Resilient Design of Discrete-Time Observers with General Criteria Using LMIs

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Abstract—Much of the recent work on robust control or observer design has focused on preservation of stability of the controlled system or the convergence of the observer in the presence of parameter perturbations in the plant equations. The present work addresses the important problem of resilience or nonfragility which is the maintenance of convergence or performance when the observer is erroneously implemented due possibly to computational errors, e.g., round off errors in digital implementation or sensor errors. A linear matrix inequality approach is presented that maximizes performance in the implementation based on the knowledge of an upper bound on the error in the observer gain. Some design examples are provided for illustration purposes. © 2005 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

A controller for which the closed-loop system is destabilized by a small perturbation in the control gains is referred to as a "fragile" or "nonresilient" controller. In fact, the fragility problem is not new. After the publication of [1], the subject of fragility has gained more attention [2-5]. In

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practice, more and more controllers and observers are implemented digitally. Thus, implementation is subject to finite word length round off errors in numerical computations. Moreover, in some implementations, it is necessary to make manual tuning to obtain the desired performance of a closed-loop system. Therefore, the overall system must be able to tolerate some perturbations in the controller and observer coefficients. This means that any useful design procedure should generate a controller or observer, which also has sufficient room for readjustment of its coefficients.

In this paper, we introduce a novel design of resilient observers in discrete time for various performance criteria. Linear matrix inequalities (LMIs) [6] are used as the main mathematical tool. In the next section, system and signal models and performance criteria are introduced. Then, an LMI formulation of various observer design problems is presented for these criteria. Some illustrative examples conclude the paper.

The following notation is utilized in this work: \( x \in \mathbb{R}^n \) denotes an \( n \)-dimensional vector with real elements and with the associated norm \( |x| = (x^T x)^{1/2} \) where \((\cdot)^T\) represents the transpose. \( A \in \mathbb{R}^{m \times n} \) denotes an \( m \times n \) matrix with real elements. \( A^{-1} \) is the inverse of matrix \( A \). \( A > 0(A < 0) \) means \( A \) is a positive (negative) definite matrix, and \( I_m \) is an identity matrix of dimension \( m \). \( \lambda_{\min}(A)(\lambda_{\max}(A)) \) denotes the minimum (maximum) eigenvalue of the symmetric matrix \( A \). \( \ell_2 \) is the space of all infinite sequences of vectors with finite energy.

2. MODELS AND PERFORMANCE CRITERIA

Consider the following discrete time system and measurement equation

\[
\begin{align*}
x_{k+1} &= Ax_k + Bu_k + Fw_k, \\
y_k &= Cx_k + Du_k + Gw_k,
\end{align*}
\]

(1)

where \( x_k \in \mathbb{R}^n \) is the state to be estimated from the knowledge of the applied control \( u_k \in \mathbb{R}^m \) and the measurements \( y_k \in \mathbb{R}^p \). \( w_k \) is an \( \ell_2 \) disturbance input.

Consider the following equation in the Luenberger observer form,

\[
\hat{x}_{k+1} = A\hat{x}_k + Bu_k + (K + A)(y_k - C\hat{x}_k - Du_k),
\]

(2)

where \( A \) represents the error made in computing the observer gain \( K \), with the bound

\[
\Delta^T \Delta \leq \gamma I, \text{ for } \gamma > 0.
\]

(3)

Let \( e_k = x_k - \hat{x}_k \) denote the estimation error.

Substituting equation (1),(2), we find the error dynamics obey

\[
e_{k+1} = (A - (K + \Delta)C)e_k + (F - (K + \Delta)G)w_k.
\]

(4)

Let \( Z_k \) denote the performance output where

\[
Z_k = C_ze_k + D_zw_k.
\]

(5)

Consider the general performance objective

\[
V_{k+1} - V_k + \delta \|Z_k\|^2 + \epsilon \|w_k\|^2 - \beta Z_k^TW_kw_k \leq 0,
\]

(6)

for a \( V_k = e_k^TPe_k \), real numbers and \( \delta, \epsilon, \) and \( \beta \), where \( P > 0 \). Notice that upon summation, inequality (6) yields

\[
e_N^TPe_N \leq e_0^TPe_0 - \sum_{k=0}^{N-1} \left( \delta \|Z_k\|^2 + \epsilon \|w_k\|^2 - \beta Z_k^TW_kw_k \right),
\]

(7)
or by using Rayleigh’s inequalities,

\[ \lambda_{\min} (P) \| e_N \|^2 \leq \lambda_{\max} (P) \| e_0 \|^2 - \sum_{k=0}^{N} \left( \delta \| Z_k \|^2 + \epsilon \| w_k \|^2 - \beta Z_k^T w_k \right), \tag{8} \]

that allows several optimization choices in a unified eigenvalue problem [6] framework. We can design different observers for a variety of performance criteria for this system.

First of all, we let \( F = 0 \), \( G = 0 \), and \( D_z = 0 \) to eliminate the additive noise dependence. In this case, if we take \( \delta = 0 \), \( \beta = 0 \), and \( \epsilon = 0 \), (8) yields

\[ \| e_N \|^2 \leq \frac{\lambda_{\max} (P)}{\lambda_{\min} (P)} \| e_0 \|^2 \]

This means that by minimizing \( \lambda_{\max} (P) \) and maximizing \( \lambda_{\min} (P) \), we can lower the bound on the norm of the estimation error, which will guarantee a faster response for the observer. Note that this implies boundedness of the estimation error (stability).

By taking \( \delta > 0 \), \( \beta = 0 \), and \( \epsilon = 0 \), (8) will yield a bound on the energy of the performance output in terms of the initial estimation error \( e_0 \)

\[ \sum_{k=0}^{N} \| Z_k \|^2 \leq \frac{1}{\delta} \lambda_{\max} (P) \| e_0 \|^2. \]

Minimizing \( \lambda_{\max} (P) \) and maximizing \( \delta \) will give us a smaller bound on the energy of the performance output. This is a suboptimal \( H_2 \) observer.

In the case of additive noise \( w_k \), and for general choices of \( F \), \( G \), and \( D_z \), by setting \( \delta = 1 \), \( \beta = 0 \), and \( \epsilon < 0 \), for \( e_0 = 0 \), gives the result

\[ \sum_{k=0}^{N} \| Z_k \|^2 \leq -\epsilon \sum_{k=0}^{N} \| w_k \|^2, \]

which means a bound on the \( \ell_2 \) to \( \ell_2 \) gain of the estimator (suboptimal \( H_\infty \) design). By maximizing \( \epsilon \), the \( H_\infty \) gain of the estimator can be minimized.

When \( e_0 = 0 \), if we use this formulation, we can design several dissipative controllers by using different values of \( \delta \), \( \beta \), and \( \epsilon \).

If we take \( \delta = 0 \), \( \beta = 1 \), and \( \epsilon = 0 \), it yields the input strict passivity result,

\[ \sum_{k=0}^{N} Z_k^T w_k \geq \epsilon \sum_{k=0}^{N} \| w_k \|^2. \]

By maximizing \( \epsilon \), the dissipation rate of the estimator can be maximized.

Similarly, other dissipativity results can be obtained by changing \( \delta \), \( \beta \), and \( \epsilon \) values. For example, taking \( e_0 = 0 \), \( \delta = 0 \), \( \beta = 1 \), and \( \epsilon = 0 \) gives passivity

\[ \sum_{k=0}^{N} Z_k^T w_k \geq 0. \]

If we set \( \delta > 0 \), \( \beta = 1 \), and \( \epsilon = 0 \), we get output strict passivity,

\[ \sum_{k=0}^{N} Z_k^T w_k \geq \delta \sum_{k=0}^{N} \| Z_k \|^2. \]

By maximizing \( \delta \), the dissipation rate can be maximized.
Table 1. Different performance criteria in a common framework.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\beta$</th>
<th>$\epsilon$</th>
<th>Inequality</th>
<th>Performance Criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$|e_N|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}|e_0|^2$</td>
<td>boundedness of the estimation error</td>
</tr>
<tr>
<td>&gt;0</td>
<td>0</td>
<td>0</td>
<td>$\sum_{k=0}^{N} |Z_k|^2 \leq \frac{1}{\beta} \lambda_{\max}(P)|e_0|^2$</td>
<td>suboptimal $H_2$ observer</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>&lt;0</td>
<td>$\sum_{k=0}^{N} |Z_k|^2 \leq -\epsilon \sum_{k=0}^{N} |w_k|^2$</td>
<td>bound on the $L_2$ to $L_2$ gain of the estimator</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>&gt;0</td>
<td>$\sum_{k=0}^{N} Z_k^T w_k \geq \epsilon \sum_{k=0}^{N} |w_k|^2$</td>
<td>input strict passivity</td>
</tr>
<tr>
<td>&gt;0</td>
<td>1</td>
<td>0</td>
<td>$\sum_{k=0}^{N} Z_k^T w_k \geq 0$</td>
<td>passivity</td>
</tr>
<tr>
<td>&gt;0</td>
<td>1</td>
<td>&gt;0</td>
<td>$\sum_{k=0}^{N} Z_k^T w_k \geq \epsilon \sum_{k=0}^{N} |w_k|^2 + \delta \sum_{k=0}^{N} |Z_k|^2$</td>
<td>output strict passivity</td>
</tr>
<tr>
<td>&gt;0</td>
<td>1</td>
<td>&gt;0</td>
<td>$\sum_{k=0}^{N} Z_k^T w_k \geq \epsilon \sum_{k=0}^{N} |w_k|^2 + \delta \sum_{k=0}^{N} |Z_k|^2$</td>
<td>very strict passivity</td>
</tr>
</tbody>
</table>

Very strict passivity, which is the strict passivity both in the terms of the input and the output, can be obtained if we set $\delta > 0$, $\beta = 1$, and $\epsilon > 0$,

$$\sum_{k=0}^{N} Z_k^T w_k \geq \epsilon \sum_{k=0}^{N} \|w_k\|^2 + \delta \sum_{k=0}^{N} \|Z_k\|^2.$$

By maximizing both $\delta$ and $\epsilon$, the dissipation rate can be maximized.

As described above, and summarized in Table 1, this LMI formulation enables us to design various observers according to different performance criteria in a common framework.

### 3. LMI FORMULATION

The nonnoisy and noisy cases will be treated separately in the following development. First, consider inequality (6) with $\epsilon = \beta = 0$. Substituting for $V_k$ and from (4),(5), we obtain

$$e_k^T \left[ - (A - (K + \delta)C)^T P (A - (K + \delta)C) + P - \delta D_z^T D_z \right] e_k \geq 0. \quad (9)$$

Using Schur’s complement result [6] on the inequality (9) and the following fact,

$$\begin{bmatrix} 0 & C^T \delta^T P \\ P \delta C & 0 \end{bmatrix} \leq \begin{bmatrix} \alpha \gamma C^T C & 0 \\ 0 & \alpha^{-1} P^2 \end{bmatrix} \quad (10)$$

for any $\alpha > 0$, a sufficient condition for (9) is

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ * & q_{22} & q_{23} \\ * & * & q_{33} \end{bmatrix} \succeq 0 \quad (11)$$
for $Q = Q^\top$ and

\[
\begin{align*}
q_{11} &= P - \delta C_z^T C_z - C^T C, \\
q_{12} &= A^T P - C^T Y^T, \\
q_{13} &= 0, \\
q_{22} &= P, \\
q_{23} &= P, \\
q_{33} &= \alpha I,
\end{align*}
\]

where $Y = PK$ and $\alpha$ is picked as $\gamma^{-1}$.

In the noisy case, similar substitutions will lead to

\[
\begin{bmatrix}
[\eta_k^T w_k^T] \\
\begin{bmatrix}
P - \delta C_z^T C_z \\
* \begin{bmatrix}
-\delta C_z^T D_z + 0.5\beta C_z^T \\
-\delta D_z^T D_z - \epsilon I + 0.5\beta (D_z + D_z^T)
\end{bmatrix}
\end{bmatrix}
\end{bmatrix} \geq \begin{bmatrix}
(A - (K + \Delta) C)^T \\
*(A - (K + \Delta) C) \left( F - (K + \Delta) G \right)^T
\end{bmatrix}
\]

(12)

Using the Schur's complement result and the inequality

\[
\begin{bmatrix}
0 & BC \\
C^TB^T & 0
\end{bmatrix} \leq \begin{bmatrix}
\alpha BB^T & 0 \\
0 & \alpha^{-1} C^T C
\end{bmatrix},
\]

(13)

for any $\alpha = \gamma^{-1} > 0$, a sufficient condition for (12) is obtained as

\[
Q = \begin{bmatrix}
q_{11} & q_{12} & q_{13} & q_{14} \\
* & q_{22} & q_{23} & q_{24} \\
* & * & q_{33} & q_{34} \\
* & * & * & q_{44}
\end{bmatrix} \succeq 0,
\]

(14)

for $Q = Q^\top$ and

\[
\begin{align*}
q_{11} &= P - \delta C_z^T C_z - C^T C, \\
q_{12} &= A^T P - C^T Y^T, \\
q_{13} &= 0, \\
q_{22} &= P, \\
q_{23} &= P, \\
q_{33} &= \alpha I,
\end{align*}
\]

where $Y = PK$ and $\alpha$ is picked as $\gamma^{-1}$. So, to sum up the resilient observer design procedure, the LMI (7) needs to be solved for $P > 0$, $\gamma$, and the minimum positive $\alpha$ value (that will maximize $\gamma$ or the resilience). The necessary $K$ is found from $K = P^{-1}Y$. 


Table 2. Design parameter values.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Cₐ</th>
<th>D</th>
<th>Dₐ</th>
<th>F</th>
<th>G</th>
<th>α</th>
<th>δ</th>
<th>β</th>
<th>ε</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bounded Estimation Error</td>
<td>0.5</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>47.912</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>H₂-Observer</td>
<td>0.5</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>41.837</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Input Strict Passivity</td>
<td>0.1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0.1</td>
<td>1</td>
<td>0.1</td>
<td>0</td>
<td>4.5866</td>
<td>0</td>
<td>1</td>
<td>1.01,0.1,0.2</td>
</tr>
</tbody>
</table>

4. EXAMPLES

This section contains an investigation into the regions in the \( P \) and \( Y \) coordinates in which those LMIs (7),(10) have solutions for a one-dimensional system and various design parameters. These parameters are given in Table 2 for three different cases of performance criteria involving bounded estimation error, \( H₂ \) suboptimal, and m.s. input strict passivity results.

![Figure 1. Feasibility region for bounded estimation error.](image1)

![Figure 2. Feasibility region for \( H₂ \) suboptimal design.](image2)
Figure 1 shows the feasibility region of the LMI (11) when $\alpha$ parameter is a solution of the LMI. In simulations, it was found that the minimum value of $\alpha = 1.0001$ which gives a maximum value slightly less than one for the upper bound on the gain perturbations.

The feasibility regions corresponding to various parameter values and performance criteria are shaded differently in Figures 2 and 3 to indicate how the shape of the regions change as the design parameters $\delta$ and $\epsilon$, respectively change. Large areas should be interpreted as enclosing the smaller areas. The range of $\delta$ is found to be $\delta \in (0, 40.8369)$ in Figure 2 that will result in a feasible LMI. Higher values in this range will result in lower bounds on the energy of the performance output. In Figure 3, LMI (14) is initially solved for $\alpha$ as well as $P$ and $Y$ and then this $\alpha$ value is kept fixed, as $\epsilon$ is changed.

5. CONCLUSIONS

The purpose of this paper was to present a simple solution to the problem of nonfragile observer design for discrete time systems. We have presented an LMI based technique to design observers with guaranteed performance and/or stability. Examples have been provided to show the feasibility of this approach. Current results do not include any directional information regarding the computational error $\Delta$ in (3). Therefore, any discussion of the conservativeness of these results would be analogous to the same discussion for the unstructured parameter perturbations for linear time invariant systems [7]. Therefore, future work will involve resilient design with structured perturbations. Other future work will involve extension of current results using the game theoretic approach [8].

REFERENCES