A Riemannian Weighted Filter for Edge-sensitive Image Smoothing

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Abstract

This paper describes a new method for image smoothing. We view the image features as residing on a differential manifold, and we work with a representation based on the exponential map for this manifold (i.e., the map from the manifold to a plane that preserves geodesic distances). On the exponential map we characterise the features using a Riemannian weighted mean. We show how both gradient descent and Newton's method can be used to find the mean. Based on this weighted mean, we develop an edge-preserving filter that combines Gaussian and median filters of gray-scale images. We demonstrate our algorithm both on direction fields from shape-from-shading and tensor-valued images.

1. Introduction

Recently there has been increasing interest in processing image features which reside on non-flat manifolds. Examples include orientations data and diffusion tensor magnetic resonance images (DT-MRI) [8, 12, 10, 1, 11]. Tang, et al. [1] and Chan, et al. [10] model the distribution of orientations using a unit sphere and use methods based on both variational analysis and the harmonic map to restore directional features. Fletcher, et al. [7] consider diffusion tensors as a manifold and develop techniques to analyse the principle geodesic modes of the feature set. Lenglet et al [2] use a similar method to segment DT-MRI data. Pennec et al. [11] has developed a framework for the analysis of statistical data residing on manifolds, and has generalised the operations of interpolation, Gaussian and anisotropic regularisation for DT-MRI.

The aim in this paper is to extend these ideas by introducing the concept of the Riemannian weighted mean. We work with the exponential map, i.e. the map from the manifold to a plane that preserves geodesic distance. Using the exponential map, we show how the weighted mean can be computed efficiently using Newton’s method. This numerical method is faster than gradient descent [7, 11] and converges quadratically. We use this method to develop an edge-preserving filter that inherits the advantages of both Gaussian and median filters of gray-scale images. As an application, we demonstrate the effectiveness of our algorithm for filtering orientation fields from shape-from-shading and DT-MRI data. The resulting smoothing method offers some advantages over alternatives described elsewhere in the literature. For instance, Trahanias, et al. [8] have generalised median filters [3] to orientation fields by minimizing the sum of angles. Their algorithm is only for orientation features, and does not use the differential geometry of the orientation field. Welk, et al. [5] have applied the median filter to DT-MRI images. However, they use the Frobenius inner product as a distance measure which is equivalent to the Euclidean inner product in the linear matrix space. Moreover, their gradient descent scheme is not positive-definite preserving, and has required subsequent improvement [6].

2 Preliminaries

Let the image feature space be a $n$-dimensional manifold $M$ with a local co-ordinate system $x_1, ..., x_n$. At each point $x \in M$ the tangent space $T_xM$ is spanned by the basis of directional derivatives $\partial_1, ..., \partial_n$ where $\partial_i := \frac{\partial}{\partial x_i}$. An inner product in any tangent-space $T_xM$ between two tangent vectors $\xi, \eta \in T_xM$ defines the Riemannian metric $g$, i.e., $g(\xi, \eta) = \sum_{ij} g_{ij}\xi_i\eta_j$, where $(g_{ij})_{i,j=1}^{n}$ is a symmetric positive-definite $n \times n$ matrix with element $g_{ij} = \langle \partial_i, \partial_j \rangle$.

Using the metric $g$, we measure the length of a smooth curve $c : [a, b] \rightarrow M$ on $M$ as $l^g(a, b) = \int_a^b \sqrt{c'(t)g(c'(t))}dt$. The distance between two points $x, y \in S^{n-1}$ is the infimum of lengths of curves connecting them, i.e. the distance is given by $d(x, y) := \inf \{ l(c) \mid c(a) = x, c(b) = y \}$. The curve satisfying this infimum condition is a geodesic.

Given any point $x \in M$ and a tangent vector $\xi \in T_xM$ with $|\xi| < \epsilon$, there exists a unique geodesic $\gamma(\xi, t) : t \in [-\delta, \delta] \rightarrow M$ satisfying the condition $\gamma(0) = p$, $\gamma'(0) = \xi$. Thus, we can relate an open subset of the tangent space $T_xM$ with a local neighbourhood of $x$ in $M$ using the radial geodesics originating from $x$. The exponential map $\exp : \Omega \subset T_xM \rightarrow M$ is defined as

$$\exp_x(\xi) = \gamma(\xi, 1).$$

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Geometrically, \( \exp_x(\xi) \) is a point of \( M \) obtained by marking out a length equal to \( |\xi| \) commencing from \( x \), along a geodesic which passes through \( x \) with velocity equal to \( \frac{\xi}{|\xi|} \). Since \( \exp_x \) is a local diffeomorphism, it has an inverse map, the so-called logarithmic map \( \log_x : M \to T_xM \) where \( \log_x(\gamma(x,\xi)(t)) = tx \). So the distance between the points \( x \) and \( y \) is equal to the length of the tangent vector \( \log_x(y) \in T_xM \), i.e., \( d(x,y) = \sqrt{\log_x(y), \log_x(y)} \). We assume that \( \exp_x \) and \( \log_x \) are well defined in order to avoid difficulties associated with the cut locus.

### 3 Riemannian Weighted Mean

Consider a set of \( m \) points \( p_1, \ldots, p_m \in M \) residing on the manifold, with corresponding weights \( w_1, \ldots, w_m \in \mathbb{R} \) such that \( \sum_{k=1}^m w_k = 1 \). An efficient way of characterising the points is to compute their weighted mean, or center of mass. If the points are in \( \mathbb{R}^n \), recall that the arithmetic Euclidean weighted mean \( q \) can be defined as the linear combination \( q = \sum_{k=1}^m w_k p_k \). From a variational standpoint the Euclidean weighted mean minimizes the sum of the weighted squared distances \( f(q) \) to the given points \( p_k \). In other words,

\[
    f(q) = \frac{1}{2} \sum_{k=1}^m w_k d_e(q, p_k)^2
\]

where \( d_e(q, p_k) = \|q - p_k\| \) is the Euclidean distance between the weighted mean \( q \) and the point \( p_k \).

To generalise the least squares minimization to the manifold \( M \), one intuitive way is to replace \( d_e(q, p_k) \) by the shortest geodesic distance \( d_g(q, p_k) \) on \( M \) between the mean \( q \) and the point \( p_k \). Thus, we can define the Riemannian weighted mean as the point \( q \) that minimizes the sum of geodesic distances

\[
    f(q) = \frac{1}{2} \sum_{k=1}^m w_k d_g(q, p_k)^2.
\]

The mean \( q \) is a critical point of \( f \), i.e. the location where the directional first derivatives of \( f \) are all zero. In the following, we assume that \( f \) always attains a unique minimum and that it is twice differentiable, i.e., \( f \in C^2(M) \).

#### 3.1 Numerical methods

To locate the minimum of \( f \), one widely used and easy to implement numerical method is to use steepest gradient descent [7, 11]. This method offers the advantage that it converges linearly. At a point \( q \in M \), the gradient of \( f \) with respect to \( q \) is

\[
    \nabla f(q) = -\sum_{k=1}^m w_k \log_q(p_k).
\]

The minimum of \( f \) is located at the critical point \( q \), \( \nabla f(q) = 0 \), where the Euclidean weighted mean of tangent vectors \( \log_q(p_k) \in T_qM \) is zero. The iteration of the classical gradient descent method can be stated as \( q^{t+1} = q^t - \nabla f(q^t) \). This iteration scheme means that the estimated mean \( q^{t+1} \) moves towards mean \( q \) in the direction of steepest descent. Thus, the intrinsic gradient descent algorithm on \( M \) gives the following iteration scheme for estimating the Riemannian weighted mean

\[
    q^{t+1} = \exp_q(\sum_{k=1}^m w_k \log_q(p_k)). \tag{5}
\]

To use this mean for image filtering, we need to evaluate it at each image location, and this is computationally burdensome. Hence, the rate of convergence of the iteration scheme is of critical importance, and the scheme given above has only linear convergence. To overcome this problem, we draw on ideas from [9, 13] and propose a Newton method which has quadratic convergence. According to Newton’s method the iteration scheme for finding a minimum of a one variable function is \( q^{t+1} = q^t - (H(q^t))^{-1} \nabla f(q^t) \), where \( (H(q^t))^{-1} \) is the inverse of Hessin matrix \( H := (h_{ij}) \) of the function \( f \) at the point \( q^t \). On a manifold \( M \) the Newton method gives the following iteration scheme for finding the Riemannian weighted mean

\[
    q^{t+1} = \exp_q \left( (H(q^t))^{-1} \sum_{k=1}^m w_k \log_q(p_k) \right). \tag{6}
\]

Here the Hessian is the second covariant derivative of \( f \) on \( M \). If at a point \( q \) on \( M \) we set up a co-ordinate system \( x_1, \ldots, x_n \) with origin at \( q \) for the tangent space \( T_qM \), we can write \( F(x) = f(\exp_q(x)), x \in T_qM \). In this co-ordinate system the elements of the Hessian matrix at \( q \) are \( h_{ij} = (\frac{\partial^2 f}{\partial x_i \partial x_j})(q) \).

To initialise the iteration scheme, we can either set \( q^0 \) to be the location of one of the sample points that has minimal value \( f \), or to be the projection point of the Euclidean weighted mean to \( M \).

### 4 Edge-Preserving Filter

In this paper our aim is to use the Riemannian weighted mean and geodesic distance to generalise the classical Gaussian and median filters to features residing on curved manifolds. The Gaussian filter convolves an image \( I_0 : \mathbb{R}^2 \) or \( \mathbb{R}^3 \rightarrow M \) with a Gaussian kernel \( G_\sigma \) as \( I(x) = \int I_0(x-\tau)G_\sigma(\tau)d\tau \) where \( x, \tau \in \mathbb{R}^2 \) or \( \mathbb{R}^3 \). To obtain a discrete approximation we can truncate the kernel to a window of width \( 3\sigma \). Suppose that this window contains the feature set \( W = \{p_1(x_1), \ldots, p_m(x_m)\} \). That is \( I(x) = \sum_{k=1}^m G_\sigma(x_k)p_k \) where the sum \( \sum_{k=1}^m G_\sigma(x_k) \) is
normalised to one. Thus, at \( x \) the Riemannian Gaussian filter (RGF) outputs the Riemannian weighted mean for the feature set

\[
I_{\text{GAUSS}} := \arg \min_{q \in M} \frac{1}{2} \sum_{k=1}^{m} G_\sigma(x_k) d_g(q, p_k)^2. \tag{7}
\]

It is important to note that the Gaussian filter defined in this way is no longer linear. The degree of smoothing can be controlled by the filter width \( \sigma \) of the Gaussian kernel. One disadvantage of the the Gaussian filter is that although it is effective for smoothing images corrupted by Gaussian and uniformly distributed noise, it has the detrimental effect of blurring edge structure.

One widely used structure-preserving filter that can potentially overcome this problem is the median filter [3]. The median filter can be generalised to our feature space using the geodesic distance on \( M \)

\[
I_{\text{MEDIAN}} := \arg \min_{p_i \in W} \sum_{k=1}^{m} d_g(p_i, p_k), \quad i = 1, ..., m, \tag{8}
\]

which we refer to as the Riemannian median filter (RMF). The median filter outputs one of the sample features in the set \( W \). It is edge preserving and can be used to remove impulse noise, such as salt-and-pepper noise. However, it is not effective at suppressing uniformly distributed noise.

To compensate the shortcomings of these two filters, we propose the following edge-preserving filter that combines the Riemannian Gaussian and Riemannian median filters

\[
I_x := \begin{cases} 
I_{\text{GAUSS}} & \text{if } \sum_{k=1}^{m} d_g(I_{\text{GAUSS}}, p_k) < \sum_{k=1}^{m} d_g(I_{\text{MEDIAN}}, p_k) \\
I_{\text{MEDIAN}} & \text{otherwise.}
\end{cases} \tag{9}
\]

We refer to it as the Riemannian Gaussian-median filter (RGMF). Like the median filter, this filter may be applied iteratively. Although it does not always output one of the input features, it performs like a median filter when \( \sigma \) is not very large. For edges, it is more likely to output the result of the median filter, and so it does not blur edges. Inside image regions, it has the effect of noise elimination as Gaussian filter.

\section{Application}

\subsection{Normals from Shape-From-Shading}

For \( n \)-dimensional directional fields, the feature space \( M \) is the unit sphere \( S^{n-1} \) embedded in \( \mathbb{R}^n \). For two feature points \( p_i \) and \( p_j \), the connecting geodesic is a great circle of \( S^{n-1} \). Thus the geodesic distance is given by \( d_g(p_i, p_j) = \arccos(p_i \cdot p_j) \). To find the mean of the feature set \( p_1, ..., p_m \) using Newton’s method, it is necessary to calculate the gradient and Hessian of the function \( f \) on the unit sphere \( S^{n-1} \). That is, we require the first- and second-order derivatives of \( f \).

At the point \( q \), we set up a local co-ordinate system \( x_1, ..., x_{n-1} \) with origin at \( q \) for the tangent-space \( T_qM \) and co-ordinate system \( x_1', ..., x_n' \) for \( \mathbb{R}^n \). The two co-ordinate systems are such that \( q \) is at point \( (0, ..., 0, 1) \) in \( \mathbb{R}^n \) and the axes \( x_i \) and \( x_i' \) are parallel for \( i = 1, ..., n-1 \). The exponential map \( \exp_q \) maps a point \( x = (x_1, ..., x_{n-1}) \in T_qM \) to \( x' = (x_1', ..., x_n') \) on \( S^n \) with \( x_1' = \frac{x_1}{\|x\|} \sin \|x\| \) when \( i = 1, ..., n-1 \), and \( x_n' = \cos \|x\| \). In the same way, the logarithmic map \( \log_q \) maps a point \( x' \in S^n \) to a point \( x \in T_qM \) as \( x = x' \theta / \sin \theta \), \( \theta = \arccos x_n' \). To find the derivatives of \( f \) appearing in equation (3), we make use of the exponential map and locate instead the derivative of \( f_i(q) = \frac{1}{2} d_g(q, p_i)^2 \) which can be described as a function in the tangent space \( T_qM \). As a result \( f_i(x) := f_i(\exp_q(x)) = \frac{1}{2} d_g(q, \exp_p(x))^2 \). To simplify the calculation of the derivative of \( f_i \), we set up another co-ordinate system \( x_1', ..., x_{n-1}' \) with origin at \( q \) and axis \( x_i \) parallel with the tangent vector \( \log_q(p_i) \). Because the exponential map \( \exp_q \) is distance preserving and the values of \( f(x_i') \), \( j \neq 1 \) are symmetric in \( p \), the first derivative of \( f_i \) is

\[
\frac{\partial f_i}{\partial x_1^i} = d_i, \quad \text{and} \quad \frac{\partial f_i}{\partial x_j^i} = 0 \text{ for } j \neq 1, \tag{10}
\]

where \( d_i = d_g(q, p_i) \). As demonstrated in [9], the second derivative is

\[
\frac{\partial^2 f_i}{\partial (x_1^i)^2} = 1, \quad \frac{\partial^2 f_i}{\partial (x_j^i)^2} = d_i \cos \theta \text{ for } j \neq 1, \quad \frac{\partial^2 f_i}{\partial x_j^i \partial x_k^i} = 0 \text{ for } j \neq k. \tag{11}
\]

Using the derivatives of \( f_i \), we can calculate the gradient \( \nabla f \) and Hessian \( H_f \) of \( f_i \) at \( q \). To obtain the gradient and Hessian of \( f \), we need to transform \( \nabla f \) and \( H_f \) to the co-ordinate system \( x_1, ..., x_n \). Once this is done the mean of \( p_1, ..., p_m \) can be found using Newton’s method as outlined in equation (6).

\subsection{DT-MRI}

DT-MRI associates with each voxel of an imaging volume a \( 3 \times 3 \) symmetric positive-definite matrix. Let \( S(n) \) be the space of all \( n \times n \) symmetric matrices and \( S^+(n) \) be the space of all symmetric positive-definite matrices. Thus, the feature space \( M \) for DT-MRI is \( S^+(3) \). The space \( S^+(n) \) is a \( \frac{1}{2} n(n + 1) \) dimensional convex cone of \( \mathbb{R}^n \). Let \( GL(n) \) be the set of non-singular matrices. \( GL(n) \) has the properties that it is both a Lie group and a differential manifold for which the group multiplication and inverse operations are both smooth. Additionally, \( S^+(n) \)
is a homogeneous space under the Lie group GL(n), i.e.,
\( \phi(G, A) = GAG^T, \ G \in GL(n), \ A \in S^+(n) \).

Let \( A = UDU^T \) be the eigendecomposition of matrix \( A \), then the exponential of \( A \) is given by
\( \exp A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = U \exp(D)U^T, \) and the inverse logarithm of \( A \) is given by
\( \log A = - \sum_{k=1}^{\infty} \frac{(I-A)^k}{k} = U \log(D)U^T, \)
where \( I \) is the identity matrix.

Recall that in the space of all \( n \times n \) matrices the Euclidean inner products, which is known as the Frobenius inner product and used by the tensor median filter [5, 6], is defined as \( (A, B)_F = \text{tr}(A^T B) \), where \( \text{tr}(\cdot) \) denotes the trace. For each \( P \in S^+(n) \) the tangent space \( T_P S^+(n) \) is identified with \( S(n) \). Following [7, 4, 11], we define the metric at \( P \) as a positive-definite inner product
\[
(A, B)_P = \text{tr}(P^{-1}AP^{-1}B), \tag{12}
\]
which is invariant under \( GL(n) \). Thus, for a smooth curve \( C : [a, b] \rightarrow S^+(n) \) in \( S^+(n) \), the length of \( C(t) \) can be computed via the invariant metric
\[
\ell(C) = \int_a^b \left\| C'(t) \right\|_C(t) = \int_a^b \sqrt{\text{tr}(C'(t)^{-1}C'(t))}, \tag{13}
\]
which is also invariant under \( GL(n) \), i.e., \( C(t) \rightarrow GC(t)G^T, \ G \in GL(n) \). The geodesic with initial point at identity matrix \( I \) and tangent vector \( W \in T_I S^+(n) \) is given by \( \exp(tW) \). Using the group action, an arbitrary geodesic \( \Gamma(t) \) such that \( \Gamma(0) = P \) and \( \Gamma'(0) = W \) is given by
\[
\Gamma_{(P,W)}(t) = P^{\frac{1}{2}} \exp(itP^{-\frac{1}{2}}WP^{-\frac{1}{2}})P^{\frac{1}{2}}. \tag{14}
\]
Thus, the geodesic distance between two points \( A \) and \( B \) in \( S^+(n) \) is
\[
d(A, B) = \left\| \log(A^{-1}B) \right\|_F = \sqrt{\sum_{i=1}^{n} (\log \lambda_i)^2}, \tag{15}
\]
where \( \lambda_i \) are the eigenvalues of \( A^{-1}B \). It also follows that the exponential and logarithmic maps at \( P \) are
\[
\exp_P(W) = P^{\frac{1}{2}} \exp(P^{-\frac{1}{2}}WP^{-\frac{1}{2}})P^{\frac{1}{2}},
\log_P(A) = P^{\frac{1}{2}} \log(P^{-\frac{1}{2}}AP^{-\frac{1}{2}})P^{\frac{1}{2}}. \tag{16}
\]
Based on the geodesic distance together with the exponential and logarithmic maps, the mean and median of a sample set on \( S^+(n) \) can be found using equation (5) and (8) respectively.

5.3 Experiments

We have applied the Riemannian filter to synthetic and real-world data. Fig.4 shows the result of RGMF and the standard vector directional filter [8] on a synthetic direction field corrupted by both Gaussian and salt-and-pepper noise. The figure shows that our method offers the advantages of both the Gaussian and median filter. In Fig.1 we apply RGMF to improve the surface depth map of a cat by smoothing the noisy normal field from the needle map of shape-from-shading 1. For tensor-valued images, Fig.5 compares RGMF with the Frobenius norm based median filter on a noisy synthetic volume of tensor image. A real application of our method to a slice of DT-MRI is shown in Fig.6. We have also quantitatively compared the performance of the RGMF against the Euclidean median filter (EMF) and the RMF on synthetic tensor images. The synthetic images are corrupted by different amounts of both Gaussian and impulse noise. The result of the comparison is shown in Fig.2, which shows the root-mean-square error (RMSE) between the smoothed noisy \( I_0 \) and ground truth \( I_0 \) images, i.e.,
\[
\text{RMSE} = \sqrt{\frac{1}{N} \sum_{x \in I_0} (\mathcal{d}(I_0(x), I_0(x)))^2},
\]
where \( \mathcal{d}(\cdot, \cdot) \) is the distance between two points in the feature space and \( N \) is the number of pixels or voxels of \( I_0 \). The main feature to note from the plot is that the RMSE is lowest for the Riemannian filter. In Fig.3 we show the percentage of the image pixels for which the Gaussian filter is selected using the decision criterion in equation (9) as a function of the filter width \( \sigma \). Subfig.(a) shows the result for the directional image in Fig.4 and Subfig.(b) the result for the tensor image in Fig.5. In the case of the tensor data the dependence on \( \sigma \) is sharpest. In both cases the median filter dominates for small \( \sigma \) and the Gaussian filter dominates for large \( \sigma \).

6 Conclusions

We have proposed a new Riemannian weighted filter that can restore images whose features reside on a curved manifold. We work with a representation of the features based on the exponential map. The main idea is to use the Riemannian weighted mean and geodesic distance to generalise, and combine, classical Gaussian and median filters for nonflat features. Experiments show that the proposed method is edge-preserving and capable of eliminating both Gaussian and impulse noise.

References


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\footnote{The authors thank Gary Atkinson to provide data of the cat from SFS.}


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**Figure 1. Application for shape-from-shading**

![Cat from noisy normals](image1)

![Cat from filtered normals](image2)

**Figure 2. Tensor Filter comparison**

![Directional case](image3)

![Tensor case](image4)

**Figure 3. Percentage of Gaussian**

![a) Synthetic direction field](image5)

![b) With Gaussian & salt noise](image6)

![c) Vector directional filter (8)](image7)

![d) Our Riemannian filter (3 x 3, 10 iterations)](image8)

**Figure 4. Smoothing synthetic direction field**

![a) Synthetic tensor field](image9)

![b) With Gaussian & salt noise](image10)

![c) Frobenius norm window : 3 x 3 x 3, 3 iterations](image11)

![d) Our Riemannian filter (3 x 3, \( \sigma = 1 \), 10 iterations)](image12)

**Figure 5. Smoothing synthetic tensor field**

![a) Slice of DT-MRI](image13)

![b) Filtered result](image14)

**Figure 6. Application for DT-MRI**