Threshold condition for global existence and blow-up to a radially symmetric drift–diffusion system

Carlos Conca\(^{a,b}\), Elio Espejo\(^{c,\ast}\)

\(^{a}\) Departamento de Ingeniería Matemática (DIM) and Centro de Modelamiento Matemático (CMM), Universidad de Chile (UMI CNRS 2807), Casilla 170-3, Correo 3, Santiago, Chile

\(^{b}\) Institute for Cell Dynamics and Biotechnology: a Centre for Systems Biology, University of Chile, Santiago, Chile

\(^{c}\) Millennium Institute for Cell Dynamics and Biotechnology, University of Chile, Santiago, Chile

**ABSTRACT**

For a class of drift–diffusion systems Kurokiba et al. \cite{Kurokiba2006} proved global existence and uniform boundedness of the radial solutions when the \(L^1\)-norm of the initial data satisfies a threshold condition. We prove in this letter that this result prescribes a region in the plane of masses which is sharp in the sense that if the drift–diffusion system is initiated outside the threshold region of global existence, then blow-up is possible: suitable initial data can be built up in such a way that the corresponding solution blows up in a finite time.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction

A mathematical model for particles interacting via the gravitational potential is the following system of PDE’s:

\[
\begin{align*}
\partial_t n - \Delta n + \nabla \cdot (n \nabla \psi) &= 0 & t > 0, x \in \mathbb{R}^2 \\
\partial_t p - \Delta p - \nabla \cdot (p \nabla \psi) &= 0 & t > 0, x \in \mathbb{R}^2 \\
-\Delta \psi &= -(p - n) & x \in \mathbb{R}^2 \\
\psi(0, x) &= n_0(x) \geq 0, \quad p(0, x) = p_0(x) \geq 0 & x \in \mathbb{R}^2.
\end{align*}
\]  

(1)

The initial value problem (1) is one of the most representative systems of the so-called drift–diffusion models. Kurokiba and Ogawa proved in \cite{Kurokiba2006} for system (1) local well-posedness, positiveness of the variables \(n\) and \(p\), mass conservation and the following blow-up result.

**Theorem 1 (Blow-up in Finite Time)**. Let \(s > 1\) and

\[L^s_2(\mathbb{R}^2) = \{f \in L^1_{\text{loc}}(\mathbb{R}^2); (1 + |x|^2)^{s/2} f(x) \in L^2(\mathbb{R}^2)\}.\]

Let \(n_0\) and \(p_0\) be given in \(L^s_2(\mathbb{R}^2)\) with \(n_0, p_0 \geq 0\) everywhere, and satisfying

\[
\frac{\left(\int_{\mathbb{R}^2} (n_0 - p_0) \, dx\right)^2}{\int_{\mathbb{R}^2} (n_0 + p_0) \, dx} > 8\pi.
\]  

(2)

Then the solution of (1) blows up in a finite time.
The possibility of having global existence in time for system (1) whenever
\[ \frac{\int_{\mathbb{R}^2} (n_0 - p_0) \, dx}{\int_{\mathbb{R}^2} (n_0 + p_0) \, dx} < 8\pi \] was suggested by Kurokiba and Ogawa in [1] and it was partially proved by Kurokiba et al. in [2]. Specifically, they proved in particular the following.

**Theorem 1** (Existence of Blow-up). Let \( s > 1 \). Suppose that the initial data \( n_0 \) and \( p_0 \in L_s^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \) are positive radially symmetric functions. If
\[ n_0 \|_{L^1(\mathbb{R}^2)}, \quad p_0 \|_{L^1(\mathbb{R}^2)} < 8\pi, \] then the corresponding radially symmetric solution \((n(t), p(t))\) exists globally in \( C(0, \infty; L^2_s(\mathbb{R}^2)) \). Moreover, there exists a constant \( C \) such that
\[ \sup_{t \geq 0} \| n(t) \|_{L^\infty(\mathbb{R}^2)} \leq C, \quad \sup_{t \geq 0} \| p(t) \|_{L^\infty(\mathbb{R}^2)} \leq C. \]

However, condition (3) results not to be sufficient to guarantee global existence as was proved by Espejo et al. in [3]. The aim of this letter is to show that although condition (3) does not represent a sufficient condition for global existence in time, the condition of global existence (4) is optimal (in the radial case). Precisely, we show that if \( \theta_1 \) and \( \theta_2 \) are arbitrary positive parameters satisfying
\[ \theta_1 > 8\pi \quad \text{or} \quad \theta_2 > 8\pi, \] then initial data \( n_0 \) and \( p_0 \) can be constructed in such a way that
\[ \theta_1 = \int_{\mathbb{R}^2} n_0 \, dx, \quad \theta_2 = \int_{\mathbb{R}^2} p_0 \, dx, \] and system (1) blows up.

We find it worth pointing out that the blow-up result of Theorem 1 is valid even in the non-radial case; meanwhile Theorem 2 holds true only under radially symmetric conditions on the initial data. As already mentioned, condition (4) involves the constant \( 8\pi \) which results to be sharp. The problem of finding a similar optimal threshold condition under non-radial initial conditions remains open. Threshold-type conditions for a similar system, mainly for the Keller–Segel model of two species with non-radial initial conditions, are currently being investigated by Conca et al. (see [4], for example).

**Notation.** We denote \( M_1(r, t) = \int_{B(0, r)} n \, dx = 2\pi \int_0^r n \rho \, d\rho \), \( M_2(r, t) = \int_{B(0, r)} p \, dx = 2\pi \int_0^r p \rho \, d\rho \), \( \theta_1 = \int_{\mathbb{R}^2} n_0 \, dx \) and \( \theta_2 = \int_{\mathbb{R}^2} p_0 \, dx \). In terms of \( M_1 \) and \( M_2 \), system (1) reduces to
\[ \begin{align*}
\partial_t M_1 &= \frac{r}{\partial r} \left( 1 \quad \frac{1}{r} \frac{\partial M_1}{\partial r} \right) - \frac{M_2 - M_1}{2\pi r} \frac{\partial M_1}{\partial r}, \\
\partial_t M_2 &= \frac{r}{\partial r} \left( 1 \quad \frac{1}{r} \frac{\partial M_2}{\partial r} \right) + \frac{M_2 - M_1}{2\pi r} \frac{\partial M_2}{\partial r}.
\end{align*} \]

**2. Optimization of the blow-up condition**

In order to simplify system (5) we prove first that under suitable conditions on the initial data, then either \( M_1 \leq M_2 \) or \( M_1 \geq M_2 \). This result will allow us to reduce our analysis to only one equation and then obtain our main result concerning blow-up.

**Theorem 3** (Mass Comparison). Suppose that the initial data \( n_0, p_0 \in L_s^2(\mathbb{R}^2) \cap C^2(\mathbb{R}^2) \) of (1) are positive radially symmetric functions. If
\[ M_1(0, r) = \int_{\mathbb{R}^2} n_0 \, dx \geq \int_{\mathbb{R}^2} p_0 \, dx = M_2(0, r), \] then for any solution in \( C([0, T); L^2_s(\mathbb{R}^2)) \cap C([0, T); C^2(\mathbb{R}^2)) \),
\[ M_1(t, r) \geq M_2(t, r). \]

**Proof.** The idea of the proof is to formulate a suitable parabolic equation in the variable \( R := \int_0^r n - p \rho \, d\rho \) and then apply the maximum principle to show that \( R \geq 0 \). With this end in mind, we first define the following variables:
\[ v(t, x) = n(t, x) + p(t, x) \]
\[ w(t, x) = n(t, x) - p(t, x). \]
It follows that \((v, w)\) satisfies the following parabolic–elliptic system:

\[
\begin{align*}
\partial_t v - \Delta v + \nabla (v \nabla \psi) &= 0 & t > 0, & x \in \mathbb{R}^2 \\
\partial_t w - \Delta w + \nabla (v \nabla \psi) &= 0 & t > 0, & x \in \mathbb{R}^2 \\
-\Delta \psi &= w & x \in \mathbb{R}^2 \\
v(0, x) &= n_0(x) + p(x), & w(0, x) &= n_0(x) - p_0(x).
\end{align*}
\]

We change now to polar coordinates and integrate on \((0, r)\). Denoting \(S = \int_0^r v \rho d\rho, R = \int_0^r w \rho d\rho\) and using \(v = \frac{1}{r} \int_0^r v \rho d\rho = \frac{1}{r} \frac{\partial S}{\partial r}\) and \(w = \frac{1}{r} \int_0^r w \rho d\rho = \frac{1}{r} \frac{\partial R}{\partial r}\) we get after simplifying the following reduced system:

\[
\begin{align*}
\partial_t S - r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial S}{\partial r} \right) - \frac{1}{r} \frac{\partial R}{\partial r} R &= 0 \\
\partial_t R - r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial R}{\partial r} \right) - \frac{1}{r} \frac{\partial S}{\partial r} R &= 0.
\end{align*}
\]

(7) (8)

By hypothesis in \(t = 0\) we have \(R \geq 0\) and \(R = 0\) on \(r = 0\), in addition the coefficient of \(R\) is negative. By means of the change of variables \(R = R^e\) in (8) we get

\[
\partial_t R - r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial R^e}{\partial r} \right) + \left( -\frac{1}{r} \frac{\partial R}{\partial r} - 1 \right) R^e = 0.
\]

Now if the minimum of \(R^e\) were negative at this point we would have \(\partial_t R^e \leq 0, \frac{\partial R}{\partial r} = 0\) and \(\frac{\partial^2 R}{\partial r^2} \geq 0\) and therefore

\[
\partial_t R^e - r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial R^e}{\partial r} \right) = \partial_t R - \frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} \leq 0 \quad \text{and} \quad (-\frac{1}{r} \frac{\partial R}{\partial r} - 1) R^e > 0 \quad \text{getting a contradiction with (8).}
\]

It follows that \(R^e \geq 0\) and consequently \(R \geq 0\). Using that

\[
R = \int_0^r w \rho d\rho = \int_0^r (n - p) \rho d\rho \\
= \frac{1}{2\pi} \left( \int_{B(0,r)} n \rho d\rho - \int_{B(0,r)} p \rho d\rho \right) = \frac{1}{2\pi} (M_1 - M_2)
\]

we conclude that

\[
M_1 \geq M_2.
\]

(9)

Therefore, variables \(M_1\) and \(M_2\) are comparable. \(\Box\)

The following result was proved in [[3], Th. 3] and it plays an essential role for our considerations. For the sake of completeness we outline the proof here.

**Theorem 4 (Conditions for the Boundedness of \(p\)).** Suppose that the initial data \(n_0, p_0 \in L^2(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)\) of (1) are positive radially symmetric functions. If the initial data of (1) satisfy

\[
n(r, 0) \geq p(r, 0),
\]

(10)

then for any solution in \(C([0, T]; L^2(\mathbb{R}^2)) \cap C([0, T]; C^2(\mathbb{R}^2))\) there exists a constant \(C\) such that

\[
p(r, t) \leq C \quad \forall t > 0, x \in \mathbb{R}^2.
\]

**Proof.** From (5) and (9) it follows that

\[
\partial_t M_2 \leq r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial M_2}{\partial r} \right).
\]

(11)

At the time \(t = 0\), the variable \(p_0\) is bounded for some constant \(C\), then we have \(M_2(r, 0) = \frac{1}{2\pi} \int_{B(0,r)} p_0 \rho d\rho \leq \frac{1}{2\pi} C \int_{B(0,r)} \rho d\rho = C^2\). Introducing the transformation \(\overline{M}(r, t) = M_2(r, t) - C^2\), it follows from (11) that \(\overline{M}\) satisfies

\[
\begin{align*}
\partial_t \overline{M} &\leq \frac{\partial^2 \overline{M}}{\partial r^2} - \frac{1}{r} \frac{\partial \overline{M}}{\partial r} \\
\overline{M}(r, 0) &\leq 0, \quad \overline{M}(0, t) = 0, \quad \overline{M}(r, t) \leq 0.
\end{align*}
\]

(12)

Thus the maximum principle yields

\[
\overline{M}(r, t) = M_2(r, t) - C^2 \leq 0
\]
and, hence, $M_2(r, t) = 2\pi \int_0^r p\rho d\rho \leq Cr^2$. Using regularity theory for parabolic equations (see [5]), we then obtain the bound $p = \frac{1}{r} \frac{\partial M_2}{\partial r} \leq C$, for a suitable constant $C > 0$. □

The last theorem will allow us to simplify system (5) and apply the moments technique (see, e.g. [6–8]) to prove blow-up. This final result shows that in the radial case condition (2) for blow-up can be improved and even more it shows that conditions (4) for global existence (2) are optimal.

**Theorem 5** (Finite Time of Existence for $p$). Suppose that the initial data $n_0, p_0 \in L^2(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ are positive radially symmetric functions and that the inequality $n(r, 0) \geq p(r, 0)$ is satisfied. If

$$\frac{\theta_1}{2\pi} (8\pi - \theta_1) + Cm_1(0) < 0, \quad (13)$$

where

$$\theta_1 = \int_{\mathbb{R}^2} n_0 \, dx, \quad \theta_2 = \int_{\mathbb{R}^2} p_0 \, dx,$$

is fulfilled, then we have $T_{\text{max}} < \infty$, where $T_{\text{max}}$ is the maximum time of existence of solution $n$.

**Proof.** Let $m_1(t) = \int_{\mathbb{R}^2} |x|^2 n(x, t) \, dx$. Multiplying the first equation of (1) by $|x|^2$ and integrating the resulting relation over $\mathbb{R}^2$, we obtain

$$\partial_t \int_{\mathbb{R}^2} n |x|^2 \, dx = \int_{\mathbb{R}^2} |x|^2 \Delta n \, dx - \int_{\mathbb{R}^2} |x|^2 \, \nabla \cdot (n \nabla \psi) \, dx. \quad (14)$$

From Green’s identity we get

$$\partial_t \int_{\mathbb{R}^2} n |x|^2 \, dx = \int_{\mathbb{R}^2} (\Delta |x|^2) n \, dx - \int_{\mathbb{R}^2} |x|^2 \, \nabla \cdot (n \nabla \psi) \, dx = 4 \int_{\mathbb{R}^2} n \, dx + 2 \int_{\mathbb{R}^2} x \cdot \nabla \psi \, dx.$$  

From $\frac{\partial \psi}{\partial r} = \frac{M_2 - M_1}{2\pi}$ and the identity for radial symmetric functions $x \cdot \nabla \psi = r \frac{\partial \psi}{\partial r}$ we get

$$\int_{\mathbb{R}^2} n (x \cdot \nabla \psi) \, dx = 2\pi \int_0^\infty nr \frac{\partial \psi}{\partial r} \, rdr = 2\pi \int_0^\infty n \left( \frac{M_2 - M_1}{2\pi} \right) \, rdr = \frac{1}{2\pi} \int_0^\infty M_2 nrdr - \int_0^\infty M_1 nrdr. \quad (15)$$

From Theorem 4 we know that $p$ is bounded. Thus we obtain the estimate $M_2 \leq Cr^2$. It follows from (15) that

$$\int_{\mathbb{R}^2} n (x \cdot \nabla \psi) \, dx \leq C \int_0^\infty nr^3 \, dr - \frac{1}{2\pi} \int_0^\infty M_1 \frac{\partial M_1}{\partial r} \, dr = \frac{C}{2\pi} \int_{\mathbb{R}^2} n |x|^2 \, dx - \frac{1}{4\pi} \theta_1^2.$$  

From (14) it follows that

$$\frac{d}{dt} m_1(t) \leq 4\theta_1 + 2 \left( -\frac{1}{4\pi} \theta_1^2 \right) + Cm_1(t)$$

$$= 4\theta_1 - \frac{1}{2\pi} \theta_1^2 + Cm_1(t)$$

$$= \frac{\theta_1}{2\pi} (8\pi - \theta_1) + Cm_1(t).$$

Suppose

$$\frac{\theta_1}{2\pi} (8\pi - \theta_1) + Cm_1(0) < 0.$$  

In consequence,

$$0 \leq m_1(t) < m_1(0) + \left( \frac{\theta_1}{2\pi} (8\pi - \theta_1) + Cm_1(0) \right) t.$$
Thus there exists $T_0 \in (0, \infty)$ such that
$$m_1(t) \to 0 \quad \text{as} \quad t \to T_0.$$  
Therefore $T_{\text{max}} \leq T_0 < \infty$. □

In a similar way we obtain the following result.

**Theorem 6 (Finite Time of Existence for $n$).** Suppose that the initial data $n_0$ and $p_0 \in C_0^\infty(\mathbb{R}^2)$ and $p(r, 0) \geq n(r, 0)$.

If
$$\frac{\theta_2}{2\pi} (8\pi - \theta_2) + Cm_2(0) < 0,$$
where
$$\theta_1 = \int_{\mathbb{R}^2} n_0 \, dx \quad \text{and} \quad \theta_2 = \int_{\mathbb{R}^2} p_0 \, dx,$$
is fulfilled, then we have $T_{\text{max}} < \infty$, where $T_{\text{max}}$ is the maximum time of existence of solution $p$.

Theorems 5 and 6 show that if $\theta_1$ and $\theta_2$ are arbitrary positive parameters satisfying
$$\theta_1 > 8\pi \quad \text{or} \quad \theta_2 > 8\pi,$$
then we can construct initial data $n_0$ and $p_0$ such that
$$\theta_1 = \int_{\mathbb{R}^2} n_0 \, dx, \quad \theta_2 = \int_{\mathbb{R}^2} p_0 \, dx,$$
and system (1) blows up. For example, take $n_0, p_0$ satisfying (6) with $\theta_1 = \int_{\mathbb{R}^2} n_0 \, dx > 8\pi$ together with an initial moment $m_1(0)$ small enough such that inequality (13) holds, then Theorem 5 implies blow-up for system (1). Consequently, the optimal blow-up region should be the square found by Kurokiba et al. in [2].

**References**