Basis functions for concave polygons

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Abstract

Polynomials suffice as finite element basis functions for triangles, parallelograms, and some other elements of little practical importance. Rational basis functions extend the range of allowed elements to the much wider class of well-set algebraic elements, where well-set is a convexity type constraint. The extension field from $R(x, y)$ to $R(x, y, \sqrt{x^2 + y^2})$ removes this quadrilateral constraint as described in Chapter 8 of [E.L. Wachspress, A Rational Finite Element Basis, Academic Press, 1975]. The basis function construction described there is clarified here, first for concave quadrilaterals and then for concave polygons. Its application is enhanced by the GADJ algorithm [G. Dasgupta, E.L. Wachspress, The adjoint for an algebraic finite element, Computers and Mathematics with Applications, doi:10.1016/j.camwa.2004.03.021] for finding the denominator polynomial common to all the basis functions.

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1. Motivation

Basis functions for globally continuous patchwork polynomial approximation play an essential role in many finite element applications. The ubiquitous elements are triangles and rectangles in either physical or isoparametric coordinates, in which case easily determined polynomial basis functions can be constructed for any degree of polynomial approximation within each element. Although this holds for parallelograms as well, polynomials do not suffice for other quadrilaterals or elements with more than four sides. The generality of isoparametric triangles and rectangles is such that little effort has been directed toward other elements. However, there are some applications for which more general element shapes offer advantages. Thirty years ago [1], rational basis functions were developed for generalization to elements having any number of sides, each of which could be an algebraic curve. Certain convexity type constraints were needed. These elements were not widely used because of difficulties in computing the basis function denominators and evaluating various integrals needed for assembling finite element equations.

Recently, the denominator construction for convex polygons was simplified in [3] and generalized to elements with curved sides in [2]. Also, Dasgupta demonstrated how, by using software like MATHEMATICA and MAPLE, one...
may resolve integration complexities by converting area integrals to contour integrals through use of the divergence theorem. These innovations greatly enhanced application of rational basis functions with a wide class of new elements.

One application of the more general elements is when isoparametric theory is used to map physical elements into triangles and rectangles. It is a simple task to develop rational basis functions over regular polygons with any number of sides and to generate quadrature formulas for these elements. In addition, the degree two isoparametric elements yield only degree one approximation in the physical space. There are applications for which degree two approximation improves accuracy. Only the rational basis functions in the physical coordinates can achieve any desired degree of accuracy.

The convexity type constraint limited these elements somewhat. It was shown in [1] that concave elements could be handled in two dimensions with irrational basis functions derived from three dimensional elements. Again, the practical value is limited by the alternative of subdividing a concave element into convex elements. Only for applications for which the smoothness of derivatives is required interior to a concave region can one derive any benefit from retaining the concave element. Such applications are rare, but may be encountered more often in biological studies with the modeling of actual cell structures. Another application is when the Jacobian of the isoparametric mapping vanishes within the element. This is especially apt to occur when the physical element is not convex. This situation can be resolved through the use of mapping into a concave quadrilateral in the isoparametric space. Degree two approximation over a concave quadrilateral is required for this application. This paper is concerned with the construction of irrational basis functions over concave polygons. Degree one approximation is first considered. Finally, theory for constructing degree two approximation over concave quadrilaterals is exposed.

2. Theory for concave quadrilaterals

Let points 1, 2, 3 and 4 be the vertices of a quadrilateral with a concave corner at 2, as in Fig. 1. The origin is chosen to lie at 2. Let the midpoints of sides (1,4) and (3,4) be 5 and 6, respectively. We set $z = (x^2 + y^2)^{1/2}$ at the vertices, where the nonnegative square root is understood throughout. We construct basis functions for the “polycondron” of order five (Fig. 2) with the three planar faces (1; 4), (3, 4) and (1; 2; 3) and the fourth face the surface in the nonnegative $z$ branch of the quadric $z^2 - (x^2 + y^2)$ into which the quadrilateral is mapped. When 1, 2 and 3 are collinear, the plane (1, 2, 3) is chosen as $y_3 x - x_3 y$. This is the dividing plane between concavity and convexity. We denote the nonnegative-$z$ branch of the quadric as $G_+$ and the negative branch $z + (x^2 + y^2)^{1/2}$ as $G_-$. The 3D element has vertices 1, 3, and 4. Its fourth vertex is 7 at the intersection of the three planar faces.

The order $m$ of the 3D element is the sum of the orders of its surfaces. The “adjoint” is the surface of maximum order $m - 4$ that vanishes on all the EIP. In this case $m = 5$, and the adjoint is a plane. The EIP $A$, $B$ and $C$ which determine the adjoint plane are:

$$A = G_- \cdot (1; 4) \cdot (3; 4) \quad B = G_- \cdot (1; 2; 3) \cdot (1; 4) \quad C = G_- \cdot (1; 2; 3) \cdot (3; 4).$$

The computation of these EIPs is not difficult. Point $A = (x_4, y_4, -(x_4^2 + y_4^2)^{1/2})$. When 1, 2, and 3 are collinear, $B = (x_1, y_1, -(x_1^2 + y_1^2)^{1/2})$ and $C = (x_3, y_3, -(x_3^2 + y_3^2)^{1/2})$. Otherwise, we compute $B$ and $C$ as follows: Determine
coefficients $\alpha$ and $\beta$ for the plane

$$(1; 2; 3) = z + \alpha x + \beta y.$$  

(1)

Let

$$(1; 4) = 1 + a_1 x + b_1 y$$

(2a)

and

$$(3; 4) = 1 + a_3 x + b_3 y.$$  

(2b)

Choice of the origin at vertex 2 allows this normalization when 1, 2, and 3 are not collinear. Define

$$a_0 = (b_1 \alpha - a_1 \beta)^2 - (a_1^2 + b_1^2)$$

(3a)

and

$$b_0 = 2a_1 (\beta^2 - 1) - 2b_1 a \beta.$$  

(3b)

Then

$$x_B = -\left( \frac{b_0}{a_0} + x_1 \right),$$

(4a)

$$y_B = -\frac{1 + a_1 x_B}{b_1},$$

(4b)

and

$$z_B = -(x_B^2 + y_B^2)^{1/2}.$$  

(4c)
C is found with \((a_3, b_3, x_3)\) replacing \((a_1, b_1, x_1)\) above. The adjoint (denominator polynomial) for the 3D basis functions vanishes on the plane \((A; B; C)\). The 3D basis function numerators are:

\[
\begin{align*}
N_1(x, y, z) &= k_1(3; 4)(2; 5; B) \\
N_2(x, y, z) &= k_2(1; 4)(3; 4) \\
N_3(x, y, z) &= k_3(1; 4)(2; 6; C) \\
N_4(x, y, z) &= k_4(1; 2; 3)(5; 6; A) \\
N_5(x, y, z) &= k_5(3; 4)(1; 2; 3) \\
N_6(x, y, z) &= k_6(1; 4)(1; 2; 3) \\
N_7(x, y, z) &= k_7[z - (x^2 + y^2)^{1/2}].
\end{align*}
\]

The value for \(k_i\) normalizes \(N_i/(A; B; C)\) to unity at node \(i\). This element is ill-set when plane \((A; B; C)\) penetrates the element. However, \((A; B; C)\) does not vanish over the restriction to surface \(G_+.\) Replacing \(z\) in the \(N_i\) and in \((A; B; C)\) with \((x^2 + y^2)^{1/2}\), we obtain the quadrilateral basis functions

\[
\begin{align*}
W_1(x, y) &= (N_1 + .5N_5)/(A; B; C) \\
W_2(x, y) &= N_2/(A; B; C) \\
W_3(x, y) &= (N_3 + .5N_6)/(A; B; C) \\
W_4(x, y) &= [N_4 + .5(N_5 + N_6)]/(A; B; C).
\end{align*}
\]

These basis functions satisfy all the requirements for degree one approximation over the concave quadrilateral.

The basis function associated with vertex 7 does not affect the quadrilateral basis. In fact, vertex 7 plays no role. We may choose a fifth surface as \(-c\) for some negative \(c\) between zero and the maximum value of \(z\) among \(A, B,\) and \(C\). The reduction theorem relating adjoints of adjoining elements yields \(Q(\text{new element}) = dG + (z - c)(A; B; C)\) for some constant \(d\). Thus \(Q(\text{new element}) \equiv (z - c)(A; B; C)\) on surface \(G\). The numerators for the 3D basis functions for all nodes on \(G_+\) are now just \((z-c)\) times the old numerators. The concave quadrilateral basis functions are identical for both elements.

An anomaly must still be addressed. Vertex 2 of the quadrilateral was chosen as the apex of the cone to yield the change in sign of the square root interior to the 2D element. The edge \((1, 2, 3)\) of the 3D element contains this double-point of surface \(G\) and the element is not well-set. However, the analysis may be applied to a nearby well-set element defined by a small parameter, say \(\epsilon\), which approaches zero. The surface is now \(G = z^2 - (x^2 + y^2 + \epsilon^2)\). We add the plane face \(z\) as discussed in the last paragraph (with \(c = 0\)). The resulting 3D element is well-set. Surfaces \(G_+\) and \(G_-\) do not touch, and the quadrilateral is mapped into a section of \(G_+\) with no double-point at the side node into which vertex 2 has been mapped. As \(\epsilon\) approaches zero, the basis functions approach those previously developed. We have defined the construction at the limit to be identical to the limit of the well-set constructions, and have thus removed the singularity.

3. Examples of concave quadrilaterals

We first choose the quadrilateral in Section 8.4 of [1], relabelling the nodes to be consistent with those used here (Fig. 1): \(1 = (-1, 0), 2 = (0, 0), 3 = (0, -1), 4 = (1, 1), 5 = (0, .5), 6 = (.5, 0)\). Then plane \((1; 2; 3) = x + y + z\) so \(\alpha = \beta = 1\). Hence, vertex 7 of the 3D element (Fig. 2) is \((1, 1, -2)\). Also, \((1; 4) = 1 + x - 2y\) so \(a_1 = 1\) and \(b_1 = -2\). Line \((3; 4) = 1 + y - 2x\) so \(a_3 = -2\) and \(b_3 = 1\). EIP \(A = (1, 1, -\sqrt{2})\). For EIP \(B\): \(a_0 = (-2 - 1)^2 - (1^2 + 2^2) = 4\) and \(b_0 = -2(2) = 4\) so \(x_B = -(1 + 1) = 0, y_B = .5,\) and \(z_B = -.5\). Thus, \(B = (0, .5, -.5)\). For \(C\) the roles of \(x\) and \(y\) are interchanged and \(C = (.5, 0, -.5)\). Thus,

\[
Q = (A; B; C) = 1 + (1 + 1.5\sqrt{2})(x + y) + 3(1 + \sqrt{2}/2)z.
\]

The opposite factor at vertex 1 is \((3; 4; C) = (1 + y - 2x),\) and the adjacent factor at vertex 1 is \((2; 5; B) = x\). The normalized 3D basis function at vertex 1 is therefore:

\[
W_1(x, y, z) = -x(1 + y - 2x)/Q.
\]
The normalized basis function at vertex 2 is
\[ W_2(x, y, z) = \frac{(1 + y - 2x)(1 + x - 2y)}{Q}. \] (7b)
Interchanging \( x \) and \( y \) in Eq. (7a), we have
\[ W_3(x, y, z) = \frac{-y(1 + x - 2y)}{Q}. \] (7c)
The adjacent factor at vertex 4 is \((5, 6, A) = 1 - (1.5\sqrt{2} - 1)(x + y) - 3(1 - .5\sqrt{2})z\), and the normalized basis function at vertex 4 is
\[ W_4 = \frac{(1 + .5\sqrt{2})(x + y + z)[(1.5\sqrt{2} - 1)(x + y) + 3(1 - .5\sqrt{2})z - 1]}{Q}. \] (7d)
The normalized basis functions at side nodes 5 and 6 are
\[ W_5 = \frac{(2 + \sqrt{2})(1 + y - 2x)(x + y + z)}{Q}; \] (7e)
\[ W_6 = \frac{(2 + \sqrt{2})(1 + x - 2y)(x + y + z)}{Q}. \] (7f)
Using Eqs. (5) and (6), we obtain the quadrilateral basis function numerators with \( z = (x^2 + y^2)^{1/2}; \)
\[ N_1 = (1 + y - 2x)[.5\sqrt{2}x + (1 + .5\sqrt{2})(y + z)] \] (8a)
\[ N_2 = (1 + y - 2x)(1 + x - 2y) \] (8b)
\[ N_3 = (1 + x - 2y)[.5\sqrt{2}y + (1 + .5\sqrt{2})(x + z)] \] (8c)
\[ N_4 = (1 + .5\sqrt{2})(x + y + z)[1 + (1.5\sqrt{2} - 2)(x + y) + 3(1 - .5\sqrt{2})z]. \] (8d)
One may verify that these functions are linear on the quadrilateral sides. This is easily shown for the basis function \( W_2 \):
\[ N_2 = (1 + x - 2y)(1 + y - 2x) = (1 + x)(1 - 2x) \] on side \((1, 2)\) where \( y = 0 \), and \( x < 0 \) (9a)
\[ Q = 1 + (1 + 1.5\sqrt{2})(x + y) + 3(1 + \sqrt{2}/2)z = 1 + x - 3x = 1 - 2x, \] (9b)
\[ W_2 = \frac{N_2}{Q} = (1 + x) \] on side \((1, 2)\). (9c)
We observe that on the extension of side \((1, 2)\) into the Quadrilateral, \( x > 0 \) and \( N_2 \) is positive within the element, vanishing only at \((.5, 0)\) on side \((3, 4)\). \( Q \) is also positive within the element. The same result is obtained on side \((2, 3)\) with \( y \) and \( x \) interchanged.
We now consider the linearity of \( W_1 \) on side \((1, 2)\) where \( y = 0 \) and \( z = -x: \)
\[ N_1 = 1.5(1 - x)[.5\sqrt{2}x + .5(1 + .5\sqrt{2})(1 + x) + (1 + .5\sqrt{2})z] \]
\[ = .5(1 - x)[1.5(1 + .5\sqrt{2}) + 1.5(1 + .5\sqrt{2})x + 3(1 + .5\sqrt{2})z] \] (10a)
\[ Q = 1 + .5(1 + 1.5\sqrt{2})(1 + 3x) + 3(1 + .5\sqrt{2})z \]
\[ = 1.5[1 + .5\sqrt{2} + (1 + 1.5\sqrt{2})x + 2(1 + .5\sqrt{2})z] \] (10b)
\[ W_1 = \frac{N_1}{Q} = .5(1 - x). \] (10c)
That \( N_3 \) is linear on sides \((2, 3)\) and \((3, 4)\) follows from symmetry. We consider \( W_4 \) on side \((1, 4)\), where \( y = .5(1 + x)\):
\[ N_4 = (1 + .5\sqrt{2})[(x + y) + (x^2 + y^2) + (3\sqrt{2} - 4)xy + (1 + x + y)z] \]
\[ = .5(1 + .5\sqrt{2})(1 + x)[x + .5(1 + x) + (3\sqrt{2} - 4)x + 3z] \]
\[ = .75(1 + .5\sqrt{2})(1 + x)[1 + 2z + (2\sqrt{2} - 1)x]. \] (11a)
Moreover, on side (1, 4):

\[ Q = 1 + (1 + 1.5\sqrt{2})(x + y) + 3(1 + \sqrt{2}/2)z \]
\[ = 1 + .5(1 + 1.5\sqrt{2}) + 1.5(1 + 1.5\sqrt{2})x + 3(1 + .5\sqrt{2})z \]
\[ = 1.5(1 + .5\sqrt{2})[1 + 2z + (2\sqrt{2} - 1)x]. \]

Thus, \( W_4 = N_4/Q = .5(1 + x) \) on side (1, 4). Interchanging the roles of \( x \) and \( y \), we find that \( W_4 \) is also linear on side (1, 3). We have established the linearity of all four basis functions on the quadrilateral sides.

We next examine a concave vertex at which the adjacent sides do not meet at a right angle (Fig. 3). The vertices are: 1 = \((-1, 1, \sqrt{2})\), 2 = \((0, 0, 0)\), 3 = \((0, -1, 1)\), and 4 = \((1, 0, 1)\). The side nodes are 5 = \((0, .5, .5)\) and 6 = \((.5, -5, -5\sqrt{2})\). The sides are \((1; 4) = 1 - x - 2y, (3; 4) = 1 + y - x, \) and \((1; 2; 3) = y + z + (1 + \sqrt{2})x\). The three EIPs are \( A = (1, 0, -1), B = (0, .5, -5), \) and \( C = (.5, -5, -5\sqrt{2})\). The adjoint is

\[ Q = (A; B; C) = 1 + (1.5\sqrt{2} - 1)x + (1.5\sqrt{2} - 2)y + 1.5\sqrt{2}z. \]

Rather than repeat the tedious algebra in the previous example, we consider only the linearity of the basis function at node 4 on side (1, 4) where \( W_6 = 0 \). On this side \( B_4(x, y) = W_4(x, y, z) + .5W_5(x, y, z)\). The 3D adjacent factor at vertex 4 is

\[ R_4 = (5; 6; A) = (4 - 1.5\sqrt{2})x + 1.5(2 - \sqrt{2})z - (1 - 1.5\sqrt{2})y - 1 \]
\[ \equiv (1 + z - 3y) \text{ (mod(1, 4)) where } x = 1 - 2y. \]

The opposite factor \((1; 2; 3)\) on side (1, 4) is:

\[ F_4 = [(1 + \sqrt{2}) + z - (1 + 2\sqrt{2})y]. \]

The numerator of the 3D basis function at vertex 4 is on side (1, 4):

\[ N_4(x, y, z) = k_4[(1 + \sqrt{2}) + z - (1 + 2\sqrt{2})y](1 + z - 3y). \]

We note that on (1, 4)

\[ Q = 1.5\sqrt{2}(1 - y + z) \]

so that \( W_4 \) normalized to unity at vertex 4 is:

\[ W_4(x, y, z) = 1.5(\sqrt{2} - 1)[(1 + \sqrt{2}) + z - (1 + 2\sqrt{2})y](1 + z - 3y)/Q. \]

In similar fashion, we find that the normalized basis function at node 5 is on side (1, 4):

\[ W_5 = 3\sqrt{2}y[(1 + \sqrt{2}) + z - (1 + 2\sqrt{2})y]/Q. \]
It follows that the 2D basis function at node 4 on side (1, 4) is:

\[ B_4 = W_4 + \frac{1}{2} W_5 = 1.5 \left[ (1 + \sqrt{2}) + z - (1 + 2\sqrt{2})y \right] \left( \sqrt{2} - 1 \right) (1 + z - 3y) + \sqrt{2}y \right] / Q. \]  

(19)

Noting that \((\sqrt{2} - 1)^2 = 3 - 2\sqrt{2}\), we simplify Eq. (19) to

\[ B_4 = 1.5(\sqrt{2} - 1) \left[ (1 + \sqrt{2}) + z - (1 + 2\sqrt{2})y \right] \left( 1 + z + (\sqrt{2} - 1)y \right) / Q, \]

and replacing \(z^2\) by \(x^2 + y^2\), we collect terms to obtain

\[ B_4 = 1.5(\sqrt{2} - 1)(2 + \sqrt{2})(1 - y)(1 - y + z) / Q = (1 - y)[1.5\sqrt{2}(1 - y + z)] / Q = 1 - y. \]  

(20)

We have shown that \(B_4\), the 2D basis function at node 4, has the asserted linear variation on side (1, 4). This must be true since the 3D basis functions are linear in \((x, y, z)\) on the quadrilateral sides. We omit further verification. The foregoing analysis indicates how one may proceed to verify linearity of all the 2D basis functions.

This analysis was implemented with the MATLAB program CONQUAD. Basis functions for the two sample problems discussed here were constructed and the analysis was verified for these elements. These basis functions did achieve degree one approximation over the elements.

4. The GADJ algorithm

The GADJ algorithm exposed in [2] generates the element adjoint (denominator common to all its basis functions) as a linear combination of the more readily found numerators. For this algorithm, one must determine EIPs only of adjacent sides in two space, and for the three faces which meet at each vertex in three space. For elements with more than a few sides or faces this is a significant reduction in the effort required to generate the adjoint from all the EIPs as originally described in [1]. For the concave quadrilateral, the adjoint is simply the plane \((A,B,C)\), where EIPs \(A\), \(B\) and \(C\) are computed to determine the adjacent factors in any event. Here, GADJ is not needed. This is also true for convex rectangles where the adjoint is the exterior diagonal of the quadrilateral. However, it is instructive to demonstrate the validity of the algorithm for the concave quadrilateral before proceeding to concave polygons of order greater than four. As in Eq. 12 of [2], let \(L_i\) denote any plane pierced at vertex \(i + 1\) by the line \((i, i + 1)\), and let \(N_i\) denote the numerator of the basis function at \(i\). The algorithm succeeds when the ratio \(L_i N_i / L_i N_{i+1}\) is constant along edge \((i, i + 1)\).

Consider edge (1, 2) of a concave quadrilateral with basis functions of Eq. (5):

\[ N_1 = (3; 4)(2; 5; B) \equiv (3; 4)(2; 5) \text{ mod}(1; 2), \]
\[ L_1 = (2; 3; B) \equiv (2; 3) \text{ mod}(1; 2), \]
\[ L_2 = (1; 4), \]
\[ N_2 = (1; 4)(3; 4). \]

Thus, the ratio required to be constant is

\[ \frac{L_1 N_1}{L_1 N_2} = \frac{(1; 4)(3; 4)(2; 5)}{(2; 3)(1; 4)(3; 4)} = \frac{(2; 5)}{(2; 3)} \equiv 1 \text{ mod}(1; 2). \]

That such conditions occur follows from the construction as a mapping from a 3D element for which linearity along the edges is guaranteed.

5. Concave polygons

Polygons with only one concave vertex may be treated in a similar fashion. The concave vertex may be chosen as the origin so that the quadric surface remains \(z^2 - (x^2 + y^2)\) with the face on the nonnegative component. This may be clarified by means of a simple example. Let the 2D concave element be of order five with vertices: \(1 = (-1, -1), 2 = (0, 0), 3 = (1, -1), 4 = (1, 1), \) and \(5 = (-1, 1)\) (Fig. 4). The 3D element is of order \(m = 6\) with four planar and one quadric face. The planar faces are \((1; 2; 3) = z + \sqrt{2}y, (3; 4; A) = 1 - x (4; 5; B) = 1 - y (5; 1; C) = 1 + x)\.

The intersection of opposite planes \((1; 2; 3)\) and \((4; 5; B)\) is the edge on which \(y = 1\) and \(z = -\sqrt{2}\). All the EIPs fall on the plane on which \(z + \sqrt{2} = 0\), and the adjoint of order \(m - 4 = 2\) is \(Q = (z + \sqrt{2})^2\). Linearity on side (1,
2) is easily shown for the basis function $W_2 = 2(1 - x^2)(1 - y)/Q$. On this side, $y = x$ and $z = -\sqrt{2}x$. Thus, $W_2 = \frac{2(1-x^2)(1-x)}{2(1-x)^2} = 1 + x \mod(1,2)$. Linearity on side (2, 3) is also verified easily. Now $y = -x$ and $z = \sqrt{2}x$ so that $W_2 = \frac{2(1-x^2)(1+x)}{2(1+x)^2} = 1 - x \mod(2, 3)$. Linearity has Thus been verified for all four basis functions.

A special case is of practical interest. When part of a partition is refined to improve accuracy over a subregion, interfaces between refined and unrefined regions may introduce hybrid elements in which a vertex of the refined elements is a side node of the coarser elements. Various interpolation schemes have been introduced to allow for such elements. For example, one may restrict the fine element value to be the average of the two adjacent vertex values along the common edge with the coarse element. The theory for concave polygons provides an alternative. The side node of the coarse element may be considered as a vertex with a $180^\circ$ angle. This simplifies the concave element basis function construction in that all the 3D planar side linear forms do not depend on $z$. The resulting basis functions yield a smooth transition from coarse to fine elements without limiting the side node to an average of the adjacent vertex nodes.

The situation is more complex when the polygon has more than one concave vertex. The basis functions must be singular at all of these vertices. Let $P$ be the product of $[(x - x_j)^2 + (y - y_j)^2]$ for all concave vertices $(x_j, y_j)$. For $r$ concave vertices, the surface on which $z^2 - P = 0$ is of order $2r$.

We choose all faces of the 3D element which do not contain concave vertices of the polygon as planes parallel to the $z$-axis. The 3D element may be constructed with a plane $z = c = 0$, as discussed in Section 1. The choice of this plane does not affect the analysis when the adjoint is determined by GADJ and the factor $z - c$ is not introduced into the numerators of the polygon basis functions.

We consider a case where there are no adjacent concave vertices. Let the concave polygon have $n$ sides and $r$ concave nodes. The $n - 2r$ 3D faces of sides without concave vertices may be chosen to be parallel to the $z$ axis. The $r$ faces with concave 2D vertices as edge nodes are determined by the concave vertex and the two adjacent convex vertices. The construction of basis functions then becomes quite complex. An indication of the complexity is indicated by partial analysis of a relatively simple configuration. We consider a polygon with six edges and symmetry in $x$ and $y$ (Fig. 5). The vertices are:

- $1 = (-1, -1)$
- $2 = (0, -1/2)$
- $3 = (1, -1)$
- $4 = (1, 1)$
- $5 = (0, 1/2)$
- $6 = (-1, 1)$. 
Vertices 2 and 5 are concave. The 3D element is of order \( m = 8 \) with four planar and one quartic surface. The quartic surface is

\[
z^2 - [(1/2 + y)^2 + x^2][(1/2 - y)^2 + x^2]. \tag{22}
\]

The common value for \( z \) on \( G_+ \) for the four convex vertices is \( \sqrt{65}/4 \). The four planar faces are: (1; 6; \( A \)) = 1 + \( x \), (3; 4; \( B \)) = 1 - \( x \), (1; 2; 3) = 1 + 2\( y \) + 4\( z \)/\( \sqrt{65} \), (4; 5; 6) = 1 - 2\( y \) + 4\( z \)/\( \sqrt{65} \). Here \( A \), \( B \) and \( C \) are arbitrary points chosen so that the planes are parallel to the \( z \) axis.

The adjacent \( Q \) is of order \( m - 4 = 4 \). The numerators of the basis functions are of order five. The opposite factor for vertex 1 is \((1 + \( x \))(1 - 2\( y \) + 4\( z \)/\( \sqrt{65} \))\), of order two. The adjacent factor is of order three. A surface of order three has 19 degrees of freedom. The 19 points which determine this surface include five points on each of the three edges meeting at vertex 1 plus four non coplanar points on the quartic surface. The location of these points is discussed in Chapter 7 of reference [1]. The actual computation of this adjacent factor is best done with a computer. This discussion was introduced only to indicate the increase in complexity resulting from introduction of more than one concave vertex.

6. Isoparametric concave quadrilaterals

The construction of isoparametric basis functions for concave quadrilaterals is not a simple task. In addition to the four vertex nodes, there must be a node on each side to yield degree two interpolation along the side for the isoparametric mapping of the physical side. It will be shown that the eight boundary nodes do not suffice for degree two (or even degree one) approximations within the concave quadrilateral. A ninth node not on the boundary must be introduced.

We first construct degree two basis functions for the 3D element. There are twelve nodes on the surface \( G_+ \) into which the quadrilateral is mapped. Three of these are the 3D vertices, and there are another three on each side. One of these is vertex 2 of the 2D element. In addition, nodes 13–15 are introduced on sides (1, \( B \)), (3, \( C \)) and (4, \( A \)) to enable degree two approximation on these edges. EIPs \( A \), \( B \) and \( C \) are not nodes but are denoted by indices 16–18. Each vertex adjacent factor vanishes at one of these. The opposite factors of the basis functions are easily found. The adjacent factors require more subtle analysis. Each side node has a linear adjacent factor passing through the other two side nodes on that side. Since three nodes determine a unique linear factor in 3D, it is apparent that another node must be located. Each of the three vertex nodes on \( G_+ \) has a quadratic adjacent factor passing through the six side nodes on the two adjacent sides, the EIP on the edge of the planar sides adjacent to the vertex, and the added node along that edge. This adds up to eight nodes for each vertex. Nine nodes are required to define a unique quadratic factor.

The error in the approximation to polynomial \( P(x, y, z) \) is

\[
\sum_{k=1}^{K} \left[ P(x_k, y_k, z_k) \frac{N_k(x, y, z)}{Q(x, y, z)} \right] - P(x, y, z). \tag{23}
\]

over the \( K \) nodes on the mapped quadrilateral. Degree two approximation is attained when for any polynomial of maximal degree two

\[
\sum_{k=1}^{K} \left[ P_2(x_k, y_k, z_k)N_k(x, y, z) \right] - P_2(x, y, z)Q(x, y, x) = 0. \tag{24}
\]

This sum is of maximal degree three. The twelve 3D nodes on the mapped quadrilateral boundary all lie on the product of the three planar factors of the 3D boundary surfaces. Thus Eq. (24) may be satisfied for all points on the quadrilateral boundary and not be true within the element. We must ensure satisfaction at an interior node on the mapped quadrilateral. This added node, which we label 19, also provides the additional point needed to define unique adjacent factors. We choose this node as the midpoint of the line connecting vertices 2 and 4. The numerator of this added basis function is just the product of the three planar sides. Its adjacent factor is unity. The twelve 3D boundary nodes must be reduced to eight 2D nodes by eliminating two of the side nodes on each side. Appropriate linear combinations of these added to the remaining nodes yields the required 2D degree two basis along each 2D linear side. This analysis has been implemented with the MATLAB program CON2. The program CONTEX has been used to verify degree two approximation within rounding limits.
7. Integration

The derivatives of basis functions along sides (1, 2) and (2, 3) do not exist at vertex 2. Numerical quadrature of functions involving derivatives of basis functions must not use vertex 2 as a node. Analytic integration based on the divergence theorem seems preferable. The program DIVINT first computes the indefinite integral \( g \) with respect to \( y \) of the integrand \( f \) and then computes the contour integral of \( g \) with respect to \( x \) around the element boundary. For the concave quadrilateral, each element side is linear. Special treatment must be given for the integrals along sides (1, 2) and (2, 3). Although the derivative is not defined at vertex 2, the directional derivatives are defined. If either side is vertical, it does not contribute to the integral. If either side is horizontal, we replace \( g(x, y, z) \) by \( g(x, 0, |x|) \) for evaluation of the integral. (Here \( z \) is any occurrence of \( (x^2 + y^2)^{1/2} \). The value of \( g \) at vertex 2 is taken as the limit of this function as \( x \) approaches zero. Otherwise, \( y = mx \) along the side and \( g(x, y, z) \) is replaced by \( g(x, mx, \sqrt{1 + m^2}|x|) \). Again, any ambiguity at vertex 2 is resolved by use of the limit as \( x \) approaches zero. In general the limits at vertex 2 have one value for side (1, 2) and another value for side (2, 3).

References