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Topological Relations in Spatial Databases

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Abstract. Various models for the representation of topological relations have been developed. The aim
of this chapter is to propose a comprehensive view of state-of-the-art models for topological relations in
spatial databases. A prominent role is assumed by the CBM (calculus-based method), which offers a small
set of topological relations with high expressiveness and suitability to be embedded as operators of a spatial
query language. The CBM is applicable to all kinds of complex geometric features and constitutes the
user-oriented level (top level) of a hierarchy of topological operators. At lower levels (with more geometric
details), other models are proposed, which are also applicable to geometric features with a broad boundary
(features that are able to model spatial uncertainty). Eventually, a classification of all topological invariants
of a spatial relation between two features offers the tools to describe the finest topological details.

3.1 Introduction

Spatial database systems have constantly increased in popularity during the last decade. The major
motivation pushing research in spatial database systems is the large number of application domains
they involve. An incomplete list of these domains includes: geographical information systems (GIS),
scientific databases, pictorial databases, and CAD. Spatial data management imposes a number of
new requirements on database systems when compared with traditional ones. Among the many
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requirements, the formalization of spatial relations among physical objects has a central role, since they occur in the majority of spatial queries. The following three examples are typical spatial queries borrowed from the geographical context:

- Retrieve all the states bordering on Italy.
- Retrieve the names of the lakes bigger than 40 square miles and located northwest of Venice.
- Retrieve the three-star hotels close to the railway station in Rome.

The usual way of classifying spatial relations is in terms of two basic categories, namely, topological and metric (orientation and distance):

- Topological relations (e.g., adjacent, overlap, ...) describe whether two objects intersect or not and, in the former case, how they intersect.
- Orientation relations (e.g., north-of, south-of, ...) describe where an object is located with respect to a reference.
- Distance relations (e.g., very close, close, ...) describe how far an object is with respect to a reference.

Within these categories, topological relations have been studied in more depth. Among the available models for classifying topological relations, in this chapter, we will review the evolution and the application of the so-called calculus-based method (CBM). The CBM allows us to model the topological relation between two spatial objects represented as 2D geometric elements (namely, points, lines, areas, and collections of those) in terms of five relations (i.e., in, overlap, cross, touch, and disjoint) and three boundary operators. The CBM enjoys three formal properties:

- Completeness—The five relations make a full covering of all the possible topological situations.
- Exclusiveness—It cannot be the case that two different relations hold between the same two objects.
- Expressiveness—The CBM is better able to distinguish among finer topological configurations than an entire category of previous methods, all based on point-set topology, namely: the four-intersection model, the dimension extended method, and the 9-intersection model. The expressiveness of the CBM was investigated in Clementini and Di Felice.

The previous properties, combined with the small size of the set of relations and operators, make the CBM a natural candidate for being part of a spatial query language. It is worthwhile to note that existing spatial query languages offer some topological operators (for example, on, adjacent, and within in the case of MAPQUERY), however none of the above query languages discusses issues like the expressive power of the topological operators offered or their completeness with respect to some predefined formal criteria. Recently, the CBM has been adopted as the standard way of defining topological operators inside a spatial extension of SQL by the OpenGIS Consortium.

Most of the literature on topological relations used to deal with simple areas and lines (simple areas are two-dimensional point sets homeomorphic to a disk, and simple lines are one-dimensional features embedded in the plane with only two endpoints). Unfortunately, the variety and complexity of spatial entities cannot be modeled with simple geometric features. In order to become operational within a real environment, spatial models should be general enough to include complex objects. Based on dimensionality, a complex feature is either a complex area feature (that is, an area made up of several components possibly containing holes), a complex line feature (that is, a line with separations, self-intersections, and an arbitrary number of endpoints), or a complex point feature (that is, a set of points thought of as a single entity).

Let us put various contributions in a historical perspective. Egenhofer et al. extended the four-intersection model to cover the case of topological relations between areas with holes. Worboys and Bofakos proposed a tree-based model for the representation of areas with holes and islands nested to any finite level. Clementini et al. proposed a hierarchical model, based on the CBM, able
to treat composite areas in topological queries. In Clementini and Di Felice,14 an extension of the CBM for the case of complex features was given, and it was formally proved that the topological relations are mutually exclusive and complete. The CBM also applies for modeling topological relations between 3D features (a 3D implementation of the CBM was described by van Oosterom et al.28). The spatial model adopted by the OpenGIS Consortium27 is based on complex features (multipoints, multipolylines, and multipolygons).

Recently, another extension to spatial data models was considered with the goal of treating uncertainty of spatial data. In essence, a new geometric model is needed that overcomes the limits of the current models of spatial databases, which traditionally are a collection of features with a crisp boundary, which are a rough approximation of geographic reality. The proposal of many research papers is to introduce broad boundaries replacing crisp ones.15,16,18,29−31 The advantage of this approach is that it can be implemented on existing database systems at reasonable cost: the new model can be seen as an extension of existing geometric models. More effort is needed to define new operators applying to the new category of features in order to use them with profit. A first step in this direction took place defining approximate topological relations.16,18 Clementini and Di Felice18 verified that the CBM can also be applied to features with a broad boundary by replacing the definition of set intersection between crisp areas with set intersection between areas with a broad boundary. The CBM relations applied to features with a broad boundary maintain the formal properties of being mutually exclusive and providing a complete coverage of all the realizable geometric configurations. The CBM relations can be seen as the upper level of a hierarchy of topological operators that allow users to query uncertain spatial data independently of the underlying geometric data model.

Topological relations that have been considered by the CBM, the 9-intersection, and all other models discussed so far are based on the analysis of few topological properties (also called topological invariants) of the two features involved in the scene: essentially, the void or non-void intersection between parts of the features (the so-called content invariant). The topological relation between two features can be refined by analyzing all topological invariants that characterize the relation. Topological properties do not change after topological transformations, which include very common GIS transformations such as rotation, translation, scaling, and rubber sheeting. If two different scenes have the same topological invariants, they are topologically equivalent. Formal criteria to establish topological equivalence are extremely important both for recognizing a particular configuration of features and for checking if a scene has been transformed consistently. We will propose a set of topological invariants that is necessary and sufficient to characterize topological equivalence, taken from Clementini and Di Felice.17 The proposed set of invariants to fully describe a topological relation is relevant for the definition of a spatial query language that is able to consider different levels of granularity in topological queries. By considering all topological invariants, the query language would be able to operate at the most detailed level of granularity. Operators at higher levels can be defined starting from the most basic. This process also involves the meaning of natural language spatial expressions (e.g., the road crosses the park), which can be defined by aggregating several similar topological equivalence classes that have a common property.

This chapter is organized as follows: in Section 3.2, we define complex geometric features. Section 3.3 introduces the CBM. In Section 3.4, the 9-intersection model for topological relations between features with a broad boundary is described. Section 3.5 introduces a hierarchy of topological operators. In Section 3.6, the topological invariants are discussed. Section 3.7 gives short conclusions.

### 3.2 Complex Features

The aim of this section is to introduce basic concepts of point-set topology related to the definition of geometric two-dimensional features, which is the usual model for representing 2D projections of
3D scenes. The definition of geometric features is purely topological since we disregard all the shape and metric properties and concentrate on the study of topological relations.

A topological space is generally described as a set of arbitrary elements (points) in which a concept of continuity is specified. A mapping of a topological space \( X \) onto a subset of a topological space \( Y \) in which the neighborhood relations between mapped points are preserved in both spaces is called a continuous mapping. These mappings are also called topological transformations and include translation, rotation, and scaling. Topological relations are those remaining invariant under topological transformations. The study of topological relations between features also depends on the embedding space, which we assume to be \( \mathbb{R}^2 \).

First, we extend the definitions of area, line, and point given in Clementini, Di Felice, and van Oosterom \(^2\) in order to take into account complex areas having both separations and holes, complex lines having separations, more than two endpoints, possibly self-intersections, and, also, sets of points as a single complex point feature. Complex features are far more common than simple ones in reality.

In the following, we introduce the notion of complex geometric features (complex area, complex line, and complex point, respectively); for each feature, the notions of boundary, interior, closure, and exterior are given. All the definitions in this section are based on the concepts of open sets, closed sets, and continuity of point-set topology. \(^3\) Since all the point-sets we refer to in this chapter are subsets of \( \mathbb{R}^2 \), usually we do not repeat it.

### 3.2.1 Complex Area Features

Let \( A \) be a two-dimensional point-set:

- The interior of \( A \), \( A^\circ \), is defined as the union of all open sets contained in \( A \).
- The closure of \( A \), \( \overline{A} \), is defined as the intersection of all closed sets containing \( A \). If \( A \) is a closed set, then \( \overline{A} = A \).
- The boundary of a closed set \( A \), \( \partial A \), is defined as the intersection between its closure and its interior.
- The exterior of a closed set \( A \), \( \text{ext} A \), is defined as \( \text{ext} A = (\mathbb{R}^2 - \overline{A}) \).

Throughout this chapter, we assume that areas are regular closed sets.

**Definition 1:** A point-set \( A \) is called regular closed if and only if \( A = \overline{A}^\circ \).

Regular closed point-sets are usually assumed in spatial data modelling \(^1\) because they allow getting rid of strange cases such as those shown at the top level of Figure IV.3.1.

The notion of complex area relies on the topological notions of connectedness and component. \(^3\) A separation of \( A \) is a pair of disjoint non-empty open sets \( A_1 \) and \( A_2 \) whose union is \( A \). \( A \) is connected if there exists no separation of \( A \), disconnected otherwise, and \( A_1 \) and \( A_2 \) are called the components of \( A \). Separations and components refer to interiors of areas, while spatial features are represented as closed sets, in order to permit an area to also contain its boundary.

**Definition 2:** A simple area is a regular, closed (non-empty), two-dimensional point-set, \( A \), with a connected interior and connected exterior.

Let us now proceed to extend the above definition by taking holes into account (Figure IV.3.2). The exterior of an area with holes may be separated. Separations of the exterior imply that there exists one outer exterior (unbounded set) and \( n > 0 \) inner exteriors (bounded sets). The outer exterior will be denoted by \( A_0^- \) and the inner exteriors by \( A_1^- \ldots A_n^- \). Their union makes up the entire exterior, i.e.,

\[
A^- = \bigcup_{i=0}^{n} A_i^-
\]
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Definition 3: An area with holes is a regular, closed (non-empty), two-dimensional point-set, $A$, with a connected interior such that the intersections of the closures of any two different is empty or equal to a finite set of points:

$$\forall i, j = 0 \ldots n, i \neq j : (\overline{A_i} \cap \overline{A_j} = \emptyset) \lor (\overline{A_i} \cap \overline{A_j} = \{p_1 \ldots p_k\})$$

According to Definition 3, case (a) in Figure IV.3.3 is allowed, while cases (b) and (c) are both not allowed since they do not correspond to regular closed point-sets.

If we relax the constraint that the interior of $A$, $A^\circ$, is connected, we can define the components $A_1 \ldots A_n$ of $A$ as the closure of the corresponding components of $A^\circ$.

Definition 4: A complex area is a closed (non-empty) two-dimensional point-set $A$ with components $A_1 \ldots A_n$, such that:

1. Each $A_i$ is either a simple area or an area with holes.
2. $A_i^\circ \cap A_j^\circ = \emptyset \forall i \neq j$.
3. $(\partial A_i \cap \partial A_j = \emptyset) \lor (\partial A_i \cap \partial A_j = \{p_1 \ldots p_k\})$. 

\[ \text{FIGURE IV.3.1 } \text{Examples of non-regular closed point-sets (cases (a), (b), and (c)) and their counterparts after “regularization” (cases (d), (e), and (f), respectively).} \]

\[ \text{FIGURE IV.3.2 } \text{A valid (a) and three invalid ((b), (c) and (d)) simple areas.} \]

\[ \text{FIGURE IV.3.3 } \text{A valid (a) and two invalid ((b) and (c)) areas with holes.} \]
A complex area made up of four components.

A simple line (a) and four lines with self-intersections (b–e).

Notice that a complex area is a regular closed point-set because, by definition, it is the union of a finite number of regular closed point-sets. Figure IV.3.4 shows an example of a complex area.

3.2.2 Complex Line features

Definition 5: A simple line is a closed (non-empty) one-dimensional point-set, \( L \), defined as the image of a continuous mapping \( f : [0, 1] \rightarrow \mathbb{R}^2 \), such that
\[
f(t_i) \neq f(t_j), \ \forall t_i, t_j \in [0, 1], t_i \neq t_j.
\]
The mappings of 0 and 1 through \( f \) are the two endpoints of \( L \); these two points make up the boundary of a simple line in the plane; i.e., \( \partial L = \{f(0), f(1)\} \).

By relaxing the constraint of no self-intersections in the interior, we get the notion of line with self-intersections. Figure IV.3.5 shows a simple line and four lines with self-intersections. The number of endpoints for a line with self-intersections can be either 2, 1, or 0. In detail, the number of endpoints of \( L \) is equal to:

- Two, if \( f(0) \neq f((0, 1)) \land f(1) \neq f((0, 1)); \partial L = \{f(0), f(1)\} \)
- One, if \( f(0) \neq f((0, 1)) \land f(1) \neq f((0, 1)); \partial L = \{f(1)\} \) (vice versa if \( f(0) \neq f((0, 1)) \land f(1) \in f((0, 1)) \)); \( \partial L = \{f(0)\} \)
- Zero, if \( f(0) = f(1); (f(0) \in f((0, 1)) \land f(1) \in f((0, 1))); \partial L = \emptyset \)

Definition 6: Let \( f_1, f_2, \ldots, f_n \) be continuous mappings from the interval \([0, 1]\) to the plane. We call a complex line any closed (non-empty) one-dimensional point-set, \( L \), defined as the union of the image of the functions \( f_1, f_2, \ldots, f_n \):
\[
f_1([0, 1]) \cup f_2([0, 1]) \cup \cdots \cup f_n([0, 1])
\]
Figure IV.3.6 shows a complex line made up of four components.
The boundary of $L$, i.e., the set of its endpoints, can be built subtracting from the set of endpoints of single $f_i$ all the endpoints that connect with the lines, that is:

$$\partial L = \{f_1(0), f_1(1), f_2(0), f_2(1), \ldots, f_n(0), f_n(1)\}$$

$$- \{f_i(0) | i \in 1..n, f_i(0) \in f_i((0, 1])\}$$

$$\cup \{f_i(1) | i \in 1..n, f_i(1) \in f_i((0, 1])\}$$

$$\cup \{f_i(0) | i, \exists j \in 1..n, i \neq j, f_i(0) \in f_i([0, 1])\}$$

$$\cup \{f_i(1) | i, \exists j \in 1..n, i \neq j, f_i(1) \in f_i([0, 1])\}\}$$

The closure of a line $\overline{L}$, is the set of all points of $L$, endpoints included; therefore, $\overline{L} = L$. The interior of a line, $L^\circ$, is the set difference between the closure of the line and its boundary: $L^\circ = \overline{L} - \partial L$. The exterior of a line, $L^-$, is the set difference between the embedding space and the closure of the line: $L^- (\mathbb{R}^2 - \overline{L}) = (\mathbb{R}^2 - L)$.

3.2.3 Complex Point Features

Definition 7: A point is a (non-empty) zero-dimensional point-set, $P$, consisting of only one element.

As an obvious generalization of the previous concept, we introduce the notion of complex point features as follows.

Definition 8: A complex point is a (non-empty) zero-dimensional point-set, $P$, consisting of a finite number of distinct elements.

Throughout this chapter, we consider the boundary of a complex point feature $P$ empty (i.e., $\partial P = \emptyset$); as a consequence, the interior of a complex point feature $P$ is equal to the union of all the elements in $P$ (i.e., $P^\circ = \bigcup_{i=1}^n P_i$).

3.2.4 Features with Broad Boundaries

Below we give the definition of features with broad boundaries, which differ from crisp ones with regard to the boundary definition. For an area with a broad boundary, we can define an inner boundary and an outer boundary, where the inner boundary is surrounded by the outer boundary. The closed annular area between the inner and outer boundary is the broad boundary of the original area.

Definition 9: An area with a broad boundary $A$ is made up of two areas $A_1$ and $A_2$, with $A_1 \subseteq A_2$, where $\partial A_1$ is the inner boundary of $A$ and $\partial A_2$ is the outer boundary of $A$.

Definition 10: The broad boundary $\Delta A$ of an area with a broad boundary $A$ is the closed subset between the inner boundary and the outer boundary of $A$, i.e., $\Delta A = A_2 - A_1$ or, equivalently, $\Delta A = A_2 - A_1^\circ$. 
Definition 11: The interior, closure, and exterior of an area with a broad boundary $A$ are defined as

$$A^o = A_2 - \Delta A, \quad \overline{A} = A^o \cup \Delta A, \quad A^- = \mathbb{R}^2 - \overline{A},$$

respectively. The interior and exterior of an area with a broad boundary are open sets, while the broad boundary is a closed set. Notice that both the exterior and broad boundary of an area $A$ may have several components, because areas $A_1$ and $A_2$ may have holes. Figure IV.3.7 illustrates some cases that can arise: case (a) is an area without holes; case (b) is an area with two holes; case (c) is an area where $A_1$ has one hole and $A_2$ has two holes; case (d) is an area where $A_1$ has two holes and $A_2$ has one hole.

Now, we consider the most general case of composite areas with a broad boundary:

**Definition 12:** A composite area with a broad boundary $A$ is made up of two composite areas $A_1$ and $A_2$, with $A_1 \subseteq A_2$, where $\partial A_1$ is the inner boundary of $A$ and $\partial A_2$ is the outer boundary of $A$.

**Definition 13:** The broad boundary $\Delta A$ of a composite area with a broad boundary $A$ is the closed subset between the inner boundary and the outer boundary of $A$, i.e., $\Delta A = A_2 - A_1$ or, equivalently, $\Delta A = A_2 - A^o_1$.

**Definition 14:** The interior, closure, and exterior of a composite area with a broad boundary $A$ are defined as $A^o = A_2 - \Delta A, \quad \overline{A} = A^o \cup \Delta A, \quad A^- = \mathbb{R}^2 - \overline{A}$, respectively.

Figure IV.3.8 illustrates some configurations of composite areas: case (a) is an area with two components; case (b) is an area where $A_1$ has two components and $A_2$ has one component; case (c) is an area where $A_1$ has one component and $A_2$ has two components.

For lines embedded in $\mathbb{R}^2$, we can distinguish two kinds of broad boundaries, modeling either the position of the line or the position of its endpoints (Figure IV.3.9). The first kind of broad boundary is an area surrounding the whole line, while the second kind is made up of an area for each endpoint. We will refer to the first kind as a broad line and to the second one as a line with a broad boundary. A broad point is a special case of a broad line. The following definitions hold:
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**Definition 15:** A broad line $\Lambda$ is a simple area representing a family of positions that a simple line $L_1$ can assume under a continuous deformation. The interior of $\Lambda$ is empty, while $\Delta \Lambda = \Lambda$.

**Definition 16:** A line with a broad boundary $L$ is made up of a simple line $L_1$ and two simple areas $A_1$ and $A_2$, surrounding the two endpoints $P_1$ and $P_2$ of the line $L_1$, respectively.

**Definition 17:** The broad boundary $\Delta L$ of a line with a broad boundary $L$ is the union of the two simple areas $A_1$ and $A_2$, that is, $\Delta L = A_1 \cup A_2$.

**Definition 18:** The interior, closure, and exterior of a line with a broad boundary $L$ are defined as $L^\circ = L_1 - \Delta L$, $L = L_1 \cup \Delta L$, and $L^- = \mathbb{R}^2 - L$, respectively.

3.3 The Topological Relations of the CBM

In this section, we refer to generic complex features without the need to distinguish among areas, lines, and points. $\lambda$ will be used to denote these features. Therefore, if $\lambda$ is a point-set, then $\partial \lambda$, $\lambda^\circ$, $\lambda^-$, and $\text{dim} (\lambda)$ will denote the boundary, the interior, the closure, and the dimension of $\lambda$, respectively. The latter is a function, which returns the dimension of a point-set (0, 1, or 2) or nil (–) for the empty set. In case the point-set consists of multiple parts, then the highest dimension is returned. The CBM, as it appeared in Clementini, Di Felice, and van Oosterom\(^2\) for the case of simple features, was made up of five relations and three boundary operators. For the case of complex features, the CBM needs slight modifications to one relation (the cross) and to the boundary operators. For features with a broad boundary, as proven in Clementini and Di Felice,\(^18\) the definitions are the same. In the following, we give all the definitions.

**Definition 19:** The touch relation (applies to area/area, line/line, line/area, point/area, and point/line groups of relations, but not to the point/point group):

\[
\langle \lambda_1, \text{touch}, \lambda_2 \rangle \Leftrightarrow (\lambda_1^\circ \cap \lambda_2^\circ = \emptyset) \land (\lambda_1 \cap \lambda_2 = \emptyset).
\]

**Definition 20:** The in relation (applies to every group):

\[
\langle \lambda_1, \text{in}, \lambda_2 \rangle \Leftrightarrow (\lambda_1 \cap \lambda_2 = \lambda_1) \land (\lambda_1^\circ \cap \lambda_2^\circ = \emptyset).
\]

**Definition 21:** The cross relation (applies to line/line, line/area, point/area, and point/line groups):

\[
\langle \lambda_1, \text{cross}, \lambda_2 \rangle \Leftrightarrow (\text{dim} (\lambda_1^\circ \cap \lambda_2^\circ) < \max (\text{dim} (\lambda_1^\circ), \text{dim} (\lambda_2^\circ)))
\]

\[
\land (\lambda_1 \cap \lambda_2 \neq \lambda_1) \land (\lambda_1 \cap \lambda_2 \neq \lambda_2).
\]
Definition 22: The overlap relation (applies to area/area, line/line, and point/point groups):

\[ \langle \lambda_1, \text{overlap}, \lambda_2 \rangle \Leftrightarrow (\dim(\lambda_1 \cap \lambda_2) = \dim(\lambda_1) = \dim(\lambda_2)) \land (\lambda_1 \cap \lambda_2 \neq \lambda_1) \land (\lambda_1 \cap \lambda_2 \neq \lambda_2). \]

Definition 23: The disjoint relation (applies to every group):

\[ \langle \lambda_1, \text{disjoint}, \lambda_2 \rangle \Leftrightarrow \lambda_1 \cap \lambda_2 = \emptyset. \]

Definition 24: The boundary operator \( b \) for a complex area \( A \): the pair \( (A, b) \) returns the complex line \( \partial A \), which is the union of several (either disjoint or intersecting in a finite number of points) circular lines (with no endpoints).

Definition 25: The boundary operators \( f, t \) for a complex line \( L \): the pairs \( (L, f) \) and \( (L, t) \) return the two complex points \( \{f_i(0), \forall i\} \cap \partial L \) and \( \{f_i(1), \forall i\} \cap \partial L \).

3.4 The Topological Relations of the 9-Intersection

Binary topological relations between two features, \( A \) and \( B \), in \( \mathbb{R}^2 \) can be classified according to the intersection of \( A \)'s interior, boundary, and exterior with \( B \)'s interior, boundary, and exterior. The nine intersections between the six feature parts describe a topological relation and can be concisely represented by the following \( 3 \times 3 \) matrix \( M \), called the 9-intersection:

\[
M = \begin{pmatrix}
A^0 \cap B^0 & A^0 \cap \partial B & A^0 \cap B^-\\
\partial A \cap B^0 & \partial A \cap \partial B & A \partial \cap B^-\\
A^- \cap B^0 & A^- \cap \partial B & A^- \cap B^- \\
\end{pmatrix}.
\]

By considering the values empty (0) and nonempty (1), we can distinguish between \( 2^9 = 512 \) binary topological relations. For two simple areas with a one-dimensional boundary, only eight of them can be realized, and those are disjoint, meet, overlap, coveredBy, inside, covers, contains, equal. For two composite areas with a one-dimensional boundary, there are eight additional matrices that can be realized, totaling 16 relations.\(^4\) Each set of relations provides complete coverage and is mutually exclusive.\(^4\)

The 9-intersection model has been extended to simple areas with a broad boundary (non-composite and without holes).\(^15\)\(^16\) The matrix \( M \) needs to be redefined as follows, having a broad boundary in place of a sharp boundary:

\[
M = \begin{pmatrix}
A^0 \cap B^0 & A^0 \cap \Delta B & A^0 \cap B^- \\
\Delta A \cap B^0 & \Delta A \cap \Delta B & \Delta A \cap B^- \\
\Delta A^- \cap B^0 & \Delta A^- \cap \Delta B & \Delta A^- \cap B^- \\
\end{pmatrix}.
\]

Clementini and Di Felice\(^15\) showed that there are 44 realizable matrices for simple areas with a broad boundary.\(^\star\)\(^\star\) In Figure IV.3.10, the 44 topological relations are illustrated. Notice that the numbering of relations is kept the same as in the mentioned paper\(^15\) for compatibility.

\(^4\)Currently, there are no names in the literature for these additional relations.

\(^\star\)\(^\star\)The interested reader may refer to that paper\(^15\) for understanding the geometric process that leads to exclude intersection matrices that do not correspond to a physical realization of a topological relation.
We now discuss the extension of the previous result to composite areas with a broad boundary. Clementini and Di Felice, discussed the geometric constraints that lead to 44 relations for simple areas with a broad boundary. Such constraints need to be replaced by a set of less restrictive constraints that allow for 56 possible relation matrices for composite areas with a broad boundary. In this analysis, areas with a broad boundary. Also, we assume that each component of an area has a non-empty interior.
In detail, the following geometric constraints that must hold for simple areas with a broad boundary are no longer valid for composite areas (see Clementini and Di Felice\textsuperscript{15}):

- If $A$'s interior intersects with $B$'s interior and exterior, then it must also intersect with $B$'s boundary, and vice versa.
- If both boundaries intersect with the opposite interiors, then the boundaries must also intersect with each other.
- If $A$'s broad boundary intersects with $B$'s interior and exterior, then it must also intersect with $B$'s broad boundary, and vice versa.

By removing the four constraints above, there are 12 additional relation matrices, besides the 44 already known, for composite areas with a broad boundary, whose geometric interpretations are given in Figure IV.3.11.

In Clementini and Di Felice,\textsuperscript{16} spatial relations were organized in a graph having a node for each relation and an arc for each pair of matrices at a minimum topological distance. Such a distance is measured in terms of the number of different values in the corresponding matrices. This kind of graph has been called the closest topological relation graph.\textsuperscript{33} Another way of interpreting the graph is to consider the arc between two relations as a smooth transition that can transform one relation to the other and vice versa. This kind of graph under the latter interpretation has been called the conceptual neighborhood\textsuperscript{34} and is very useful for the analysis of deformations that can affect topological relations.
during motion or changes over time. A sequence of elementary deformations corresponds to a path in the graph. The graph is extremely useful for spatial reasoning and for the process of grouping relations in clusters (as it will be clear in Section 3.5). Such clusters are meaningful to build more general operators, enabling users to communicate with the information system. The clustering process can be done in a complementary manner with a formal basis and with human subjects testing; it has been shown that both approaches reach a substantial agreement over the significant groupings.

We have completed the graph for the new model for composite areas with a broad boundary: Figure IV.3.12 presents the new graph for the 56 relations, as well as a way of clustering them, which will be explained in Section 3.5.

3.5 Hierarchy of Topological Operators

In this section, we define a hierarchy of topological operators to be integrated in an SQL spatial extension such as that proposed by the OpenGIS Consortium. The topological operators are hierarchically organized in three levels: the bottom level offers operators the ability to check for

*The same graph has also been called a continuity network.
detailed topological relations between areas with a broad boundary using the model of Section 3.4. The intermediate and top level offer more abstract operators that allow users to query uncertain spatial data independently of the underlying geometric data model. The results of the application of an operator at one level can be refined by the corresponding operators at a lower level.

The OpenGIS model currently supports two kinds of topological operators: (1) a set of high level user-oriented operators (disjoint, touch, overlap, etc.) that have been extracted from the theoretical work of Clementini and Di Felice; and (2) a low level geometric-oriented operator (relate) that is able to test for the specific topological relation holding among two features in terms of the corresponding 9-intersection matrix.

The model developed in Section 3.4 is able to extend the second kind of topological operators (the geometric-oriented ones) towards features with a broad boundary. Such a model has the advantage that users can test for a large number of spatial relations and fine tune the particular relation being tested. It has the disadvantage that it is a lower level building block and does not have a corresponding natural language equivalent. Users of spatial databases include, among others, SQL interactive users who may wish, for example, to select all the features “nearly overlapping” a query polygon. To address the needs of such users, a set of named spatial relation predicates for simple features has been recalled in Section 3.3, i.e., the calculus-based method (CBM).

The relations and the boundary operators of the CBM can be seen as the upper level of a hierarchy of topological operators having the relation matrices of the 9-intersection at the bottom level. The relations of the CBM are mutually exclusive and provide a complete coverage of all the realizable geometric configurations; as was proven by Clementini and Di Felice, every topological relation expressed by a 9-intersection matrix can be translated into a boolean expression over the CBM relations and boundary operator.

We verified that the definitions of the CBM relations can be also applied to features with a broad boundary by replacing the intersection between features with a crisp boundary with the intersection between areas with a broad boundary. The set intersection between two features with a broad boundary is a new feature with a broad boundary

\[ C = A \cap B, \]

which is defined analogously to ordinary point set intersection as follows:

\[
\begin{array}{c|ccc}
\cap & B^0 & \Delta B & B^-\\
A^0 & C^0 & \Delta C & C^-\\
\Delta A & \Delta C & \Delta C & C^- \\
A^- & C^- & C^- & C^- \\
\end{array}
\]

The CBM relations applied to features with a broad boundary maintain the formal properties of being mutually exclusive and providing a complete coverage of all the realizable geometric configurations. The proof of these properties would be straightforward, being similar to the work of Clementini and Di Felice, and, therefore, it will not be repeated. The correspondence between the CBM relations and the 56 relations of the 9-intersection for composite areas with a broad boundary is shown in Figure IV.3.12: the CBM partitions the set of 56 relations into four clusters.

With only one exception (concerning the cluster induced by the relation disjoint), the number of geometric configurations falling inside a cluster is quite large. The CBM provides the user with a degree of resolution that is not able to capture the geometric details related to the presence of broad boundaries and separate components. It follows that an intermediate level of clustering of topological relations is needed to reach more detail than that offered by the CBM without going down to the bottom level of 56 relations. We give to each intermediate level cluster a name that corresponds to a topological operator.

Each of the previous CBM induced clusters can be subdivided into smaller ones following different criteria such as geometric patterns and topological distance. Specifically, we propose the splitting...
of the touch cluster by identifying pattern matrices that characterize visually similar geometric configurations. The in cluster splits into three smaller clusters derived by aggregating the relations which are at distance 1 from specific relations that are valid also for simple areas with a crisp boundary. The overlap cluster is split into several smaller clusters following a combination of the criteria above. The intermediate level clusters are shown in Figure IV.3.13.

Below, each intermediate cluster is described by listing the numbers of relations (subsequently also called cases) they are composed of and the adopted criteria. The names given to the clusters are obtained by identifying a prototypical geometric configuration representing the cluster (see later in this section). However, we do not exclude that other possible names might be more descriptive for the operators in particular application contexts.

The intermediate level clusters coming from the touch relation are the following:

- **nearlyMeet** (nM) (cases 2, 3, 6, 9): \[
\begin{pmatrix}
0 & \delta & 1 \\
\delta & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\];

- **coveredByBoundary** (cBB) (cases 4, 5, 10, 11): \[
\begin{pmatrix}
0 & 1 & 0 \\
\delta & 1 & \delta \\
1 & 1 & 1 \\
\end{pmatrix}
\]
The intermediate level clusters coming from the overlap relation are six, three of which concern composite areas; notice also that there are two cases (44 and 56) that, being symmetric, are shared by two different clusters:

- **nearlyOverlap** (nO) (cases 18–26): Notice that this matrix gives rise to sixteen different configurations, seven of which are not realizable.
- **interiorCoveredByInterior** (iCBi) (cases 28, 29, 31, 44): case 29 plus other relations at distance 1 from it being part of the overlap cluster (except case 23)
- **interiorCoversInterior** (iCvi) (cases 34, 35, 37, 44): case 35 plus other relations at distance 1 from it being part of the overlap cluster (except case 23)
- **partlyInside** (pI) (cases 46, 47, 48, 52, 56): all configurations being part of the overlap cluster that need to have an area A with at least two components to be realized
- **partlyContains** (pCt) (cases 49, 50, 51, 53, 56): all configurations being part of the overlap cluster that need to have an area B with at least two components to be realized
- **crossContainment** (cCt) (case 45).

The in cluster splits into three intermediate level clusters derived by aggregating the relations which are at distance 1 from the relations 27 (coveredBy), 33 (covers), and 41 (equal) (valid also for simple areas with a crisp boundary), respectively:

- **nearlyInside** (nI) (cases 39, 27, 30, 55): case 27 plus other relations at distance 1 from it being part of the in cluster
- **nearlyContains** (nCt) (cases 40, 33, 36, 54): case 33 plus other relations at distance 1 from it being part of the in cluster
- **nearlyEqual** (nE) (cases 41, 42, 43, 32, 38): case 41 plus other relations at distance 1 from it.

On average, the size of the clusters is 56 (relations)/14 (clusters) = 4. This size represents a reasonable trade-off from two opposite points of view: the need of enhancing the degree of granularity granted by the CBM and the need of keeping the number of operators low also at the intermediate level in order to encourage the user to use them. Table IV.3.1 collects information about the three-level hierarchy of topological operators.

The top level and the intermediate level can be used for queries that are independent of the geometric model used at the bottom level. While the top level is useful for a fast screening of the database, the intermediate level offers the possibility of refining the answer. The usability of the intermediate level operators is guaranteed by the fact that each corresponding cluster can be associated to a prototypical geometric configuration expressing, in a visual manner, the kind of results that the user can expect from that operator. In this way, queries on uncertain data are implemented as a small deviation from the reference prototype.

Prototypes for the intermediate level operators are shown in Figure IV.3.14. They are designed making use of the simplest and more intuitive shapes characterizing the clusters. Hence, simple crisp features are used for the operators disjoint, nearlyMeet, nearlyOverlap, and nearlyEqual. The prototypes for the operators coveredByBoundary, coversWithBoundary, boundaryOverlap, interiorCoveredByInterior, and interiorCoversInterior make explicit the broad boundary because the
corresponding clusters refer to geometric configurations which do not exist in the case of features with crisp boundaries. Similarly, the prototypes for the operators nearlyInside, nearlyContains, crossContainment, partlyInside, and partlyContains make explicit the existence of multiple components since the corresponding clusters include geometric configurations that are realizable for composite areas.

### 3.6 Topological Invariants

In the models for representing topological relations described so far, only a few general topological invariants are considered (mainly, the content invariant); such invariants provide a broad classification of topological relations. The introduction of other invariants allows finer topological distinctions.
3.6.1 Content Invariants

In the relation between two regular closed geometric elements, there are various point-sets to be considered, which can be empty or nonempty. This is generally called the content invariant and can be calculated for several sets: intersections, set differences, and symmetric differences. The most convenient one seems to be the intersection, since it gives a comprehensive categorization of topological relations; further, it is extensively used in the literature. Content invariants are not independent from each other but can be hierarchically structured to form classes of topological relations. At the root of this hierarchy there is the following invariant:

**Definition 26:** Given two geometric features \( \lambda_1 \) and \( \lambda_2 \), the feature intersection content invariant is the intersection \( \lambda_1 \cap \lambda_2 \).

The feature intersection content invariant may assume the two values, empty (\( \emptyset \)) or nonempty (\( \neg \emptyset \)), and therefore leads to two classes of relations: disjoint and non-disjoint. The category of relations in which the features have some common parts can be further refined by considering intersections of boundary and interiors:

**Definition 27:** The 4-intersection content invariant is a \( 2 \times 2 \) matrix of the intersections of the interiors and boundaries of the two features \( \lambda_1 \) and \( \lambda_2 \):

\[
M = \begin{pmatrix}
\lambda_1^0 \cap \lambda_2^0 & \lambda_1^0 \cap \partial \lambda_2 \\
\partial \lambda_1 \cap \lambda_2^0 & \partial \lambda_1 \cap \partial \lambda_2
\end{pmatrix}
\]

Each intersection may be empty (\( \emptyset \)) or nonempty (\( \neg \emptyset \)), resulting in a total of \( 2^4 = 16 \) combinations. Each case is represented by a matrix of values. It is possible to apply some simple geometric constraints to assess that not all combinations make sense for all areas, where the 4-intersection content invariant is able to recognize 8 different categories of relations. All 16 combinations are, instead, possible for lines.

**Definition 28:** The 9-intersection content invariant is a \( 3 \times 3 \) matrix containing the empty/nonempty values for interior, boundary, and exterior intersections:

\[
\begin{pmatrix}
\lambda_1^0 \cap \lambda_2^0 & \lambda_1^0 \cap \partial \lambda_2 & \lambda_1^0 \cap \lambda_2^2 \\
\partial \lambda_1 \cap \lambda_2^0 & \partial \lambda_1 \cap \partial \lambda_2 & \partial \lambda_1 \cap \lambda_2^2 \\
\lambda_1^2 \cap \lambda_2^0 & \lambda_1^2 \cap \partial \lambda_2 & \lambda_1^2 \cap \lambda_2^2
\end{pmatrix}
\]
FIGURE IV.3.15  (a) zero-dimensional intersection with one component, (b) one-dimensional intersection with one component, (c) zero-dimensional intersection with two components, and (d) zero-dimensional intersection with three components.

It is an extension of the 4-intersection that considers also the exterior of the features, besides interior and boundary. Excluding the impossible cases, we have 33 possible cases for line features. The 9-intersection allows for finer refinements of topological relations than the 4-intersection.

3.6.2 Dimension
A refinement of the content invariant is given by the dimension of each intersection set.

Definition 29: The **dimension invariant** is a function that returns the dimension of a given set $S$: $\dim(S)$. If $S$ has disconnected components, then the highest dimension is considered.

The dimension is useful for distinguishing whether the set intersection is 2, 1, or 0-dimensional. The dimension of the intersection cannot be higher than the lowest dimension of the two operands of the intersection. For line features, this means that the dimension of the intersection may be only 0 or 1 (Figure IV.3.15). The dimension is significant for the closures’ intersection (or equivalently for the interiors’ intersection), since the intersections involving boundaries can only be 0-dimensional.

The extension of the 4 and 9 intersections with the dimension has been also considered. The topological configurations distinguishable with the 9-intersection extended with the dimension are 36.

3.6.3 Number of Intersections
Different configurations may have the same content and dimension of intersection sets, but each intersection set may be disconnected and made up of a different number of connected parts (Figure IV.3.15).

Definition 30: The **number of connected components** of a set $A$ is a topological invariant and will be denoted as $\#(A)$.

$\#(A)$ may be any positive integer. We exclude cases of an infinite number of intersections.

Until now, we have discussed invariants that are meaningful for every geometric feature embedded in $\mathbb{R}^2$. To exploit all invariants for lines, we will concentrate on specific topological invariants characterizing the relation between pairs of simple lines. For the case of a non-void intersection between two simple lines, we are going to consider all connected intersection components. To find out all invariants, it is necessary to establish an order on the points belonging to the line, hence considering the line as an oriented feature.

3.6.4 Intersection Sequence
The intersection sequence describes the order in which the various components of the intersection between two lines $L_1$ and $L_2$ occur. Under a topological transformation, the intersection sequence must be preserved. Following the line $L_1$ from its first point and assigning numeric labels to the
intersections until the last point is reached, the intersection sequence is a sequence of numbers established traversing the line $L_2$ and recording the labels that were previously assigned to $L_1$.

**Definition 31:** Let us consider two intersecting lines $L_1$ and $L_2$, images of the mappings $f_1$ and $f_2$, respectively. Their intersection has a number of connected components $m = \#(L_1 \cap L_2)$; let us label them with $0, 1, \ldots, m-1$, following the order given by the line $L_1$ (from $f_1(0)$ to $f_1(1)$). The *intersection sequence* is an $m$-tuple $S(L_2) = \langle k_0, k_1, \ldots, k_{m-1} \rangle$, which is a permutation of the $m$-tuple $(0, 1, \ldots, m-1)$ and is obtained by traversing the line $L_2$ from $f_2(0)$ to $f_2(1)$ and recording the labels previously assigned when traversing the line $L_1$. The intersection sequence is a topological invariant.

For example, the sequence of intersection of the configuration in Figure IV.3.16 is $S(L_2) = \langle 0, 1, 3, 2 \rangle$. In general, given two lines with $m$ intersections, all the possible different sequences are $m!$.

### 3.6.5 Intersection Type

Each intersection has “local” characteristics that make it different from the others. Let us consider a neighborhood of an intersection component big enough to properly contain the component itself (Figure IV.3.17). In general, each line with respect to such a neighborhood has an incoming arc $i$ and an outcoming arc $o$, however, if the intersection involves some endpoint, some of those arcs may be empty.

**Definition 32:** For each intersection $k$, let us consider a disc $I(k)$ as a neighborhood. We define in this way at most four arcs, incident at the intersection, two incoming $i_1, i_2$, and two outcoming $o_1, o_2$, belonging to the lines $L_1$ and $L_2$, respectively. Establishing a clockwise orientation on the boundary of $I(k)$, we find a sequence of arcs $T(k) = (a_1, a_2, a_3, a_4)$ that we call the *intersection type*. The intersection type is a topological invariant.
As a convention, we start recording the arcs from \( i_1 \), but this is not restrictive since sequences of arcs are invariant to cyclic permutations. The arc \( a_i, i = 1, \ldots, 4 \), may assume the values \((i_1, o_1, o_2)\), \((i_1, o_1, i_2)\), \((i_1, o_2)\), \((i_1, i_2)\). When some arcs are missing, the sequence can contain from zero to three values (Figure IV.3.18). If the sequence is empty, i.e., \( T(\emptyset) = () \), the lines are equal since the intersection has no incoming or outcoming arcs. If the intersection type contains only one arc, it means that a line is entirely contained in the other one with a common endpoint (Figure IV.3.18f). In the case of two arcs, each arc can belong to a different line (Figure IV.3.18c,d) or to the same line (Figure IV.3.18e). If a line is entirely contained in the other one and the intersection component is zero-dimensional, then the first line degenerates into a point.

Two important categories of intersections easy to differentiate are the crossing (Figure IV.3.19a), when the sequence of arcs of the two lines is alternated, and the touching (Figure IV.3.19b), when the two lines remain in the same part of the plane. The intersection type is able to distinguish the two categories.

The number of different intersection types in the case of four arcs can be evaluated by fixing the first arc in the sequence as \( i_1 \) and considering the permutations of the remaining three arcs; as a result we have \( 3! = 6 \) possible values. In the case of three arcs, we have to consider the combinations of four values taken in groups of three; in this way, we obtain four groups of three arcs. For each group, which is invariant to cyclic permutations, we have to fix an arc and take into account the permutations of the remaining two. Therefore, the number of different intersection types is given by:

\[
\binom{4}{3} \cdot 2! = 8.
\]
In the case of two arcs, all possible combinations are:
\[
\binom{4}{2} = 6.
\]

For only one arc, the number of intersection types is 4.

### 3.6.6 Collinearity Sense

It is also necessary for one-dimensional intersections to distinguish whether the segments that make these components are traversed following the same orientation in the two lines (Figure IV.3.20a) or the reverse orientation (Figure IV.3.20b).

**Definition 33:** Given an intersection component \( k \), with \( \text{dim}(k) = 1 \), between two lines \( L_1 \) and \( L_2 \), images of the mappings \( f_1 \) and \( f_2 \), respectively, there exists an interval \([t_1, t_2]\) ⊆ [0, 1] and an interval \([u_1, u_2]\) ⊆ [0, 1], such that \( f_1([t_1, t_2]) = f_2([u_1, u_2]) \). If \( f_1(t_1) = f_2(u_1) \) and \( f_1(t_2) = f_2(u_2) \), then the **collinearity sense** is positive (\( CS(k) = +1 \)). If \( f_1(t_1) = f_2(u_2) \) and \( f_1(t_2) = f_2(u_1) \), then the collinearity sense is negative (\( CS(k) = -1 \)). The collinearity sense is a topological invariant.

For convenience, we define \( CS(k) = 0 \) if \( \text{dim}(k) = 0 \). In this way, the CS invariant replaces the dimension.

### 3.6.7 Link Orientation

Given a sequence of intersections between two lines \( L_1 \) and \( L_2 \), for each pair of consecutive intersections \( \langle h, k \rangle \), the part of line \( L_2 \) that is between the two intersections will be called a **link** (denoted with \( L_2(h, k) \)).

**Definition 34:** Given two lines \( L_1 \) and \( L_2 \) and a link \( L_2(h, k) \) between two consecutive intersections \( \langle h, k \rangle \), consider the cycle obtained traversing the link \( L_2(h, k) \) and coming back to \( h \) traversing the line \( L_1 \). If such a cycle is counterclockwise, then the **link orientation** assumes the value \( l \) (left). If it is clockwise, \( LO_{L_2}(h, k) \) assumes the value \( r \) (right). The link orientation is a topological invariant.

Notice that the orientation of the cycle is determined by \( L_2(h, k) \) and disregards the orientation of the line \( L_1 \). The values \( l \) and \( r \) come from the following consideration: if the cycle is counterclockwise, \( L_2(h, k) \) leaves the region bounded by the cycle on the left; if it is clockwise, \( L_2(h, k) \) leaves the region on the right. Figure IV.3.21 shows a counterclockwise and a clockwise link orientation. Analogously, we could also consider the orientation of links belonging to the line \( L_1 \) with respect to \( L_2 \).

### 3.6.8 The Classifying Invariant for Simple Lines

To describe a scene involving two simple lines, we can use all the invariants discussed so far. As a matter of fact, not all invariants are necessary since some of them are implicit in others. The content invariants are able to express conditions on the location of the endpoints of the two lines: the same information comes from the knowledge of the number of intersections and each intersection type.
The number of intersections is implicitly given by the intersection sequence. The dimension of an intersection component falls in the definition of collinearity sense.

Given two simple lines $L_1$ and $L_2$, with $m$ intersections, the classifying invariant (CI) for the topological relation between them is defined as a table $CI(L_1, L_2)$ made up of four columns and $m$ entries. The four columns give the intersection sequence, the collinearity sense, the intersection type, and the link orientation, respectively. The generic entry contains the label of the intersection component $k_i$, the collinearity sense $CS(k_i)$, the type $T(k_i)$, and the link orientation $LO_{L_2}(k_i, k_{i+1})$.

The link orientation is undefined in correspondence of the intersection $k_{m-1}$.

The general structure of $CI(L_1, L_2)$ is the following:

<table>
<thead>
<tr>
<th>$S(L_2)$</th>
<th>$CS$</th>
<th>$T$</th>
<th>$LO_{L_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_0$</td>
<td>$CS(k_0)$</td>
<td>$T(k_0)$</td>
<td>$LO_{L_2}(k_0, k_1)$</td>
</tr>
<tr>
<td>$k_1$</td>
<td>$CS(k_1)$</td>
<td>$T(k_1)$</td>
<td>$LO_{L_2}(k_1, k_2)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$k_{m-1}$</td>
<td>$CS(k_{m-1})$</td>
<td>$T(k_{m-1})$</td>
<td>$LO_{L_2}(k_{m-2}, k_{m-1})$</td>
</tr>
</tbody>
</table>

If there are no intersections between the two lines ($m = 0$), then the classifying invariant is an empty table and will be denoted $CI(L_1, L_2) = \emptyset$.

For example, the scene in Figure IV.3.16 has the following $CI$:

<table>
<thead>
<tr>
<th>$S(L_2)$</th>
<th>$CS$</th>
<th>$T$</th>
<th>$LO_{L_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>(i_1, i_2, o_1, o_2)</td>
<td>$l$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>(i_1, o_2, o_1, i_2)</td>
<td>$r$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>(i_1, i_2, o_1, o_2)</td>
<td>$r$</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>(i_1, o_1, i_2)</td>
<td>—</td>
</tr>
</tbody>
</table>

Clementini and Di Felice\textsuperscript{17} proved the necessity and the completeness of the classifying invariant. This result allows to conclude that if two scenes have the same classifying invariant, then they are topologically equivalent, while if two scenes have two distinct classifying invariants, they are topologically different. We refer to that paper also for a description of the classifying invariant for complex lines.

### 3.7 Conclusions

In this chapter, we have provided a global view of the models for representing topological relations in spatial databases. Such models have been developed during the last decade by the research community and, only in part, have been absorbed by database technology. The standard for spatial data recently
established by the OpenGIS Consortium incorporates the CBM relations and the 9-intersection relations for complex features with crisp boundary. The models for topological relations between features with broad boundary and the topological invariants still have to be translated inside spatial database technology.

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