G \in \text{Sat}(n, L(C_k))$ and $v \in V(G)$ is a vertex of degree of $\delta(G)$. Let us consider three cases.

Case 1: $\delta(G) = 1$. Since $\text{diam}(G) = 2$ it follows that the vertex $x \in N(v)$ has $n - 1$ neighbours. Owing to the maximality of $G$, $G[N(x)] \in M(n - 1, C_k)$ and $e(G) \geq n - 1 + \text{sat}(n - 1, C_k)$.
Case 2: \( \delta(G) = 2 \). From Proposition 3.1 it follows that

\[
    n - 1 + \text{sat}(n - 1, \mathcal{C}_1) = \begin{cases} 
        \frac{3n}{2} - \frac{3}{2} & \text{for } n \text{ odd,} \\
        \frac{3n}{2} - 2 & \text{for } n \text{ even.}
    \end{cases}
\]

and

\[
    n - 1 + \text{sat}(n - 1, \mathcal{C}_2) = \begin{cases} 
        \frac{3n}{2} - \frac{3}{2} & \text{for } n \text{ odd,} \\
        \frac{3n}{2} & \text{for } n \text{ even.}
    \end{cases}
\]

Let \( u, w \) be the neighbours of \( v \) and let \( S = V(G) - N[v] \).

Case 2.1: \( uw \in E(G) \). Since \( \text{diam}(G) = 2 \), each \( s \in S \) has at least one neighbour in \( N(v) \). Thus

\[
e(G) = 3 + \sum_{s \in S} d_{N(v)}(x) + \frac{1}{2} \sum_{s \in S} (d(x) - d_{N(v)}(x)) \\
= 3 + \frac{1}{2} \sum_{s \in S} (d(x) + d_{N(v)}(x)) \geq 3 + \frac{1}{2}(n - 3)(2 + 1) = \frac{3}{2}n - \frac{3}{2}.
\]

Thus we have

\[
\frac{3}{2}n - \frac{3}{2} \geq n - 1 + \text{sat}(n - 1, \mathcal{C}_1) \quad \text{for all } n
\]

and

\[
\frac{3}{2}n - \frac{3}{2} \geq n - 1 + \text{sat}(n - 1, \mathcal{C}_2) \quad \text{for } n \text{ odd.}
\]

Let \( n \) be even and \( G \in \text{Sat}(n, L(\mathcal{C}_2)) \). Suppose that there is a vertex \( s \in S \) which is adjacent to \( u \) and \( w \). Then the vertices \( v, u, s \) form a connected subgraph in \( N(w) \). But \( d(v) = 2 \) which contradicts Lemma 3.4.

We may therefore assume that \( u \) and \( w \) have no common neighbour in \( S \). Since \( \delta(G) = 2 \) by assumption each vertex in \( S \) has at least one neighbour in \( S \). Since \( |S| \) is odd, it follows that \( S \) has a vertex of degree at least \( 3 \) in \( G \).

Let \( N(s) = \{x, y, u\} \) where \( x, y \in S \) and \( u \in N(v) \). Suppose that \( d(x) = d(y) = 2 \). If \( x \) and \( y \) are adjacent to \( u \), then \( x, y, u \) induce the connected graph in \( N(s) \), which contradicts Lemma 3.4. Assume that one of the vertices \( x, y \), say \( y \), is adjacent to \( w \).

First we show that every \( z \in S \) belongs to a connected subgraph of \( G[N(u) \cap S] \) or \( G[N(w) \cap S] \) of order at least two. Since \( vz \notin E(G) \), it follows that one of the assertions of Proposition 3.2 holds.

If (i) of Proposition 3.2 holds, then there is \( t \in N(v) \) such that \( z \) belongs to some connected subgraph of \( G[N(t)] \) on at least 2 vertices.

If (ii) of Proposition 3.2 holds, then the subgraph \( G[N(z) \cup \{v\}] \) has a connected component with at least 4 vertices. Since each vertex of \( S \) has only one neighbour in \( N(v) \), it follows that there is \( t \in N(v) \) such that \( z \) belongs to a connected subgraph of \( G[N(t) \cap S] \) of order at least two.
Since \( d(y) = 2 \) and \( y, wy \in E(G) \) and \( sw \not\in E(G) \), we obtain that \( y \) is not contained in any connected subgraph of \( G[N(u) \cap S] \) and \( G[N(w) \cap S] \) with at least two vertices. Thus there are at least two vertices of degree at least 3 in \( S \).

\[
e(G) = 3 + \frac{1}{2} \sum_{x \in S} (d(x) + d_{N(v)}(x)) \geq 3 + \frac{1}{2}((n - 5)(2 + 1) + 2(3 + 1))
\]

\[
= \left[ \frac{3}{2}n - \frac{1}{2} \right] \geq n - 1 + \text{sat}(n - 1, \mathcal{C}_2).
\]

**Case 2.2:** \( uw \not\in E(G) \). Assume that \( G \in \text{Sat}(n, L(\mathcal{C}_1)) \). First we show that there is a vertex \( s \in S \) such that \( us \in E(G) \) and \( ws \in E(G) \). Since \( uw \not\in E(G) \) it follows that one of the assertions of Proposition 3.2 holds.

If (i) of Proposition 3.2 holds, then there is \( t \in N(u) \cap N(w) \) such that \( G[N(t)] \) has two different components, \( H_1 \) and \( H_2 \), with \( u \) in \( H_1 \) and \( w \) in \( H_2 \) and \( v(H_1) + v(H_2) \geq 3 \). Since \( d(v) = 2 \) we have \( t \neq v \) and \( t \in S \).

If (ii) of Proposition 3.2 holds, then the subgraph \( G[N(u) \cup \{w\}] \) or \( G[N(w) \cup \{u\}] \) has a connected component with at least 3 vertices. Then there exists a vertex \( s \in S \) that is adjacent to both \( u \) and \( w \). Thus

\[
e(G) = 2 + \frac{1}{2} \sum_{x \in S} (d(x) + d_{N(v)}(x)) \geq 2 + \frac{1}{2}((n - 4)(2 + 1) + 4)
\]

\[
= \left[ \frac{3}{2}n - 2 \right]
\]

and finally

\[
\left[ \frac{3}{2}n - 2 \right] \geq n - 1 + \text{sat}(n - 1, \mathcal{C}_1).
\]

Now we consider the case when \( G \in L(\mathcal{C}_2) \). Since \( vs \not\in E(G) \) for each \( s \in S \) it follows that one of the assertions of Proposition 3.2 holds.

If (i) of Proposition 3.2 holds, then there is \( t \in N(v) \) such that \( s \) belongs to some connected subgraph of \( G[N(v)] \) on at least 3 vertices. By Lemma 3.3, we have \( d(s) = 3 \).

If (ii) of Proposition 3.2 applies, then \( d(s) = 3 \).

In the same way as for property \( L(\mathcal{C}_1) \) we can show that there exists a vertex \( s \) in \( S \) that is adjacent to both \( u \) and \( w \). Thus

\[
e(G) = 2 + \frac{1}{2} \sum_{x \in S} (d(x) + d_{N(v)}(x)) \geq 2 + \frac{1}{2}((n - 4)(3 + 1) + 5)
\]

\[
= \left[ 2n - \frac{7}{2}n \right] \geq n - 1 + \text{sat}(n - 1, \mathcal{C}_2) \quad \text{for} \quad n \geq 4.
\]

**Case 3:** \( \delta(G) = 3 \). \( e(G) \geq \frac{3}{2}n \geq n - 1 + \text{sat}(n - 1, \mathcal{C}_k) \) for \( k = 1, 2 \).

The next theorem gives the minimum possible number of edges in \( S_k \)-maximal graphs with \( n \) vertices.
Theorem 3.6 (Kászonyi and Tuza [7]).

\[
\text{sat}(n, \mathcal{S}_k) = \begin{cases} 
\left( \left\lfloor \frac{(k+1)/2}{2} \right\rfloor \right) 
+ \frac{k}{2} (n - \left\lfloor (k+1)/2 \right\rfloor) & \text{for } n \geq k + \left\lfloor (k+1)/2 \right\rfloor + 1, \\
\left( \binom{k+1}{2} \right) + \left( \frac{n-k-1}{2} \right) & \text{for } k + 2 \leq n \leq k + \left\lceil \frac{(k+1)}{2} \right\rceil + 1.
\end{cases}
\]

All \( \mathcal{S}_k \)-maximal graphs with \( \text{sat}(n, \mathcal{S}_k) \) edges consist of the disjoint union of a \( k \)-regular graph and \( K_p \), \( p = \left\lfloor (k+1)/2 \right\rfloor \) or \( p = \left\lceil (k+1)/2 \right\rceil \) if \( n \geq k + \left\lfloor (k+1)/2 \right\rfloor \) (if both \( k+1 \) and \( n - \left\lfloor (k+1)/2 \right\rfloor \) are odd then these two components are joined by just one edge). If \( k + 2 \leq n \leq k + \left\lfloor (k+1)/2 \right\rfloor \) then the only \( \mathcal{S}_k \)-maximal graph with \( \text{sat}(n, \mathcal{S}_k) \) edges has two components \( K_{k+1} \) and \( K_{n-k-1} \).

Corollary 3.7. Let \( n \geq 4 \). Then

\[ \text{sat}(n, \mathcal{S}_2) = n - 1. \]

If \( G \in M(L(\mathcal{S}_k)) \), then for any edge \( e = uv \in E(\tilde{G}) \), the graph \( G + e \) contains a forbidden subgraph “located” in the neighbourhood of \( u \) or \( w \), or in the neighbourhood of a vertex, which is adjacent to both \( u \) and \( w \). Hence we have the following.

Proposition 3.8. Let \( G \in M(L(\mathcal{S}_k)) \). Two vertices \( u, w \in V(G) \) are not adjacent if and only if one of the following assertions holds:

(i) There exists a vertex \( t \) in \( G \) that is adjacent to both \( u \) and \( w \), such that
   \[ |N(t) \cap N(u)| = k \] or \[ |N(t) \cap N(w)| = k. \]
(ii) \[ |N(u) \cap N(w)| \geq k + 1. \]

Proposition 3.8 immediately follows from the definitions. However, as a consequence of Proposition 3.8 we have

Lemma 3.9. Let \( G \in M(L(\mathcal{S}_k)) \). Then \( \text{diam}(G) = 2 \) for \( v(G) \geq k + 3 \).

Lemma 3.10. Let \( G \in M(L(\mathcal{S}_k)) \). If \( u, w \) are two vertices of \( G \) such that \( d(u) \leq k \) and \( d(w) \leq k \), then \( uv \in E(G) \).

Since \( \mathcal{S}_1 = \mathcal{O}_1 \), it follows from Theorem 3.5 that \( \text{sat}(n, L(\mathcal{S}_1)) = n - 1 + \text{sat}(n-1, \mathcal{S}_1) \).

In the next theorem we determine the number \( \text{sat}(n, L(\mathcal{S}_2)) \).

Lemma 3.11. Let \( 2 \leq n \leq 5 \). Then

\[ \text{sat}(n, L(\mathcal{S}_2)) = n - 1 + \text{sat}(n-1, \mathcal{S}_2). \]
Proof. If $2 \leq n \leq 4$ then all $L(n, \mathcal{S}_2)$-maximal graphs are complete. Thus $\text{sat}(n, L(\mathcal{S}_2)) = n - 1 + \text{sat}(n - 1, \mathcal{S}_2)$ for $2 \leq n \leq 4$.

Let $G \in \text{Sat}(5, L(\mathcal{S}_2))$. If there is $v \in V(G)$ such that $d(v) = 4$ then $G[N(v)] \in \text{Sat}(4, \mathcal{S}_2)$ and $e(G) = n - 1 + \text{sat}(n - 1, \mathcal{S}_2)$.

Assume that $\Delta(G) \leq 3$. Let $u, w \in V(G)$ and $uw \notin E(G)$. Then one of the assertions of Proposition 3.8 holds.

If (i) of Proposition 3.8 holds, then there is $t \in N(u) \cap N(w)$ such that $|N(t) \cap N(u)| = 2$ or $|N(t) \cap N(w)| = 2$. Then $d(t) \geq 4$ which contradicts that $\Delta(G) \leq 3$. Thus for $u, w$ the assertion (ii) of Proposition 3.8 holds. Since $|N(u) \cap N(w)| = 3$ it follows that $V(G) = (N(u) \cap N(w)) \cup \{u, w\}$. Since $G \in \text{Sat}(5, L(\mathcal{S}_2))$, it follows from Proposition 3.8 that $e(G[N(u) \cap N(w)]) = 2$. Thus $e(G) = 8 > n - 1 + \text{sat}(n - 1, \mathcal{S}_2)$. \hfill \Box

Theorem 3.12. Let $n \geq 2$. Then

$$\text{sat}(n, L(\mathcal{S}_2)) = n - 1 + \text{sat}(n - 1, \mathcal{S}_2).$$

Proof. For $2 \leq n \leq 5$, Theorem follows from Lemma 3.11. Let $G = K_1 + H$, where $H \in \text{Sat}(n - 1, \mathcal{S}_2)$. Clearly, $G \in M(L(\mathcal{S}_2))$ and $e(G) = n - 1 + \text{sat}(n - 1, \mathcal{S}_2)$. Thus $\text{sat}(n, L(\mathcal{S}_2)) \leq n - 1 + \text{sat}(n - 1, \mathcal{S}_2)$.

We shall show that $\text{sat}(n, L(\mathcal{S}_2)) \geq n - 1 + \text{sat}(n - 1, \mathcal{S}_2)$ for $n \geq 6$. Suppose that $G \in \text{Sat}(n, L(\mathcal{S}_2))$ and $v \in V(G)$ is a vertex of degree of $\delta(G)$. Let us consider three cases.

Case 1: $\delta(G) = 1$. Since $\text{diam}(G) = 2$ it follows that the vertex $x \in N(v)$ has $n - 1$ neighbours. Owing to the maximality of $G$, $G[N(x)] \in M(n - 1, \mathcal{S}_2)$ and $e(G) \geq n - 1 + \text{sat}(n - 1, \mathcal{S}_2)$.

Case 2: $\delta(G) = 2$. From Corollary 3.7 it follows that $n - 1 + \text{sat}(n - 1, \mathcal{S}_2) = 2n - 3$.

Let $u, w$ be the neighbours of $v$ and let $S = V(G) - N[v]$. From Lemma 3.10 it follows that for any $s \in S$ we have $d(s) \geq 3$. Since $\text{diam}(G) = 2$, each $s \in S$ has at least one neighbour in $N(v)$. Let us consider two cases.

Case 2.1: $uw \in E(G)$. Thus we have

$$e(G) = 3 + \sum_{x \in S} d_{N(v)}(x) + \frac{1}{2} \sum_{x \in S} (d(x) - d_{N(v)}(x))$$

$$= 3 + \frac{1}{2} \sum_{x \in S} (d(x) + d_{N(v)}(x))$$

$$\geq 3 + \frac{1}{2} (n - 3)(3 + 1) = 2n - 3 = n - 1 + \text{sat}(n - 1, \mathcal{S}_2).$$

Case 2.2: $uw \notin E(G)$. First we show that there is a vertex $s \in S$ such that $us \in E(G)$ and $ws \in E(G)$. Since $uw \notin E(G)$ it follows that one of the assertions of Proposition 3.8 holds.
If (i) of Proposition 3.8 holds, then there is \( t \in N(u) \cap N(w) \) such that \(|N(t) \cap N(u)| = 2\) or \(|N(t) \cap N(w)| = 2\), this implies \(d(t) \geq 4\). Since \(d(v) = 2\) it follows that \(t \neq v\) and \(t \in S\).

If (ii) of Proposition 3.8 holds then \(|(N(u) \cap N(w))| \geq 3\). Then there are at least two vertices \(s_1, s_2 \in S\) that are adjacent to both \(u\) and \(w\). Thus

\[
\begin{align*}
e(G) &= 2 + \sum_{x \in S} d_{N(v)}(x) + \frac{1}{2} \sum_{x \in S} (d(x) - d_{N(v)}(x)) \\
&= 2 + \frac{1}{2} \sum_{x \in S} (d(x) + d_{N(v)}(x)) \geq \left[ 2 + \frac{1}{2}(n - 4)(3 + 1) + 5 \right] \\
&= \left[ 2n - \frac{7}{2} \right] \geq 2n - 3 = n - 1 + \text{sat}(n - 1, \mathcal{S}_2).
\end{align*}
\]

**Case 3:** \(\delta(G) = 3\). Since \(\delta(G) = 3\), it follows that for any \(s \in S\) we have \(d(s) \geq 3\). Since \(\text{diam}(G) = 2\), each \(s \in S\) has at least one neighbour in \(N(v)\).

**Case 3.1:** \(G[N(v)]\) is the complete graph. Thus we have

\[
\begin{align*}
e(G) &= 6 + \sum_{x \in S} d_{N(v)}(x) + \frac{1}{2} \sum_{x \in S} (d(x) - d_{N(v)}(x)) \\
&= 6 + \frac{1}{2} \sum_{x \in S} (d(x) + d_{N(v)}(x)) \geq 6 + \frac{1}{2}(n - 4)(3 + 1) \\
&= 2n - 2 > n - 1 + \text{sat}(n - 1, \mathcal{S}_2).
\end{align*}
\]

**Case 3.2:** The graph \(G[N(v)]\) has two edges. Since \(G[N(v)]\) has two edges, it follows that there are two vertices \(u, w\) which are not adjacent. In the same way as in Case 2.2 we can show that there exists a vertex \(s \in S\) adjacent to both \(u\) and \(w\).

\[
\begin{align*}
e(G) &= 5 + \frac{1}{2} \sum_{x \in S} (d(x) + d_{N(v)}(x)) \geq \left[ 6 + \frac{1}{2}(n - 4)(3 + 1) \right] \\
&= \left[ 2n - \frac{5}{2} \right] \geq 2n - 2 > n - 1 + \text{sat}(n - 1, \mathcal{S}_2).
\end{align*}
\]

**Case 3.3:** The graph \(G[N(v)]\) has at most one edge. We may assume \(G[N(x)]\) has at most one edge for any \(x \in V(G)\) such that \(d(x) = 3\). Otherwise we have Case 3.1 or Case 3.2.

Since \(vw \notin E(G)\) for \(s \in S\), then one of the assertions of Proposition 3.8 holds.

If (i) of Proposition 3.8 holds then there is \(t \in N(u) \cap N(s)\) such that \(|N(t) \cap N(s)| = 2\). If \(d(s) = 3\) then the graph \(G[N(s)]\) has at least two edges, a contradiction. Thus \(d(s) \geq 4\).

If (ii) of Proposition 3.8 holds then there exists \(s \in S\) that is adjacent to three vertices of \(N(v)\). Thus

\[
\begin{align*}
e(G) &= 3 + \frac{1}{2} \sum_{x \in S} (d(x) + d_{N(v)}(x)) \geq \left[ 3 + \frac{1}{2}(n - 4)(4 + 1) \right] \\
&= \left[ \frac{5}{2}n - 7 \right] \geq n - 1 + \text{sat}(n - 1, \mathcal{S}_2) \quad \text{for } n \geq 8.
\end{align*}
\]
For \( n = 7 \) there are only three vertices in \( S \). Then any \( s \in S \) has at least two neighbours in \( N(v) \). Thus \( e(G) \geq 12 > n - 1 + \text{sat}(n - 1, \mathcal{S}_2) \). For \( n = 6 \) there are only two vertices in \( S \). Then any \( s \in S \) has at least three neighbours in \( N(v) \). Thus
\[
e(G) \geq 9 > n - 1 + \text{sat}(n - 1, \mathcal{S}_2).\]

Case 4: \( \delta(G) \geq 4. e(G) \geq 2n \geq n - 1 + \text{sat}(n - 1, \mathcal{S}_2) \). □

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References