Toughness and Vertex Degrees

D. Bauer  
Department of Mathematical Sciences  
Stevens Institute of Technology  
Hoboken, NJ 07030, U.S.A.

H.J. Broersma*  
School of Engineering and Computing Sciences  
Durham University  
South Road, Durham DH1 3LE, U.K.

J. van den Heuvel  
Department of Mathematics  
London School of Economics  
Houghton Street, London WC2A 2AE, U.K.

N. Kahl  
Department of Mathematics and Computer Science  
Seton Hall University  
South Orange, NJ 07079, U.S.A.

E. Schmeichel  
Department of Mathematics  
San José State University  
San José, CA 95192, U.S.A.

Abstract

We study theorems giving sufficient conditions on the vertex degrees of a graph $G$ to guarantee $G$ is $t$-tough. We first give a best monotone theorem when $t \geq 1$, but then show that for any integer $k \geq 1$, a best monotone theorem for $t = \frac{1}{k} \leq 1$ requires at least $f(k) \cdot |V(G)|$ nonredundant conditions, where $f(k)$ grows superpolynomially as $k \to \infty$. When $t < 1$, we give two simple theorems for $G$ to be $t$-tough, in terms of its vertex degrees. We conclude with a theorem giving the best possible minimum degree condition for $G$ to be $t$-tough, for any $t > 0$.

1 Introduction

We consider only simple graphs without loops or multiple edges. Our terminology and notation will be standard except as indicated, and a good reference for any undefined terms or notation is [12].

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For a positive integer \( n \), an \( n \)-sequence (or just a sequence) is an integer sequence \( \pi = (d_1, d_2, \ldots, d_n) \), with \( 0 \leq d_j \leq n - 1 \) for all \( j \). In contrast to \([12]\), we will usually write the sequence in nondecreasing order (and may make this explicit by writing \( \pi = (d_1 \leq \cdots \leq d_n) \)). We will employ the standard abbreviated notation for sequences, e.g., \( (4, 4, 4, 4, 5, 5, 6) \) will be denoted \( 4^5 \cdot 5^2 \cdot 6^1 \). If \( \pi = (d_1, \ldots, d_n) \) and \( \pi' = (d'_1, \ldots, d'_n) \) are two \( n \)-sequences, we say \( \pi' \) majorizes \( \pi \), denoted \( \pi' \geq \pi \), if \( d'_j \geq d_j \) for all \( j \).

A degree sequence of a graph is any sequence \( \pi = (d_1, d_2, \ldots, d_n) \) consisting of the vertex degrees of the graph. A sequence \( \pi \) is graphical if there exists a graph \( G \) having \( \pi \) as one of its degree sequences, in which case we call \( G \) a realization of \( \pi \). If \( P \) is a graph property (e.g., hamiltonian, \( k \)-connected, etc.), we call a graphical sequence \( \pi \) forcibly \( P \) graphical (or just forcibly \( P \)) if every realization of \( \pi \) has property \( P \).

Let \( \omega(G) \) denote the number of components of \( G \). For \( t \geq 0 \), we call \( G \) t-tough if \( t \cdot \omega(G - X) \leq |X| \), for every \( X \subseteq V(G) \) with \( \omega(G - X) > 1 \). The toughness of \( G \), denoted \( \tau(G) \), is the maximum \( t \geq 0 \) for which \( G \) is t-tough (taking \( \tau(K_n) = n - 1 \), for all \( n \geq 1 \)). So if \( G \) is not complete, then \( \tau(G) = \min \left\{ \frac{|X|}{\omega(G - X)} \right\} \) \( X \subseteq V(G) \) is a cutset of \( G \).

For two graphs \( G, H \) on disjoint vertex sets, we denote their union by \( G \cup H \). The join \( G + H \) of \( G \) and \( H \) is the graph formed from \( G \cup H \) by adding all edges between \( V(G) \) and \( V(H) \).

Historically, the degree sequence of a graph has been used to provide sufficient conditions for a graph to have certain properties, such as hamiltonian or \( k \)-connected. In particular, sufficient conditions for \( \pi \) to be forcibly hamiltonian were given by several authors, culminating in the following theorem of Chvátal \([5]\).

**Theorem 1.1** (\([5]\)). Let \( \pi = (d_1 \leq \cdots \leq d_n) \) be a graphical sequence, with \( n \geq 3 \). If \( d_i \leq i < \frac{1}{2} n \) implies \( d_{n-i} \geq n - i \), then \( \pi \) is forcibly hamiltonian.

Unlike its predecessors, Chvátal’s theorem has the property that if it does not guarantee that \( \pi \) is forcibly hamiltonian because the condition fails for some \( i < \frac{1}{2} n \), then \( \pi \) is majorized by \( \pi' = i^i \cdot (n - i - 1)^{n-2i} \cdot (n - 1)^i \), which has a unique non-hamiltonian realization \( K_i + (K_i \cup K_{n-2i}) \). As we will see below, this implies that Chvátal’s theorem is the strongest of an entire class of theorems giving sufficient degree conditions for \( \pi \) to be forcibly hamiltonian.

Sufficient conditions for \( \pi \) to be forcibly \( k \)-connected were given by several authors, culminating in the following theorem of Bondy \([4]\) (though the form in which we present it is due to Boesch \([3]\)).
Theorem 1.2 ([3, 4]). Let \( \pi = (d_1 \leq \cdots \leq d_n) \) be a graphical sequence with \( n \geq 2 \), and let \( 1 \leq k \leq n-1 \). If \( d_i \leq i+k-2 \) implies \( d_{n-k+1} \geq n-i \), for \( 1 \leq i \leq \frac{1}{2}(n-k+1) \), then \( \pi \) is forcibly \( k \)-connected.

Boesch [3] also observed that Theorem 1.2 is the strongest theorem giving sufficient degree conditions for \( \pi \) to be forcibly \( k \)-connected, in exactly the same sense as Theorem 1.1.

A graph property \( P \) is called increasing if whenever a graph \( G \) has \( P \), so does every edge-augmented supergraph of \( G \). In particular, “hamiltonian”, “\( k \)-connected”, “\( t \)-tough”, and “\( \alpha(G) \leq \kappa(G) \)” are all increasing graph properties. In the remainder of this section, the term “graph property” will always mean an increasing graph property.

Given a graph property \( P \), consider a theorem \( T \) which declares certain degree sequences to be forcibly \( P \), rendering no decision on the remaining degree sequences. We call such a theorem \( T \) a forcibly \( P \) theorem (or just a \( P \) theorem, for brevity). Thus Theorem 1.1 would be a forcibly hamiltonian theorem. We call a \( P \) theorem \( T \) monotone if, for any two degree sequences \( \pi, \pi' \), whenever \( T \) declares \( \pi \) forcibly \( P \) and \( \pi' \geq \pi \), then \( T \) declares \( \pi' \) forcibly \( P \). We call a \( P \) theorem \( T \) optimal (resp., weakly-optimal) if whenever \( T \) does not declare \( \pi \) forcibly \( P \), then \( \pi \) has a realization without property \( P \) (resp., then there exists \( \pi' \), so that \( \pi' \geq \pi \) and \( \pi' \) has a realization without property \( P \)). A \( P \) theorem which is both monotone and weakly-optimal is a best monotone \( P \) theorem, in the following sense.

Theorem 1.3. Let \( T, T_0 \) be monotone \( P \) theorems, with \( T_0 \) weakly-optimal. If \( T \) declares a degree sequence \( \pi \) to be forcibly \( P \), then so does \( T_0 \).

Proof of Theorem 1.3. Suppose to the contrary that there exists a degree sequence \( \pi \) so that \( T \) declares \( \pi \) forcibly \( P \), but \( T_0 \) does not. Since \( T_0 \) is weakly-optimal, there exists a degree sequence \( \pi' \geq \pi \) having a realization \( G' \) without property \( P \); in particular, \( T \) will not declare \( \pi' \) forcibly \( P \). But if \( T \) declares \( \pi \) forcibly \( P \), \( \pi' \geq \pi \), and \( T \) does not declare \( \pi' \) forcibly \( P \), then \( T \) is not monotone, a contradiction.

If \( T_0 \) is Chvátal’s hamiltonian theorem (Theorem 1.1), then \( T_0 \) is clearly monotone, and we noted above that \( T_0 \) is weakly-optimal. So by Theorem 1.3 Chvátal’s theorem is the best monotone hamiltonian theorem.

Our goal in this paper is to consider forcibly \( t \)-tough theorems, for any \( t \geq 0 \). In Section 2, we will consider best monotone \( t \)-tough theorems. We first give a best monotone \( t \)-tough theorem, for any \( t \geq 1 \). When \( t = 1 \), this theorem reduces to Chvátal’s theorem, which is also therefore a best monotone 1-tough theorem. But we then show that for any integer \( k \geq 1 \), a best monotone \( 1/k \)-tough theorem for \( n \)-sequences contains at least \( f(k) \cdot n \) nonredundant conditions of a kind similar to
those in Theorem 1.1 where \( f(k) \) grows superpolynomially as \( k \to \infty \) (see the next subsection for a precise description of the kind of degree conditions we consider). A similar superpolynomial growth in the complexity of the best monotone \( k \)-edge-connected theorem in terms of \( k \) was previously noted by Kriesell [9].

This superpolynomial complexity of a best monotone \( 1/k \)-tough theorem suggests the desirability of finding more reasonable \( t \)-tough theorems, when \( t < 1 \). In Section 3 we give several such theorems. The first is a monotone, though not best monotone, \( t \)-tough theorem which works for any \( t \leq 1 \), and reduces to Theorem 1.1 when \( t = 1 \). We also give some results that are based on the well-known inequality \( \tau(G) \geq \kappa(G)/\alpha(G) \). These theorems hold for every \( t > 0 \), but one of them is not monotone. We also discuss the relative strength of these theorems.

Finally, in Section 4 we give a minimum degree theorem for a graph to be \( t \)-tough, for any \( t > 0 \), and note that it is best possible in the same sense as Dirac’s well-known minimum degree theorem [7] for a graph to be hamiltonian.

### 1.1 Best Monotone Results using Chvátal Type Conditions

In this subsection we provide a theory that allows us to lower bound the number of degree sequence conditions required in a best monotone \( P \) theorem. First we need to specify what we mean by a ‘degree sequence condition’.

A **Chvátal type condition** for \( n \)-sequences \((d_1 \leq d_2 \leq \cdots \leq d_n)\) is a condition of the form

\[
d_{i_1} \geq k_{i_1} \lor d_{i_2} \geq k_{i_2} \lor \ldots \lor d_{i_t} \geq k_{i_t},
\]

where all \( i_j \) and \( k_{i_j} \) are integers, with \( 1 \leq i_1 < i_2 < \cdots < i_t \leq n \) and \( 1 \leq k_{i_1} \leq k_{i_2} \leq \cdots \leq k_{i_t} \leq n \). Since the condition in Theorem 1.1 can be written as

\[
\text{for all } i = 1, \ldots, \left\lfloor \frac{1}{2}(n-1) \right\rfloor: \quad d_i \geq i + 1 \lor d_{n-i} \geq n - i,
\]

we see that Theorem 1.1 contains \( \left\lfloor \frac{1}{2}(n-1) \right\rfloor \) Chvátal type conditions for sequences of length \( n \).

Given an \( n \)-sequence \( \pi = (k_1 \leq k_2 \leq \cdots \leq k_n) \), let \( C(\pi) \) denote the Chvátal type condition:

\[
d_1 \geq k_1 + 1 \lor d_2 \geq k_2 + 1 \lor \ldots \lor d_n \geq k_n + 1.
\]

Intuitively, \( C(\pi) \) is the minimal condition that ‘blocks’ \( \pi \). For instance, if \( \pi = 2^23^5 \), then \( C(\pi) \) is

\[
d_1 \geq 3 \lor d_2 \geq 3 \lor d_3 \geq 4 \lor d_4 \geq 4 \lor d_5 \geq 4 \lor d_6 \geq 6.
\]

(1)

Since \( n \)-sequences are assumed to be nondecreasing, \( d_1 \geq 3 \) implies \( d_2 \geq 3 \), etc. Also, we cannot have \( d_i \geq n \), so the condition \( d_6 \geq 6 \) is redundant. Hence condition (1) can be simplified to

\[
d_2 \geq 3 \lor d_5 \geq 4.
\]

(2)
Conversely, given a Chvátal type condition $c$ let $\Pi(c)$ denote the minimal $n$-sequence that majorizes any sequence which violates $c$. So if $c$ is the condition in \(2\) and $n = 6$, then $\Pi(c)$ is $2^23^35$. Of course, $\Pi(c)$ itself violates $c$. Note that $C$ and $\Pi$ are inverses: For any Chvátal type condition $c$ we have $C(\Pi(c)) = c$, and for any $n$-sequence $\pi$ we have $\Pi(C(\pi)) = \pi$.

Given a graph property $P$, we call a degree condition $c$ $P$-weakly-optimal if for any sequence $\pi$ (not necessarily graphical) which does not satisfy $c$, there exists a $\pi'$, $\pi' \geq \pi$, such that $\pi'$ has a non-$P$ realization. In particular, the conditions in Chvátal’s hamiltonian theorem are each weakly-optimal.

Next consider the poset whose elements are graphical sequences of length $n$, with the majorization relation $\pi \leq \pi'$ as the partial order relation. We call this poset the $n$-degree-poset. The degree-poset is the union of the $n$-degree-posets, for $n = 1, 2, \ldots$.

Given a graph property $P$, consider the set of graphs without property $P$ which are edge-maximal in this regard. The degree sequences of these edge-maximal, non-$P$ graphs induce a subposet of the degree-poset, called the $P$-subposet. Of course, the $P$-subposet has one or more maximal elements in each of the $n$-degree-posets. We use the term sink for such a maximal element.

We now prove the main result of this subsection, giving a lower bound on the number of $P$-weakly-optimal Chvátal type conditions in a $P$ theorem.

**Theorem 1.4.** Let $P$ be a graph property. Then any $P$ theorem whose degree conditions consist solely of $P$-weakly-optimal Chvátal type conditions must contain at least $s(n, P)$ such conditions, where $s(n, P)$ is the number of sinks in the $P$-subposet of the $n$-degree-poset.

**Proof:** Consider a $P$ theorem whose degree conditions consist solely of $P$-weakly-optimal Chvátal type conditions.

**Claim.** If a sink $\pi$ in the $P$-subposet violates some degree condition $c$ in the theorem, then $c = C(\pi)$.

**Proof of Claim:** Since $\pi$ violates $c$, $\pi \leq \Pi(c)$. Since $\Pi(c)$ violates $c$, and $c$ is $P$-weakly-optimal, there is a sequence $\pi' \geq \Pi(c)$ such that $\pi'$ has a non-$P$ realization. But $\pi' \leq \pi''$ for some sink $\pi''$, giving $\pi \leq \Pi(c) \leq \pi' \leq \pi''$. Since sinks are majorization independent, $\pi = \pi''$. This implies $\Pi(c) = \pi$, and thus $c = C(\Pi(c)) = C(\pi)$, proving the Claim.

By the Claim, a sink $\pi$ satisfies every $P$-weakly-optimal Chvátal type condition in the theorem besides $C(\pi)$. Thus all the other conditions in the theorem besides $C(\pi)$ together do not imply $C(\pi)$, since $\pi$ satisfies all the others, but not $C(\pi)$. So the theorem must include all the Chvátal type conditions $C(\pi)$, as $\pi$ ranges over the $s(n, P)$ sinks. ■
On the other hand, it is easy to see that if we take the collection of Chvátal type conditions $C(\pi)$ for all sinks $\pi$ in the $P$-subposet, then this gives a best monotone weakly-optimal $P$ theorem.

We do not have a comparable result for $P$ theorems if we do not require the conditions to be $P$-weakly-optimal, let alone if we consider conditions that are not of Chvátal type. On the other hand, all results we have discussed so far, and most of the forcibly $P$ theorems we know in the literature, involve only weakly-optimal Chvátal type degree conditions.

## 2 Best Monotone $t$-Tough Theorems

We first give a best monotone $t$-tough theorem for $t \geq 1$.

**Theorem 2.1.** Let $t \geq 1$, and let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n > \frac{(t + 1)[t]}{t}$. If

\[(*) \quad d_{\lfloor i/t \rfloor} \leq i \implies d_{n-i} \geq n - \lfloor i/t \rfloor, \text{ for } t \leq i < \frac{tn}{(t + 1)},\]

then $\pi$ is forcibly $t$-tough.

Clearly, property $(*)$ in Theorem 2.1 is monotone. Furthermore, if $\pi$ does not satisfy $(*)$ for some $i$ with $t \leq i < tn/(t + 1)$, then $\pi$ is majorized by $\pi' = i^{\lfloor i/t \rfloor} \oplus (n - \lfloor i/t \rfloor - 1)^{n-i-\lfloor i/t \rfloor} (n-1)^i$, which has a non-$t$-tough realization $K_i + (K_{\lfloor i/t \rfloor} \cup K_{n-i-\lfloor i/t \rfloor})$. Thus $(*)$ in Theorem 2.1 is also weakly-optimal, and so Theorem 2.1 is best monotone by Theorem 1.3. Finally, note that when $t = 1$, $(*)$ reduces to Chvátal’s hamiltonian condition in Theorem 1.1.

**Proof of Theorem 2.1.** Suppose $\pi$ satisfies $(*)$ for some $t \geq 1$ and $n > (t+1)[t]/t$, but $\pi$ has a realization $G$ which is not $t$-tough. Then there exists a set $X \subseteq V(G)$ with $\omega(G-X) \geq 2$ and $\tau(G) = |X|/\omega(G-X) < t$, or $\omega(G-X) > |X|/t$. Let $H_1, H_2, \ldots$ denote the components of $G - X$. By adding edges (if needed) to $G$, we may assume $\langle X \rangle$ is complete, and each $\langle H_i \rangle$ is complete and completely joined to $X$.

Setting $x = |X|$, we show that we may assume $x \geq t$. For if $x < t$, and if $|H_j| \geq 2$ for some $j$, complete a vertex $v \in H_j$ and transfer $v$ from $H_j$ to $X$ (i.e., $H_j \leftarrow H_j - v$, $X \leftarrow X + v$), iterating until either $x = \lceil t \rceil$, at which point $\omega(G-X) \geq 2 = \lceil x/t \rceil + 1 > x/t$ still holds, or else $|H_1| = |H_2| = \cdots = |H_{n-x}| = 1$. But in the latter case, the resulting degree sequence $\pi^* = x^{n-x} (n-1)^x$ fails to satisfy $(*)$ for $i = \lceil t \rceil$: simply note that $i = \lceil t \rceil < tn/(t + 1)$ since $n > (t + 1)[t]/t$, $d_{\lfloor i/t \rfloor} = d_1 = x < t \leq \lceil t \rceil = i$, but $d_{n-i} = d_{n-\lceil t \rceil} \leq d_{n-x} = x < n-1 = n - \lfloor i/t \rfloor$. Since $\pi^* \geq \pi$ and $(*)$ is monotone, this contradicts the assumption that $\pi$ satisfies $(*)$. Hence we may assume $x \geq t$. 

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Claim 1. \( x < \frac{tn}{t+1} \).

Proof: If \( x \geq tn/(t+1) \), then \( \omega(G-X) \leq n-x \leq n/(t+1) \), and so
\[
\tau(G) = \frac{x}{\omega(G-X)} \geq \frac{tn/(t+1)}{n/(t+1)} = t,
\]
contradicting \( \tau(G) < t \). \( \square \)

Form the graph \( G' \) by adding to \( G \) a set of \( x - \lfloor x/t \rfloor \) independent vertices, each joined completely to \( X \). Then \( G' \) has \( n' = n+x-\lfloor x/t \rfloor \) vertices, degree sequence
\[
\pi' = (x, \ldots, x, d_1, d_2, \ldots, d_{n-x}, n'-1, \ldots, n'-1)
\]
and minimum degree \( x \), with \( d_i = d'_{i+x-\lfloor x/t \rfloor} \). Clearly, \( G' \) is not 1-tough, since \( \omega(G'-X) > x/t + x - \lfloor x/t \rfloor \geq x \). On the other hand, we will show the following.

Claim 2. \( \pi' \) satisfies (**1). \( \bigstar \)

Since (**1) is Chvátal’s hamiltonian condition, Claim 2 implies that every realization of \( \pi' \) is hamiltonian, and a fortiori 1-tough, contradicting the realization \( G' \). So to complete the proof of Theorem 2.1, it suffices to prove Claim 2.

First we establish two additional claims.

Claim 3. If \( a \geq b \geq 1 \) are integers, and \( t \geq 1 \), then \( \lfloor a/t \rfloor - \lfloor b/t \rfloor \leq a - b \).

Proof: Suppose \( \lfloor a/t \rfloor - \lfloor b/t \rfloor \geq a - b + 1 \). Then \( (a - b)/t = a/t - b/t \geq \lfloor a/t \rfloor - (\lfloor b/t \rfloor + 1) = ([a/t] - [b/t]) - 1 \geq a - b \), a contradiction since \( (1/t) \leq 1 \). \( \square \)

Claim 4. \( \frac{1}{2}(n'-1) < \frac{tn}{t+1} \).

Proof: Since \( \lfloor x/t \rfloor + 1 > x/t \), we have by Claim 1,
\[
\frac{1}{2}(n'-1) = \frac{1}{2}(n+x-\lfloor x/t \rfloor - 1) < \frac{1}{2}(n+x-x/t)
\]
\[
= \frac{1}{2}
\left( n + \frac{t-1}{t}x \right)
\]
\[
< \frac{1}{2}
\left( n + \frac{t-1}{t} \cdot \frac{tn}{t+1} \right) = \frac{tn}{t+1}.
\]

\( \square \)

Proof of Claim 2: Suppose \( d'_i \leq i < \frac{1}{2}n' \), so that \( i \geq d'_i \geq x \geq t \geq 1 \). By Claim 3,
\[
[i/t] - \lfloor i/t \rfloor \leq i - x.
\]

Thus \( d_{i-(x-\lfloor x/t \rfloor)} = d'_i \leq i \), and by (3) and Claim 4 we find \( d_{[i/t]} \leq d_{i-x+\lfloor x/t \rfloor} \leq i \leq \frac{1}{2}(n'-1) < tn/(t+1) \). Since \( i \geq x \geq t \) and \( \pi \) satisfies (**t), we obtain \( d_{n-i} \geq n - \lfloor i/t \rfloor \). But \( d_{n-i} = d_{n-i+(x-\lfloor x/t \rfloor)} = d'_{n+x-\lfloor x/t \rfloor-i} = d'_{n'-i} \), while by (3) we

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have \( n - \lfloor i/t \rfloor = (n' - x + \lfloor x/t \rfloor) - \lfloor i/t \rfloor = n' - (x - \lfloor x/t \rfloor + \lfloor i/t \rfloor) \geq n' - i \). So \( d'_{n-i} = d_{n-i} \geq n - \lfloor i/t \rfloor \geq n' - i \), and thus \( \pi' \) satisfies (★1). This proves Claim 2 and completes the proof of Theorem 2.1.

Using the terminology from Subsection 1.1, it follows that Theorem 2.1 gives a best monotone \( t \)-tough theorem, for \( t \geq 1 \), using a linear number (in \( n \)) of weakly-optimal Chvátal type conditions. On the other hand, we now show that for any integer \( k \geq 1 \), a best monotone \( 1/k \)-tough theorem for \( n \)-sequences requires at least \( f(k) \cdot n \) weakly-optimal Chvátal type conditions, where \( f(k) \) grows superpolynomially as \( k \to \infty \).

In view of Theorem 1.4 to prove this assertion it is enough to prove the following lemma.

**Lemma 2.2.** Let \( k \geq 2 \) be an integer, and let \( n = m(k + 1) \) for some integer \( m \geq 9 \). Then the number of \((1/k\)-tough\))-subposet sinks in the \( n \)-degree-subposet is at least \( \frac{p(k-1)}{5(k+1)} n \), where \( p \) denotes the integer partition function.

Recall that the integer partition function \( p(r) \) counts the number of ways a positive integer \( r \) can be written as the sum of positive integers. Since \( p(r) \sim \frac{1}{4r\sqrt{3}} e^\pi \sqrt{2r/3} \) as \( r \to \infty \) \([8]\), \( f(k) = \frac{p(k-1)}{5(k+1)} \) grows superpolynomially as \( k \to \infty \).

**Proof of Lemma 2.2.** Consider the collection \( C \) of all connected graphs on \( n \) vertices which are edge-maximally not-(1\(k\)-tough). Each \( G \in C \) has the form \( G = K_j + (K_{c_1} \cup \cdots \cup K_{c_{k_j+1}}) \), where \( j < n/(k + 1) = m \), so that \( 1 \leq j \leq m - 1 \), and \( c_1 + \cdots + c_{k_j+1} \) is a partition of \( n - j \). Assuming \( c_1 \leq \cdots \leq c_{k_j+1} \), the degree sequence of \( G \) becomes \( \pi = (c_1 + j - 1)^{c_1} \cdots (c_{k_j+1} + j - 1)^{c_{k_j+1}}(n-1)^{j} \). Note that \( \pi \) cannot be majorized by the degrees of any disconnected graph on \( n \) vertices, since a disconnected graph has no complete vertex.

Partition the degree sequences of \( C \) into \( m - 1 \) groups, where the sequences in the \( j \)th group, \( 1 \leq j \leq m - 1 \), are precisely those containing \( j \) complete degrees. We establish two basic properties of the \( j \)th group.

**Claim 1.** There are exactly \( p_{k_j+1}((k+1)(m-j)-1) \) sequences in the \( j \)th group.

Here \( p_{\ell}(r) \) denotes the number of partitions of integer \( r \) into at most \( \ell \) parts, or equivalently the number of partitions of \( r \) with largest part at most \( \ell \).

**Proof of Claim 1:** Each sequence in the \( j \)th group corresponds uniquely to a set of \( k_j + 1 \) component sizes which sum to \( n - j \). If we subtract 1 from each of those component sizes, we obtain a corresponding collection of \( k_j + 1 \) integers (some possibly 0) which sum to \( n - j - (k_j + 1) = (k+1)(m-j)-1 \), and which therefore form a partition of \((k+1)(m-j)-1\) into at most \( k_j + 1 \) parts. \( \square \)
Claim 2. No sequence in the $j$th group majorizes another sequence in the $j$th group.

Proof: Suppose the sequences $\pi \doteq (c_1 + j - 1)^{c_1} \cdots (c_{kj+1} + j - 1)^{c_{kj+1}}(n - 1)^{j}$ and $\pi' \doteq (c'_1 + j - 1)^{c'_1} \cdots (c'_{kj+1} + j - 1)^{c'_{kj+1}}(n - 1)^{j}$ are in the $j$th group, with $\pi \geq \pi'$. Deleting the $j$ complete degrees from each sequence gives sequences $\sigma \doteq (c_1 - 1)^{c_1} \cdots (c_{kj+1} - 1)^{c_{kj+1}}$ and $\sigma' \doteq (c'_1 - 1)^{c'_1} \cdots (c'_{kj+1} - 1)^{c'_{kj+1}}$, with $\sigma \geq \sigma'$.

Let $m$ be the smallest index with $c_m \neq c'_m$; since $\sigma \geq \sigma'$, we have $c_m > c'_m$. In particular, $c_1 + \cdots + c_m > c'_1 + \cdots + c'_m$. But $c_1 + \cdots + c_{kj+1} = c'_1 + \cdots + c'_{kj+1} = n - j$, and so there exists a smallest index $\ell > m$ with $c_1 + \cdots + c_\ell \leq c'_1 + \cdots + c'_\ell$. In particular, $c_\ell < c'_\ell$. Since $c'_1 + \cdots + c'_\ell - 1 < c_1 + \cdots + c_{\ell - 1} < c_1 + \cdots + c_\ell \leq c_1 + \cdots + c'_\ell$, we have $d_{c_1 + \cdots + c_\ell} = c_\ell - 1 < c'_\ell - 1 = d'_{c'_1 + \cdots + c'_\ell}$, and thus $\sigma \not\prec \sigma'$, a contradiction.

If $K_j + (K_c \cup \cdots \cup K_{c_{kj+1}})$ has $n$ vertices, then $K_{c_{kj+1}}$ has at most $n - j - kj$ vertices. This means that the largest possible noncomplete degree in a sequence in the $j$th group is $j + (n - j - kj - 1) = n - kj - 1$. Using this observation we can prove the following.

Claim 3. If a sequence $\pi = d^{d-j+1}(n - 1)^j$ in the $j$th group has largest noncomplete degree $d \geq n - k(j + 1)$, then $\pi$ is not majorized by any sequence in the $i$th group, for $i \geq j + 1$.

In particular, such a $\pi$ is a sink, since $\pi$ is certainly not majorized by another sequence in the $j$th group by Claim 2, nor by a sequence in groups 1, 2, \ldots, $j - 1$, since any such sequence has fewer than $j$ complete vertices.

Proof of Claim 3: If $d \geq n - k(j + 1)$, then the $d + 1$ largest degrees $d^{d-j+1}(n - 1)^j$ in $\pi$ could be majorized only by complete degrees in a sequence in group $i \geq j + 1$, since the largest noncomplete degree in any sequence in group $i$ is at most $n - ki - 1 < n - k(j + 1)$. There are only $i \leq m - 1$ complete degrees in a sequence in group $i$. On the other hand, since $j + 1 \leq i < m$, we have $d + 1 \geq n - k(j + 1) + 1 > m(k + 1) - km + 1 = m + 1 > m - 1$, a contradiction.

So by Claim 3, the sequences $\pi$ in the $j$th group which could possibly be nonsinks (i.e., majorized by a sequence in group $i$, for some $i \geq j + 1$), must have largest noncomplete degree at most $n - k(j + 1) - 1$. So in a graph $G \in C$, $G = K_j + (K_c \cup \cdots \cup K_{c_{kj+1}})$, which realizes a nonsink $\pi$, each $K_c$ must have size at most $(n - k(j + 1) - 1) - j + 1 = (k + 1)(m - j) - k$. Subtracting 1 from the size of each of these components gives a sequence of $kj + 1$ integers (some possibly 0) which sum to $(n - j) - (kj + 1) = (k + 1)(m - j) - 1$, and which have largest part at most $(k + 1)(m - j) - k - 1 = (k + 1)(m - j - 1)$. Thus there are exactly $p_{(k+1)(m-j-1)}((k + 1)(m - j) - 1)$ such sequences, and so there are at most this many nonsinks in the $j$th group. Setting $N(j) \doteq (k + 1)(m - j) - 1$, so that
Theorem 2.3. \ Let \( k \geq 2 \) be an integer, and let \( n = m(k + 1) \) for some integer \( m \geq 9 \). Then a best monotone 1/k-tough theorem for \( n \)-sequences whose degree conditions consist solely of weakly-optimal Chvátal type conditions requires at least \( \frac{p(k-1)n}{5(k+1)} \) such conditions, where \( p(r) \) is the integer partition function.
3 Simple $t$-Tough Theorems

The superpolynomial complexity as $k \rightarrow \infty$ of a best monotone $1/k$-tough theorem suggests the desirability of finding simple $t$-tough theorems, when $t < 1$. We begin with such a theorem below. It will again be convenient to assume at first that $t = 1/k$, for some integer $k \geq 1$.

**Theorem 3.1.** Let $k \geq 1$ be an integer, and $\pi = (d_1 \leq \cdots \leq d_n)$ a graphical sequence. Suppose

(1) $d_i \leq i - (k - 1) \implies d_{n-i+k-1} \geq n - i$, for $k - 1 \leq i < \frac{1}{2}(n + k - 1)$, and

(2) $d_i \leq i - 1 \implies d_n \geq n - i$, for $1 \leq i \leq \frac{1}{2}n$.

Then $\pi$ is forcibly $1/k$-tough.

Note that for $k = 1, 2$, condition (2) is implied by (1). And for $k = 1$, Theorem 3.1 is implied by Theorem 1.1.

**Proof of Theorem 3.1:** Suppose $\pi$ has a realization $G$ which is not $1/k$-tough. By (2) and Theorem 1.2, $G$ is connected. So we may assume (by adding edges if necessary) that there exists $X \subseteq V(G)$, with $x = |X| \geq 1$, such that $G = K_x + (K_{a_1} \cup K_{a_2} \cup \cdots \cup K_{a_{k-1}})$, where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_{k-1}$.

Set $i = x + k - 2 + a_{k-1}$. Then we obviously have $i \geq k - 1$. We prove some further claims.

**Claim 1.** $d_i \leq i - (k - 1)$.

**Proof:** We have $a_1 + a_2 + \cdots + a_{k-1} \geq kx - 1 \geq x + k - 2$, since $kx - x - k + 1 = (k - 1)(x - 1) \geq 0$. But then $d_i = d_{x+k-2+a_{k-1}} \leq d_{a_1+a_2+\cdots+a_{k-1}+a_{k-1}} = x + a_{k-1} - 1 = i - (k - 1)$.

**Claim 2.** $i < \frac{1}{2}n + k - 1$.

**Proof:** We have $n = a_1 + \cdots + a_{k-1} + a_{k} + a_{k+1} + x \geq (kx - 1) + 2a_{k} + x$, or $a_{k} \leq \frac{1}{2}(n - (k + 1)x + 1)$. Since $x \geq 1$, we obtain

$$i = x + k - 2 + a_{k} \leq x + k - 2 + \frac{1}{2}(n - (k + 1)x + 1)$$

$$= \frac{1}{2}(n + 2k - 3 - (k - 1)x) \leq \frac{1}{2}(n + k - 2) < \frac{1}{2}(n + k - 1).$$

**Claim 3.** $d_{n-i+k-1} < n - i$.

**Proof:** As in the proof of Claim 1, $kx \geq x + k - 1$. Thus

$$d_{n-i+k-1} = d_{n-(x+k-2+a_{k})+k-1} = d_{n-x-a_{k}+1} \leq d_{n-x} = x + a_{k} - 1$$

$$= n - (a_1 + \cdots + a_{k}) - 1 \leq n - (kx - 1) - a_{k} - 1$$

$$= n - (kx + a_{k}) \leq n - (x + k - 1 + a_{k}) < n - i$$

\[\square\]
Claims 1, 2 and 3 together contradict condition (1), completing the proof of Theorem 3.1.

We can extend Theorem 3.1 to arbitrary $t \leq 1$ by letting $k = \left\lfloor \frac{1}{t} \right\rfloor$.

**Theorem 3.2.** Let $t \leq 1$, and $\pi = (d_1 \leq \cdots \leq d_n)$ a graphical sequence. If

1. $d_i \leq i - \left(\left\lfloor \frac{1}{t} \right\rfloor - 1\right) \implies d_{n-i+\left\lfloor \frac{1}{t} \right\rfloor-1} \geq n - i$, for $i < \frac{1}{2}(n + \left\lfloor \frac{1}{t} \right\rfloor - 1)$, and
2. $d_i \leq i - 1 \implies d_n \geq n - i$, for $1 \leq i \leq \frac{1}{2}n$,

then $\pi$ is forcibly $t$-tough.

**Proof:** Set $k = \left\lfloor \frac{1}{t} \right\rfloor \geq 1$. If $\pi$ satisfies conditions (1), (2) in Theorem 3.2, then $\pi$ satisfies conditions (1), (2) in Theorem 3.1 and so is forcibly $1/k$-tough. But $k = \left\lfloor \frac{1}{t} \right\rfloor \leq \frac{1}{t}$ means $\frac{1}{k} \geq t$, and so $\pi$ is forcibly $t$-tough.

In summary, if $\frac{1}{k+1} < t \leq \frac{1}{k}$ for some integer $k \geq 1$, then Theorem 3.2 declares $\pi$ forcibly $t$-tough precisely if Theorem 3.1 declares $\pi$ forcibly $1/k$-tough.

We now turn to another simple condition for $\pi$ to be forcibly $t$-tough. In contrast to Theorem 3.2, this second condition is not limited to $t \leq 1$. Let $\kappa(G)$ and $\alpha(G)$ denote respectively the vertex connectivity and independence number of $G$. Since $|X| \geq \kappa(G)$ and $\omega(G - X) \leq \alpha(G)$ for any cutset $X \subseteq V(G)$, we have at once

**Proposition 3.3.** For any graph $G$, $\tau(G) \geq \kappa(G)/\alpha(G)$.

Note that if $G$ is guaranteed to be 1-tough by Proposition 3.3, then $\kappa(G) \geq \alpha(G)$, and hence $G$ is hamiltonian by a well-known theorem of Chvátal and Erdős [6].

Proposition 3.3 can also be used to obtain a lower bound for $\tau(G)$ given only the vertex degrees of $G$. Theorem 1.2 provides a best monotone lower bound for $\kappa(G)$ in terms of the vertex degrees of $G$. And the following result gives a similar best monotone upper bound for $\alpha(G)$.

**Proposition 3.4.** Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence with $n \geq 2$, and let $1 \leq a \leq n - 1$. If $d_{a+1} \geq n - a$, then every realization $G$ of $\pi$ satisfies $\alpha(G) \leq a$.

**Proof:** Suppose $\pi$ has a realization $G$ with $\alpha(G) \geq a + 1$. Let $A = \{v_1, \ldots, v_{a+1}\}$ be an independent set in $G$ with $d(v_1) \leq \cdots \leq d(v_{a+1})$. Since $v_{a+1}$ is not adjacent to itself or to any of $v_1, \ldots, v_a$, we have $d_{a+1} \leq d(a_{a+1}) \leq n - (a + 1)$, a contradiction.

Clearly, the bound in Proposition 3.4 is monotone. Moreover, if $\pi$ does not satisfy the bound $d_{a+1} \geq n - a$ for some $a$, $1 \leq a \leq n - 1$, then $\pi$ is majorized by
\[ \pi' = (n - a - 1)^{g+1} (n - 1)^{n-a-1}, \]

which has a realization \( K_{n+1} + K_{n-a-1} \) with independence number \( a + 1 \).

Combining Theorem 1.2 and Proposition 3.4 with Proposition 3.3 immediately gives the following.

**Theorem 3.5.** Let \( \pi = (d_1 \leq \cdots \leq d_n) \) be a graphical sequence with \( n \geq 2 \), and let \( p, q \) be integers with \( 1 \leq p, q \leq n - 1 \). If
\[ (1) \quad d_i \leq i + p - 2 \implies d_{n-p+1} \geq n - i, \quad \text{for } 1 \leq i \leq \frac{1}{2} (n - p + 1), \]
and
\[ (2) \quad d_{q+1} \geq n - q, \]

then \( \pi \) is forcibly \( p/q \)-tough.

Since Theorem 2.1 is best monotone for \( t \geq 1 \), the degree bound of Theorem 2.1 is at least as good as the degree bound in Theorem 3.5. But for \( t < 1 \) the bounds in Theorem 3.2 and Theorem 3.5 are incomparable. To show this, we give degree sequences \( \pi \) where Theorem 3.2 declares \( \pi \) to be \( t \)-tough, but Theorem 3.5 does not, and sequences where the reverse occurs.

Consider first the Nash-Williams sequence \( \pi = \left(\frac{1}{2} (n - 1)\right)^n \) for \( n \equiv 1 \pmod{4}, n \geq 9 \). It is immediate that Theorem 3.2 will declare \( \pi \) to be forcibly \( 1/2 \)-tough.

But the largest \( p \) for which condition (1) of Theorem 3.5 holds is \( p = 1 \), whereas the smallest \( q \) for which (2) holds is \( q = \frac{1}{2} (n + 1) \). Thus Theorem 3.5 cannot declare \( \pi \) to be forcibly more than \((2/(n+1))\)-tough.

On the other hand, let \( d \geq 3 \) and \( n = m \cdot d \) for some \( m \geq 2 \), and consider \( \pi = (n/d)^{n-n/d} (n - 1)^{n/d} \). The largest \( p \) for which condition (1) of Theorem 3.5 holds is \( p = n/d \), and the smallest \( q \) for which (2) holds is \( q = n - n/d \). So Theorem 3.5 guarantees \( \pi \) to be forcibly \((1/(d - 1))\)-tough. But condition (1) of Theorem 3.2 for \( t = 1/(d - 1) \) fails at \( i = n/d + (d - 2) \).

It was noted in [1] that the optimal upper bound for \( \alpha(G) \) (and not just a weakly-optimal bound as in Proposition 3.4) in terms of the degree sequence \( \pi = (d_1 \leq \cdots \leq d_n) \) of \( G \) is tractable. In particular, using Theorem 3.6 below, it is easy to determine the largest integer \( k \) such that \( \pi \) has a realization \( G \) with \( \alpha(G) = k \).

**Theorem 3.6 ([10]).** A degree sequence \( \pi \) has a realization \( G \) with \( \alpha(G) \geq k \) if and only if \( \pi \) has a realization in which some \( k \) vertices with the smallest degrees form an independent set.

Iteratively consider the integers \( k = 2, 3, 4, \ldots, n \). To decide if \( \pi \) has a realization with \( k \) independent vertices, form the graph \( H = \overline{K_k} + K_{n-k} \), and let \( v_1, \ldots, v_k \) (resp., \( v_{k+1}, \ldots, v_n \)) denote the vertices of \( K_k \) (resp., \( K_{n-k} \)). Assign \( d_i \) to \( v_i \), for \( i = 1, 2, \ldots, n \), and determine if \( H \) contains a subgraph \( H' \) with the assigned degrees. If so, then \( \pi \) has a realization with \( k \) independent vertices. Otherwise, by Theorem 3.6 \( \pi \) has no realization with \( k \) independent vertices. It is well-known [11] that deciding whether \( H' \) exists is a tractable problem.
Combining, using Proposition 3.3, the best monotone lower bound for \( \kappa(G) \) (from Theorem 1.2) and the optimal bound for \( \alpha(G) \) (from the procedure in the previous paragraph), we get a tractable lower bound for \( \tau(G) \) in terms of the vertex degrees of \( G \). In the sequel, we will call this lower bound the improved degree bound of Proposition 3.3.

Since the optimal upper bound for \( \alpha(G) \) is not monotone, the improved degree bound of Proposition 3.3 is not monotone either. This suggests the intriguing possibility that for \( t \geq 1 \), the improved degree bound of Proposition 3.3 might occasionally be better than the bound in Theorem 2.1. We conjecture, however, that this cannot occur.

**Conjecture 3.7.** Let \( t \geq 1 \). For any graphical sequence \( \pi \), if the improved degree bound of Proposition 3.3 declares \( \pi \) to be forcibly \( t \)-tough, then Theorem 2.1 will also declare \( \pi \) to be forcibly \( t \)-tough.

On the other hand, for \( t < 1 \) the same degree sequences that showed that Theorem 3.2 and Theorem 3.3 are incomparable, also show that Theorem 3.2 and the improved degree bound of Proposition 3.3 are incomparable.

As before, for the Nash-Williams sequence \( \pi = \left( \frac{1}{2}(n-1) \right)^n, n \equiv 1 \mod 4, n \geq 9 \), Theorem 3.2 will declare \( \pi \) to be forcibly 1/2-tough. But Theorem 1.2 cannot declare \( \pi \) to be forcibly 2-connected, and several realizations \( G \) of \( \pi \) will satisfy \( \alpha(G) \geq 3 \). Thus the improved degree bound of Proposition 3.3 cannot declare \( \pi \) to be forcibly more than 1/3-tough.

On the other hand, consider \( \pi = (n/d)^{n-n/d} (n-1)^{n/d}, for d \geq 3 and n = md \) for some \( m \geq 2 \), again. Theorem 1.2 will guarantee \( \pi \) is forcibly \( n/d \)-connected, and the unique realization \( G = K_{n/d} + K_{n-n/d} \) satisfies \( \alpha(G) = n - n/d \). Thus the degree bound in Proposition 3.3 guarantees \( \pi \) to be forcibly 1/(\( d-1 \))-tough. But Theorem 3.2 for \( t = 1/(d-1) \) fails at \( i = n/d + (d-2) \), since \( i < \min \{n - n/d, \frac{1}{2}(n + (d-1) - 1)\} \).

### 4 Best Minimum Degree Condition for Toughness

Using Theorems 2.1 and 3.2, we obtain the following condition on the minimum degree \( \delta(G) \) to guarantee \( G \) to be \( t \)-tough. The bound for \( t \geq 1 \) has been proved before [2].

**Theorem 4.1.** Let \( t > 0 \) and let \( G \) be a graph on \( n \) vertices. If

\[
\delta(G) \geq \begin{cases} 
\frac{tn}{(t+1)}, & \text{for } t \geq 1; \\
\frac{1}{2}n, & \text{for } \frac{1}{2} < t \leq 1; \\
\frac{1}{2}(n-1), & \text{for } 0 < t \leq \frac{1}{2},
\end{cases}
\]

then \( G \) is \( t \)-tough, and the bound is best possible.
Proof: The sufficiency when \( t \geq 1 \) follows from Theorem 2.1 since \((st)\) is vacuously satisfied if \( \delta(G) \geq tn/(t+1) \). For \( \frac{1}{2} < t \leq 1 \), the sufficiency follows from Dirac’s hamiltonian theorem [7]. (This says that if \( \delta(G) \geq \frac{1}{2}n \) for \( n \geq 3 \), then \( G \) is hamiltonian. It is obvious that a hamiltonian graph is 1-tough.) Finally, the sufficiency when \( 0 < t \leq \frac{1}{2} \) follows vacuously from Theorem 3.2, since for \( i < \frac{1}{2}(n + \lfloor 1/t \rfloor - 1) \), the antecedent \( d_i \leq i - (\lfloor 1/t \rfloor - 1) \) implies

\[
d_i < \frac{1}{2}(n + \lfloor 1/t \rfloor - 1) - (\lfloor i/t \rfloor - 1) = \frac{1}{2}(n - \lfloor 1/t \rfloor + 1) \leq \frac{1}{2}(n - 1),
\]

which is false. Thus the antecedent \( d_i \leq i - (\lfloor 1/t \rfloor - 1) \) must be false.

We next show the condition is best possible in the same sense as Dirac’s hamiltonian theorem [7]. For \( 0 < t \leq \frac{1}{2} \), let \( n \) be even and consider \( G = 2K_{n/2} \). Then \( \tau(G) = 0 \) (\( G \) is disconnected), while \( \delta(G) = \frac{1}{2}(n - 2) \). For \( \frac{1}{2} < t \leq 1 \), let \( n \) be odd and consider \( G = K_1 + 2K_{(n-1)/2} \). Then \( \tau(G) = \frac{1}{2} \), while \( \delta(G) = \frac{1}{2}(n - 1) \). Finally, for \( t \geq 1 \) consider the graph \( G = K_{\lfloor tn/(t+1) \rfloor - 1} + (n - \lfloor tn/(t+1) \rfloor + 1)K_1 \). Setting \( X = K_{\lfloor tn/(t+1) \rfloor - 1} \), we find

\[
\tau(G) = \frac{|X|}{\omega(G-X)} = \frac{\lfloor tn/(t+1) \rfloor - 1}{n - \lfloor tn/(t+1) \rfloor + 1} < \frac{tn/(t+1)}{n - (tn/(t+1))} = t,
\]

while \( \delta(G) = \lfloor tn/(t+1) \rfloor - 1 \).

References


