Some new symmetric designs with $\lambda = 10$ having an automorphism of order 5

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Abstract

In this paper we determine all symmetric designs with parameters $(61,25,10)$ with an automorphism of order 5 fixing 11 points. Among them, there are exactly 24 non-isomorphic designs admitting the action of an elementary abelian group of order 25. The only previously known design with these parameters is one of the 24 designs constructed here. We have further proved that there are at least 588 symmetric $(66,26,10)$-designs. Of these, 558 admit a specific action of the dihedral group of order 10 and exactly 22 admit the only possible action of the elementary abelian group of order 25. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction and preliminaries

A symmetric design is a finite incidence structure consisting of a set of points $\mathcal{P}$, a set of blocks (lines) $\mathcal{B}$ and an incidence relation $I \subseteq \mathcal{P} \times \mathcal{B}$ with the following properties:

1. $|\mathcal{P}| = |\mathcal{B}| = v$.
2. Every block is incident with $k$ points.
3. Every pair of points is incident with $\lambda$ blocks.

We denote such an incidence structure by $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ and write the associated parameters as a triple $(v, k, \lambda)$. It is customary to denote $k - \lambda$ by $n$. By considering the complementary design, if necessary, we may assume additionally that $k > 2\lambda$.

We suppose that $\lambda = 10$, so that $k$ or $k + 1$ is divisible by 5. The smallest possible value of $k$ is $k = 21$, giving the parameter triple $(43,21,10)$ from the Hadamard series

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of designs. In this paper, however, we consider the cases $k = 25$ and 26; the next possible $k$ would be 35, but at the moment it is not known whether such a design exists.

Let $G \leq \text{Aut} \mathcal{D}$ be an automorphism group of a symmetric design $\mathcal{D}$ with parameters $(v, k, \lambda)$, so that $G$ has the same number of orbits, $t$ say, on $\mathcal{D}$ as on $\mathcal{D}$. Let us denote these point orbits by $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_t$ and the block orbits by $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. Clearly, $\sum_{r=1}^t \omega_r = \sum_{i=1}^t \Omega_i = v$. Let $\gamma_{ir}$ be the number of points of $\mathcal{P}_r$ which are incident with every block of $\mathcal{B}_i$. The integers $\gamma_{ir}$ form a tactical decomposition for the orbits $\mathcal{P}_r$ and $\mathcal{B}_i$, $1 \leq i, r \leq t$, for which the following equations must hold:

$$\sum_{r=1}^t \gamma_{ir} = k,$$

$$\sum_{r=1}^t \frac{\Omega_i}{\omega_r} \gamma_{ir} \gamma_{jr} = \lambda \Omega_j + \delta_{ij} n.$$  

The $t \times t$ matrix $(\gamma_{ir})$ is called orbit matrix of $\mathcal{D}$ (with respect to $G$).

If $G \cong \langle \rho \rangle$ is a cyclic group generated by an automorphism of odd prime order, then $\Omega_r$ and $\omega_r$ equal 1 or $|\rho|$. Denote by $F(\rho)$ the number of points and blocks fixed by $\rho$. The following facts are known for this special case:

**Theorem 1.**
1. $v \equiv F(\rho) \pmod{|\rho|}$.
2. $F(\rho) \leq k + \sqrt{n}$.
3. If $n$ is not a square, then $t$ is odd.
4. Suppose that for some prime $p$ dividing $n^*$ (the square-free part of $n$) there exists an integer $a$ such that $pa \equiv -1 \pmod{|p|}$. Then $F(\rho)$ is odd.

For fuller details of this result, the reader is referred to [3].

The first task of the construction is to find all suitable orbit matrices $(\gamma_{ir})$, assuming the action of an automorphism group $G$. The next step is to replace each number $\gamma_{ir}$ of the orbit matrices found in a suitable way to indicate exactly which points from the orbit $\mathcal{P}_r$ lie on which blocks from the orbit $\mathcal{B}_i$. Once this is done, all the incidences are defined and the construction is completed. However, because of the extremely large number of possibilities, it is often necessary to involve the computer in both steps of the construction.

2. The case $(61, 25, 10)$

According to [1,4], there is only one known symmetric design with parameters $(61, 25, 10)$. Its full automorphism group is isomorphic to

$$K = \langle \rho, \sigma, \tau \mid \rho^5 = \sigma^5 = \tau^2 = 1, \rho^\sigma = \rho, \rho^\tau = \rho, \sigma^\tau = \sigma^2 \rangle.$$
Obviously, \( K \cong D_{10} \times Z_5 \). Further, it can be seen that the numbers of fixed points of the generators of \( K \) are \( F(\rho) = 11 \), \( F(\sigma) = 1 \) and \( F(\tau) = 13 \).

We were interested to discover if more designs with these parameters exist, and in so doing we assume only the action of the group \( \langle \rho \rangle \). To this end, let us first see how an automorphism of order 5 can act on such a design.

**Lemma 1.** Let \( \gamma \) be an automorphism of order 5 acting on a symmetric \((61,25,10)\)-design. Then, \( F(\gamma) \in \{1,11\} \).

**Proof.** From Theorem 1 we get \( F(\gamma) \in \{1,11,21\} \). However, it is easy to see that when \( F(\gamma) = 21 \) it is not possible to construct any row of the orbit matrix corresponding to a non-trivial block orbit (or, more precisely, to solve the Eqs. (1) and (2) for any \( 21 \leq j \leq 29 \)).

**Lemma 2.** Up to isomorphism there is precisely one orbit matrix which describes the action of an automorphism of order 5 fixing 11 points of a symmetric \((61,25,10)\)-design. It is given by

\[
M = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 5 & 5 & 5 & 5 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 5 & 5 & 0 & 0 & 5 & 0 & 5 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 5 & 5 & 0 & 0 & 5 & 0 & 5 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 5 & 5 & 0 & 0 & 5 & 0 & 5 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 5 & 5 & 0 & 0 & 5 & 0 & 5 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 5 & 5 & 0 & 0 & 5 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 & 0 & 5 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \
\end{bmatrix}
\]

(of course, the first 11 rows and columns correspond the fixed blocks and points.)

**Proof.** Solving Eqs. (1) and (2) using the computer. \( \Box \)
To construct a design corresponding to this orbit matrix, we have to determine circulant \((0,1)\) matrices of order 5, each having row sums 2, that can be used to indicate the incidences between the non-trivial point and block orbits (so replacing the 2's in the above matrix). There are altogether \(\binom{5}{2} = 10\) possibilities for such circulant matrices. Enumerating the points of each orbit by \(0, 1, \ldots, 4\), we can name the possible choices (often called index sets) as 

\[
0 = \{2, 3\}, \quad 1 = \{3, 4\}, \quad 2 = \{0, 4\}, \quad 3 = \{0, 1\}, \quad 4 = \{1, 2\}, \quad 5 = \{1, 4\}, \quad 6 = \{0, 2\}, \quad 7 = \{1, 3\}, \quad 8 = \{2, 4\}, \quad 9 = \{0, 3\}.
\]

Thus, an index set \(\{i,j\}\) corresponds to a \(5 \times 5\) circulant matrix whose first row contains a 1 in columns \(i\) and \(j\) and has zeros elsewhere.

It is worthwhile to point out at this stage that the \(11 \times 10\) sub-matrix corresponding to the incidences between the non-fixed points and fixed blocks constitutes 10 rows of an \((11,5,2)\) design, which is, of course, unique. There is a similar remark concerning the incidences of the fixed points and non-fixed blocks. It follows from this that a solution will be obtained by constructing a \(10 \times 10\) matrix \(A\), whose entries are block circulant matrices of the type described above, and which satisfies

\[
AA^t = (15I + 5J) \otimes I + 8J \otimes (J - I),
\]

where the matrices of the first factor in the Kronecker product have order 5 and those of the second arc of order 10. It is clear that, given any solution \(A\) of (3), \(PAQ^t\) is also a solution, where \(P\) and \(Q\) are permutation matrices of order 10. To reduce the enormous number of possibilities for the intermediate stages of the computation involved in finding solutions \(A\), we made the assumption that \(A\) was lexicographically least under the action of \(S_{10} \times S_{10}\) and the additional automorphism \(\alpha \in N(Z_5)\) from the normalizer of the group \(\langle \rho \rangle\) defined as

\[
\alpha : x \mapsto 2x \ (mod\ 5).
\]

The final result was that, with the above assumptions there are precisely two matrices \(A\) satisfying (3), each being the transpose of the other. We give only one:

\[
A =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 5 & 5 & 5 & 5 & 5 \\
0 & 0 & 5 & 5 & 5 & 0 & 0 & 0 & 5 & 5 \\
0 & 5 & 0 & 5 & 5 & 0 & 5 & 5 & 0 & 0 \\
0 & 5 & 2 & 6 & 7 & 3 & 8 & 9 & 1 & 4 \\
0 & 5 & 3 & 9 & 8 & 2 & 7 & 6 & 4 & 1 \\
5 & 0 & 5 & 0 & 5 & 5 & 0 & 0 & 0 & 0 \\
5 & 0 & 6 & 3 & 1 & 9 & 4 & 2 & 8 & 7 \\
5 & 0 & 9 & 2 & 4 & 6 & 1 & 3 & 7 & 8 \\
5 & 5 & 0 & 5 & 0 & 5 & 0 & 0 & 5 & 0 \\
5 & 5 & 5 & 0 & 0 & 0 & 0 & 5 & 0 & 5
\end{bmatrix}
\]

**Theorem 2.** A symmetric \((61,25,10)\) design with an automorphism of order 5 fixing 11 points (and blocks) has an incidence matrix isomorphic to that obtained by replacing \(A\) in the above by \(PAQ^t\), \((P, Q \in S_{10})\), or its dual.
It would appear that there could be an extremely large number of such designs, a crude upper bound being $2(10!)^2$. In fact this upper bound can be reduced by considering the automorphism group of $M$ (which has order 3600), and the automorphism group of $A$, which has order 2, together with the automorphism $x^2$ acting on $A$ (which replaces $A$ by a matrix isomorphic with $A$).

Nevertheless, in the authors' opinion there are too many designs to make a complete classification feasible. A cursory examination revealed many tens of thousands of such designs.

After the experience above, we decided to take some larger subgroup of $K$ which contains $\rho$ and to attempt a complete classification for it. To this end, let $G \cong \langle \rho \rangle \times \langle \sigma \rangle$ be the elementary abelian group of order 25.

**Lemma 3.** An elementary abelian group of order 25 can act on a (61,25,10)-design only in orbits of length 1, 5, 5, 25, 25 on points and blocks. The two generators of order 5 fix 1 and 11 points and blocks respectively.

**Proof.** The only possible orbit lengths are 1, 5 and 25. Because of the commutativity, each orbit of length 5 is fixed by one of the generators of this group (and of course not by the other). Lemma 1 then gives the remainder of the proof. □

So, without loss of generality, the action of $\rho$ and $\sigma$ as given in $K$ is unique and the action of $\rho$ is completely described by the orbit matrix $M$. The additional assumption involving $\sigma$ as a group generator results in the condition that there is a cyclic action on the rows and columns 2–6, 7–11, 12–16 and 17–21. Taking this into account it was not too difficult to classify, using the computer, all the (61,25,10)-designs having $G$ as an automorphism group. We proved the following:

**Theorem 3.** Let $G$ be an elementary abelian group of order 25 acting on a symmetric (61,25,10)-design. Then there are, up to isomorphism and duality, 18 such designs, 12 of which are self-dual. In 6 cases the full automorphism group is isomorphic to $D_{10} \times Z_5$ (with an additional involution fixing 13 points and commuting with $\rho$) and in 12 cases $G$ is the full automorphism group.

**Remark 1.** There is no (61,25,10)-design having an automorphism group $D_{10} \times Z_5$ with an involution that commutes with $\sigma$ and fixes 21 points. This result has been established as well and even in greater generality, taking the dihedral subgroup $\langle \rho \rangle \langle \tau \rangle$ of $K$ as an automorphism group for $\mathcal{D}$.

Here we list the 6 designs having $\lvert \text{Aut } \mathcal{D} \rvert = 50$; the first 2 are not self-dual whereas the other 4 are. Instead of listing all the 10 rows of the $(10 \times 10)$-matrices of index sets, it is sufficient to list only the rows 1 and 6, as the rest can be obtained by the action of $\sigma$. 
3. The case \((66, 26, 10)\)

Let \(\rho\) be an automorphism of order 5 acting on a symmetric design \(\mathcal{D}\) with parameters \((66, 26, 10)\). Then, using Theorem 1, \(F(\rho) \in \{1, 6, 11, 16, 21, 26\}\), and since it is easy to eliminate the cases \(F(\rho) > 16\), we see that the greatest possible number of fixed points for \(\rho\) is 11. We shall consider particularly that case.

If \(F(\rho) = 11\), then \(t = 22\). More precisely, \(\rho\) acts in 11 orbits of length 1 and 11 orbits of length 5 on the sets of points and blocks. An easy computation leads to the unique orbit matrix (of order 22) for this orbit partition which can be written as

\[
M = \begin{bmatrix}
I & 5D \\
D' & 2J - I
\end{bmatrix},
\]

where \(D\) is the incidence matrix of the unique \((11, 5, 2)\)-design. (We have assumed that the first 11 rows and columns correspond to the fixed blocks and points).

The next task of the construction is to replace the \(11 \times 11\) matrix \(2J - I\) by a \(55 \times 55\) matrix, obtained by replacing each entry of \(2J - I\) by a circulant \((0, 1)\) matrix of order 5, and to replace \(D'\) and \(5D\) by \(j \otimes D'\) and \(j' \otimes D\), respectively, so that \(M\), after this replacement, becomes an incidence matrix for a \((66, 26, 10)\)-design. Here, \(j\) denotes the all-one vector of dimension 5. However, an exhaustive search for this procedure was too difficult in regards to the computing time involved, and for this reason we have introduced an additional assumption. Let \(\tau\) be an involution acting on \(\mathcal{D}\) in such a way that it does not commute with \(\rho\) and the orbit matrix \(M\) remains unchanged under the action of this dihedral group of order 10. In other words, let \(\tau\) also act on \(\mathcal{D}\) fixing 22 points.

As all the orbit representatives may be assumed to be \((\tau)\)-invariant, there are, without loss of generality, only three \((0, 1)\) matrices of order 5 that can replace the elements of \(2J - I\):

\[
x = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
y = \begin{bmatrix}
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0
\end{bmatrix},
\]

\[
z = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

Now, we are able to prove the following
Theorem 4. There are exactly 558 self-dual non-isomorphic symmetric designs with parameters (66,26,10) admitting the action of the dihedral group $G$ of order 10, such that the cyclic subgroup of order five fixes 11, and the cyclic subgroup of order two fixes 22, points, respectively. There is one design among these for which $\text{Aut} \mathcal{D} \cong G \times \text{Frob}_{55}$, nine designs for which $\text{Aut} \mathcal{D} \cong G \times Z_5$ and 548 for which $G$ is the full automorphism group.

Proof. We discovered using the computer that the incidence matrix of all designs with the above assumptions can be put in the form

$$\begin{bmatrix}
I & N \\
N' & PA P' \\
\end{bmatrix},$$

where $N = j \otimes D$ (the Kronecker product of the all-one-vector of dimension 5 with the incidence matrix $D$) and $P$ is a permutation matrix of order 11. The matrix $A$ is given by

\begin{align*}
x & y & y & y & y & z & z & z & z & z \\
z & x & y & y & z & z & y & y & z & z \\
z & z & x & y & z & z & z & y & y & y \\
z & z & z & x & y & z & y & y & z & z \\
z & y & z & z & x & y & y & z & z & y \\
z & y & y & z & x & z & z & y & y & y. \\
y & z & z & y & z & y & x & z & y & z \\
y & z & y & z & y & z & y & x & z & y \\
y & z & y & z & y & z & y & x & z & y \\
y & y & z & z & y & z & y & x & z & y \\
y & y & z & y & z & z & y & z & y & x
\end{align*}

This matrix has an automorphism group $K$ of order 55 and $N$ has an automorphism group $H$ of order 660. The involution $\tau$ has the effect of replacing $A$ by an isomorphic copy. More precisely, $\tau(A) = QAQ'$ where $Q$ is the permutation matrix corresponding to the involution $(2,8)(3,11)(4,9)(5,7)(6,10)$. Here, we have numbered the rows and columns of the matrix from 1 to 11. If we adjoin this involution to the group $K$ we obtain a group $K'$ of order 110. Now, all the $(66,26,10)$-designs are generated as $P$ runs through the double coset representatives of $H$ and $K'$ in $S_{11}$. Using GAP [2] we discovered that there are exactly 558 such double cosets; moreover, all the designs obtained are pairwise non-isomorphic. As the isomorphism $\alpha$ defined as $(x)(y,z)$, replaces $A$ by its transpose, all these designs are self-dual. Among the 558 designs all but ten of them have the dihedral group of order 10 as their full automorphism group. Of these 10, there are nine with full automorphism group of order 50 and one with full group of order 550. This latter design is the one that was known before (see [7, p. 83]). To end the proof, we shall list only the nine matrices $PA P'$ of order 11 for which $|\text{Aut} \mathcal{D}| = 50$.
To find some more designs with these parameters, another assumption has shown to be suitable. Let now \( G_1 \) be \( \langle \rho \rangle \times \langle \sigma \rangle \), \( \langle \rho \rangle, \langle \sigma \rangle \cong Z_5 \), the elementary abelian group of order 25.

**Lemma 4.** There is only one possible action of \( G_1 \) on \( \mathcal{D} \) and it has \( F(\rho) = 11 \) and \( F(\sigma) = 6 \). The orbit lengths are 1, 5, 5, 5, 25, 25.

**Proof.** The only possible orbit lengths are 1, 5 and 25. Because \( \rho \) and \( \sigma \) commute, each orbit of length 5 under the action of one automorphism splits into five fixed points under the action of the other. Using the fact that the maximum number of fixed points of an automorphism of order 5 equals 11, the orbit partition must be as stated. Then, without loss of generality, we may assume \( F(\rho) = 11 \) and \( F(\sigma) = 6 \). □

We know that the action of \( \rho \) is described by the orbit matrix \( M \). Without loss of generality, we may assume that \( \sigma \) fixes the first orbit of points and blocks (of length 1) and the last (of length 5) and that it performs a cyclic action on the other orbits. We
replace the incidence matrix for \((11,5,2)\)-design admitting the cyclic action of \(\sigma\) on rows 2–6 and 7–11 and columns 1–5 and 6–10

\[
D = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}.
\]

Finally, to complete the construction, we have to replace each entry of the matrix \(2J - I\) of order 11 by \(P^l x, P^l y, P^l z, l = 0, 1, \ldots, 4\), \(P\) a circulant permutation matrix of order 5 not equal to the identity, keeping in mind also the cyclic action of \(\sigma\). An exhaustive computer analysis led to the following:

**Theorem 5.** Up to isomorphism and duality there are 16 symmetric designs with parameters \((66,26,10)\) admitting the action of an elementary abelian group of order 25. Ten designs are self-dual and there are six pairs of dual designs. For four dual pairs, \(\text{Aut} \mathcal{D} \cong E_{25}\).

**Remark 2.** The 10 self-dual designs have \(D_{10} \times Z_5 \leq \text{Aut} \mathcal{D}\) and all were found also with the previous construction. Two pairs of dual designs have \(\text{Aut} \mathcal{D} \cong \langle \rho \rangle \times D_{10}\). Hence, an additional involution here fixes 18 points and these designs are non-isomorphic to any of the designs found in the previous theorem. We shall list these two designs without their duals, giving only rows 1, 6 and 11:

\[
\begin{array}{c}
P_0 x & P_3 z & P_2 z & P_3 z & P_0 z & P_1 y & P_4 y & P_4 y & P_1 y & P_0 y \\
P_0 y & P_1 y & P_4 y & P_4 y & P_1 y & P_0 x & P_2 z & P_3 z & P_3 z & P_2 z & P_0 z \\
P_0 z & P_0 z & P_0 z & P_0 z & P_0 y & P_0 y & P_0 y & P_0 y & P_0 y & P_0 y & P_0 x \\
\end{array}
\]

**Remark 3.** Altogether we have found here 570 symmetric \((66,26,10)\)-designs. One of them (with \(\text{Aut} \mathcal{D} \cong D_{10} \times \text{Frob}_{25}\)) was known before. The 18 designs with the same parameters constructed in [5] are non-isomorphic to the ones constructed here, as there 13 divides \(|\text{Aut} \mathcal{D}|\). Hence, there are at least 588 designs with parameters \((66,26,10)\).
References