A Two-Dimensional Analysis for the Coupling of Magnetic Gears

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Abstract— **A formula is presented for computing the coupling between magnetic gears. This formula, which is based on two-dimensional analytical analysis, is expressed as a finite sum of elementary functions and is well suited for parametric analysis. It is demonstrated via application to a practical device and verified using finite element analysis (FEA).**

***Index Terms*—Magnetic analysis, magnetic coupling, magnetic fields, magnetic gear, permanent magnetic.**

I. Introduction

MAGNETIC gears can be used in place of mechanical gears to reduce undesired vibrations and for applications that require torque coupling between separated members. These devices consist of two separated radially polarized cylindrical magnets constrained to rotate about their respective axes (Fig. 1). The magnets are magnetically coupled to one another, and when one of the magnets is rotated, it imparts a torque to the second magnet causing it to rotate. The coupling between the magnets is a function of several variables including their number of poles, material properties, dimensions, and separation. Furthermore, substantial torque can be realized if modern rare-earth materials such as NdFeB are used.

Various numerical techniques such as finite element analysis (FEA) can be used to design gear mechanisms. However, they tend to be awkward for the kind of parametric analysis that is often desired. In this article, a two-dimensional (2–D) formula is derived for computing the coupling between magnetic gears. This formula, which is developed using analytical methods, is based on the assumptions that the magnets are ideal with a second quadrant demagnetization relation of the form

$$\vec{B} = \mu_0(\vec{H} + \vec{M}_s) \tag{1}$$

and that the polarization is uniform in the radial direction

$$\vec{M}_s = \pm M_s \ \hat{r}$$

and that there are no other materials present that perturb or contribute to the magnetic field (e.g., the magnets are in free space). There are numerous rare earth materials for which (1) applies including NdFeB. An important feature of this work is that the torque formula, which is expressed in terms of elementary functions, is readily programmed and ideal for parametric analysis. It is useful for the design and optimization of novel gear mechanisms. The theory is demonstrated via application to a practical device and verified by use of 2-D FEA. This paper follows earlier work on synchronous magnetic couplings [1]–[5].

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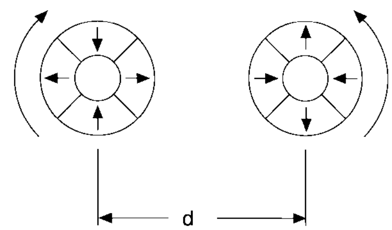

Fig. 1. Magnetic gear.

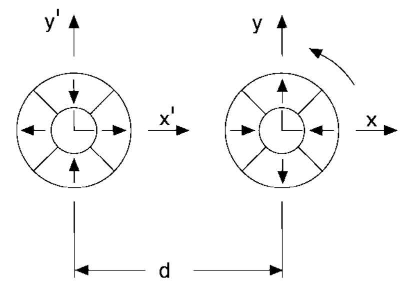

Fig. 2. Reference Frames.

II. Theory

The force on a particle with charge q moving in an external field $\vec{B}_{ext}$ is given by

$$\vec{F} = q\,\vec{v} \times \vec{B}_{ext} \tag{2}$$

which generalizes to

$$d\vec{F} = (\vec{J} \times \vec{B}_{ext})\,dV \tag{3}$$

where $\vec{J}$ is the current density vector

$$\vec{J} = \rho_v\,\vec{v} \tag{4}$$

and ρ_v is the charge per unit volume moving with velocity $\vec{v}$. It follows that the torque on this infinitesimal current is given by

$$d\vec{T} = \vec{r} \times (\vec{J} \times \vec{B}_{ext})\,dV \tag{5}$$

where $\vec{r}$ is a vector from the point about which the torque is to be computed.

Recall that a linear ($\mu_r = 1$) isotropic magnet with magnetization $\vec{M}$ can be represented as a distribution of equivalent volume and surface current densities

$$\vec{J}_M = \vec{\nabla} \times \vec{M}$$

and

$$\vec{j}_M = \vec{M} \times \hat{n}$$

respectively. It follows that the torque on a magnet in an external field is given by

$$\vec{T} = \int_V \vec{r} \times (\vec{J}_M \times \vec{B}_{ext})\, dV + \int_S \vec{r} \times (\vec{j}_M \times \vec{B}_{ext})\, da \tag{6}$$

where V and S define the volume and surface of the magnet, respectively [6]. Based on this equation, the derivation of the torque can be divided into two parts. First, an expression is obtained for the field of one of the magnets, which we denote as the source magnet. This field is taken to be $\vec{B}_{ext}$. Second, the second magnet, which we denote as the load magnet, is reduced to equivalent currents, and the torque is computed in accordance with (6). These two parts are treated in the following two sections, respectively. For the remainder of this article, we distinguish the parameters associated with the source and load magnets with superscripts s and l, respectively. Thus, for example, R_1^s and R_2^s refer to the inner and outer radii of the source magnet, whereas R_1^l and R_2^l refer to the inner and outer radii of the load magnet.

A. Field Analysis

A field solution for the source magnet in free space, ignoring the presence of the load magnet, has been developed by Lewis [7]. We give a brief summary of his results. We adopt coordinates (r', θ') centered on the axis of the source magnet (Fig. 2). The field strength is given by $\vec{H} = -\vec{\nabla}\Phi$, where Φ, the scalar potential, satisfies $\nabla^2\Phi = 4\pi\, \vec{\nabla} \cdot \vec{M}$ and the polarization $\vec{M}$ is in the radial direction with alternating polarity $\vec{M} = \pm M\,\hat{r}$. In two dimensions the solution of this equation is

$$\Phi(\vec{x'}) = \frac{1}{2\pi}\int \log|\vec{x'} - \vec{\xi}|\, \vec{\nabla} \cdot \vec{M}\, dS - \frac{1}{2\pi}\oint \log|\vec{x'} - \vec{\xi}|\, \vec{M} \cdot \hat{n}\, ds \tag{7}$$

where $\vec{\xi}$ is the variable of integration. The first integral, over the cross-sectional area of the magnet, represents the contribution of the effective magnetic charge density $\rho_M = -\vec{\nabla} \cdot \vec{M}$. The second, over the boundary of the magnet, represents the contribution of the effective magnetic surface charge density $\sigma_M = \vec{M} \cdot \hat{n}$, where $\hat{n}$ is the outward normal.

Evaluation of (7) can be simplified by replacing the vectors $\vec{x} = (x, y)$ and $\vec{\xi} = (\xi, \eta)$ by the complex variables $z = x' + iy' = r' e^{i\theta'}$ and $\zeta = \xi + i\eta = \rho e^{i\phi}$, respectively, and introducing the complex potential $F(z) = \Phi + i\Psi$. Outside the magnet, the magnetization is zero, so Φ satisfies $\nabla^2\Phi = 0$. $F(z)$ is, therefore, uniquely defined and analytic in this region. The magnetic field $\vec{H} = (H_x, H_y)$ is given by $H_x - iH_y = -F'(z)$. By inspection, $F(z)$ must be given by

$$F(z) = \frac{1}{2\pi}\int \log(z - \zeta)\, \vec{\nabla} \cdot \vec{M}\, dS - \frac{1}{2\pi}\oint \log(z - \zeta)\, \vec{M} \cdot \hat{n}\, ds \tag{8}$$

because (8) defines an analytic function whose real part satisfies (7). This solution is valid only at points z outside the magnet. For the current application, the primary interest is in calculating $F'(z)$, which is given by

$$F'(z) = \frac{1}{2\pi}\int \frac{\vec{\nabla} \cdot \vec{M}}{z - \zeta}\, dS - \frac{1}{2\pi}\oint \frac{\vec{M} \cdot \hat{n}}{z - \zeta}\, ds. \tag{9}$$

Once $F'(z)$ is evaluated, the potential $F(z)$ can be obtained, if desired, by integrating $F'(z)$ with respect to z.

Consider an arbitrary magnet segment covering the region $a \le \rho \le b$ and $\alpha \le \phi \le \beta$ with the magnetization pointing outward. By assumption, the magnetization at $\zeta = \rho e^{i\phi}$ is $\vec{M} = (M_s^s \cos\phi, M_s^s \sin\phi)$ where M_s^s is constant, so $\nabla \cdot \vec{M} = \frac{M_s^s}{\rho}$ and $\vec{M} \cdot \hat{n}$ is given by M_s^s, 0, $-M_s^s$, and 0 along the boundaries $r' = b$, $\theta' = \beta$, $r' = a$, and $\theta' = \alpha$, respectively. Equation (9) reduces to

$$F'(z) = \frac{1}{2\pi}\int_a^b \int_\alpha^\beta \frac{M_s^s/\rho}{z - \rho e^{i\phi}}\, \rho\, d\phi\, d\rho - \frac{1}{2\pi}\left[\int_\alpha^\beta \frac{M_s^s}{z - \rho e^{i\phi}}\, \rho\, d\phi\right]_a^b. \tag{10}$$

Evaluation of these integrals yields

$$F'(z) = \frac{M_s^s}{2\pi i}\left[e^{-i\phi}\log\frac{z - be^{i\phi}}{z - ae^{i\phi}}\right]_\alpha^\beta. \tag{11}$$

It is necessary to take some care with complex logarithms in results such as this. Equation (11) is written so that if the branch cut of the function $\log z$ is taken, as usual, to lie along the negative real axis ($\log z = \log r' + i\theta'$ with $-\pi < \theta' \le \pi$), then $F'(z)$ will have branch cuts only along the boundaries of the magnet segment. In general, it should always be possible to rearrange solutions so that this is the case. Some more complicated examples are given in [7].

The radial and tangential components of the field are given by

$$H_{r'} - iH_{\theta'} = e^{i\theta'}(H_x - iH_y) = -e^{i\theta'}F'(z). \tag{12}$$

The magnetic field produced by one sector of the inner magnet can therefore be written in the form

$$H_{r'}(r', \theta') - iH_{\theta'}(r', \theta') = -\frac{M_s^s}{2\pi i}\left[e^{i(\theta' - \phi)}\frac{r' e^{i(\theta' - \varphi)} - b}{r' e^{i(\theta' - \varphi)} - a}\right]_\alpha^\beta. \tag{13}$$

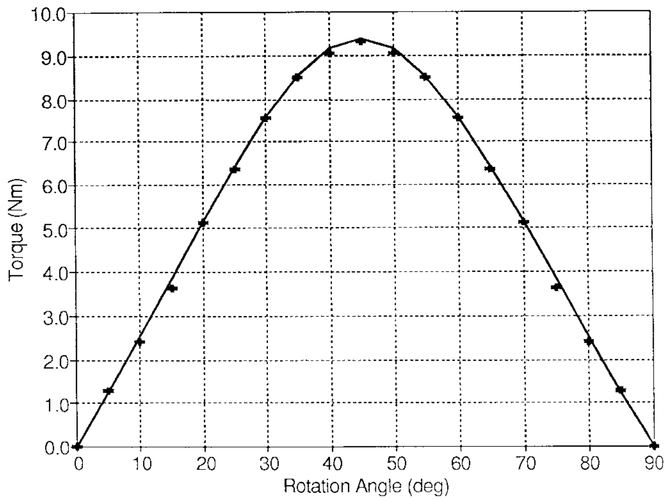

Fig. 3. Torque (per unit length) versus θ (+ = FEA).

The field produced by the entire magnet, consisting of N^s_{pole} sectors, is obtained by adding N^s_{pole} results of this form. If sector k is located between $\alpha = \theta'_{k-1}$ and $\beta = \theta'_k$, where

$$\theta'_k = \frac{(2k-1)\pi}{N^s_{pole}}$$

and if the odd-numbered poles ($k = 1, 3, \cdots, N^s_{pole} - 1$) point outward and the even-numbered poles ($k = 2, 4, \cdots, N^s_{pole}$) point inward, then the total magnetic field can be found to be

$$\begin{aligned} &H_{r'}(r', \theta') + iH_{\theta'}(r', \theta') \\ &= \frac{M^s_s\, i}{\pi} \sum_{k=1}^{N^s_{pole}} (-1)^{k+1} e^{i(\theta'_k - \theta')} \log \frac{r' e^{i(\theta'_k - \theta')} - b}{r' e^{i(\theta'_k - \theta')} - a}. \end{aligned} \tag{14}$$

To compute the torque, it is convenient to express these components as functions of the coordinates (r, θ) centered on the axis of the load magnet. The relation between the two coordinate systems is

$$r' = \sqrt{r^2 + 2rd\cos(\theta) + d^2} \tag{15}$$

and

$$\theta' = \arctan\left(\frac{r\sin(\theta)}{r\cos(\theta) + d}\right). \tag{16}$$

Substituting (15) and (16) into (14), and making use of the relation $\vec{B} = \mu_0 \vec{H}$ yields (17) and (18) shown at the bottom of the page. These are labeled as external field components because they are due to the source magnet. The Cartesian components, which are common to both coordinate systems, are given by

$$\begin{aligned} B^{ext}_x(r, \theta) =& B^{ext}_{r'}(r, \theta) \cos\left[\arctan\left(\frac{r\sin(\theta)}{r\cos(\theta) + d}\right)\right] \\ &- B^{ext}_{\theta'}(r, \theta) \sin\left[\arctan\left(\frac{r\sin(\theta)}{r\cos(\theta) + d}\right)\right] \end{aligned} \tag{19}$$

and

$$\begin{aligned} B^{ext}_y(r, \theta) =& B^{ext}_{r'}(r, \theta) \sin\left[\arctan\left(\frac{r\sin(\theta)}{r\cos(\theta) + d}\right)\right] \\ &+ B^{ext}_{\theta'}(r, \theta) \cos\left[\arctan\left(\frac{r\sin(\theta)}{r\cos(\theta) + d}\right)\right]. \end{aligned} \tag{20}$$

These expressions are used to compute the torque.

B. Torque Analysis

As indicated above, the torque can be evaluated using

$$\begin{aligned} \vec{T} =& \int_V \vec{r}(r, \theta) \times (\vec{J}_M(r, \theta) \times \vec{B}_{ext}(r, \theta))\, r\, dr\, d\theta \\ &+ \int_S \vec{r}(r, \theta) \times (\vec{j}_M(r, \theta) \times \vec{B}_{ext}(r, \theta))\, da \end{aligned} \tag{21}$$

where the current densities in this expression refer to the load magnet. For this magnet $\vec{J}_M = \nabla \times \vec{M} = 0$ and, therefore, the first term in (21) is zero. As for the second term, there are N^l_{pole} sectors to consider, and each sector has two surfaces with current densities that contribute to the torque. These surfaces constitute the radial sides of the sector at angular positions θ_1 and θ_2, respectively. If the magnet is rotated by an angle θ, then the surface current densities for the pth sector

$$\begin{aligned} B^{ext}_{r'}(r, \theta) = \mu_0 \mathrm{Re}\Bigg\{ &\frac{M^s_s\, i}{\pi} \sum_{k=1}^{N^s_{pole}} (-1)^{k+1} \exp\left[i\left(\frac{(2k-1)\pi}{N^s_{pole}} - \arctan\left(\frac{r\sin(\theta)}{r\cos(\theta)+d}\right)\right)\right] \\ &\times \log\left[\frac{\sqrt{r^2 + 2rd\cos(\theta) + d^2}\exp\left[i\left(\frac{(2k-1)\pi}{N^s_{pole}} - \arctan\left(\frac{r\sin(\theta)}{r\cos(\theta)+d}\right)\right)\right] - b}{\sqrt{r^2 + 2rd\cos(\theta) + d^2}\exp\left[i\left(\frac{(2k-1)\pi}{N^s_{pole}} - \arctan\left(\frac{r\sin(\theta)}{r\cos(\theta)+d}\right)\right)\right] - a}\right]\Bigg\} \end{aligned} \tag{17}$$

$$\begin{aligned} B^{ext}_{\theta'}(r, \theta) = \mu_0 \mathrm{Im}\Bigg\{ &\frac{M^s_s\, i}{\pi} \sum_{k=1}^{N^s_{pole}} (-1)^{k+1} \exp\left[i\left(\frac{(2k-1)\pi}{N^s_{pole}} - \arctan\left(\frac{r\sin(\theta)}{r\cos(\theta)+d}\right)\right)\right] \\ &\times \log\left[\frac{\sqrt{r^2 + 2rd\cos(\theta) + d^2}\exp\left[i\left(\frac{(2k-1)\pi}{N^s_{pole}} - \arctan\left(\frac{r\sin(\theta)}{r\cos(\theta)+d}\right)\right)\right] - b}{\sqrt{r^2 + 2rd\cos(\theta) + d^2}\exp\left[i\left(\frac{(2k-1)\pi}{N^s_{pole}} - \arctan\left(\frac{r\sin(\theta)}{r\cos(\theta)+d}\right)\right)\right] - a}\right]\Bigg\} \end{aligned} \tag{18}$$

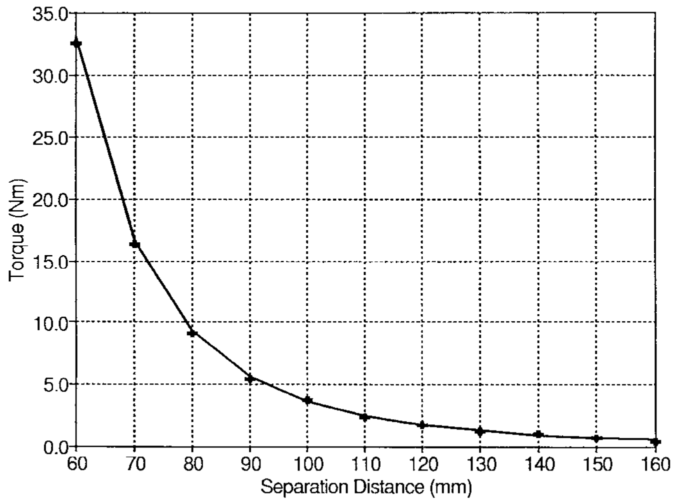

Fig. 4. Torque (per unit length) versus $d(+ = \text{FEA})$.

are given by (22), shown at the bottom of the page, where $p = 0, 1, 2, \cdots, N^l_{\text{pole}} - 1$. Notice that $\theta = 0$ occurs when the middle of the zeroth sector coincides with the x axis and its polarization is radially outward. It it is assumed that the source magnet is held stationary with one of its sectors in a similar orientation, and, therefore, θ reflects the angular offset of the two magnets.

Now, substituting (22) into (21) and simplifying yields

$$T(\theta) = 2\, M^l_s L \sum_{p=0}^{N^l_{pole}-1} (-1)^p \times \int_{R^l_1}^{R^l_2} r \left[\cos(\theta_{edge}(p))\, B^{ext}_x(r, \theta_{edge}(p)) + \sin(\theta_{edge}(p))\, B^{ext}_y(r, \theta_{edge}(p))\right] dr \quad (23)$$

where L is the length of the coupling

$$\theta_{edge}(p) = \theta + \frac{\pi}{N^l_{pole}} (1 + 2p) \quad (24)$$

and $B^{ext}_x(r, \theta_{edge}(p))$ and $B^{ext}_y(r, \theta_{edge}(p))$ are given by (19) and (20). The coefficient $2M^l_s$ takes into account the fact that there are two surfaces at the interface between neighboring sectors (one for each sector). The integral in (23) can be evaluated numerically using Simpson's method. The resulting torque formula is

$$T(\theta) = \frac{2\, M^{out}\, L\, \tilde{R}}{N_r} \sum_{p=0}^{N^l_{pole}-1} \sum_{q=0}^{N_r} (-1)^p S_r(q)\, r(q) \times \left(\cos(\theta_{edge}(p))\, B^{ext}_x(r(q), \theta_{edge}(p)) + \sin(\theta_{edge}(p))\, B^{ext}_y(r(q), \theta_{edge}(p))\right) \quad (25)$$

where $\tilde{R} = R^l_2 - R^l_1$, $S_r(q)$ are the Simpson coefficient terms

$$S_r(q) = \begin{cases} \frac{1}{3}, & (q = 0) \\ \frac{4}{3}, & (q = 1, 3, 5 \cdots) \\ \frac{2}{3}, & (q = 2, 4, 6 \cdots) \\ \frac{1}{3}, & (q = N_r) \end{cases} \quad (26)$$

and the integration points are as follows:

$$r(q) = R^l_1 + \frac{q}{N_r} (R^l_2 - R^l_1) \qquad (q = 0, 1, 2, \cdots, N_r). \quad (27)$$

An application of (25) is described in the next section.

III. Results

Equation (25) was implemented in BASIC and applied to a hypothetical gear with the following parameters:

$$\begin{aligned} M^s_s &= 7.1613 \times 10^5 \text{ A / m} \\ M^l_s &= 7.1613 \times 10^5 \text{ A / m} \\ d &= 80 \text{ cm} \quad \text{(separation)} \quad (28) \\ R^s_1 &= 10 \text{ cm} \quad \text{(inner radius of source magnet)} \\ R^s_2 &= 20 \text{ cm} \quad \text{(outer radius of source magnet)} \\ R^l_1 &= 15 \text{ cm} \quad \text{(inner radius of load magnet)} \\ R^l_2 &= 30 \text{ cm} \quad \text{(outer radius of load magnet)} \\ N^s_{pole} &= 4 \quad \text{(source magnet)} \\ N^l_{pole} &= 4 \quad \text{(load magnet).} \quad (29) \end{aligned}$$

The values used for M^s_s and M^l_s are characteristic of sintered NdFeB material. The torque per unit length was computed with the source magnet fixed and the load magnet rotated through a series of angular values $\theta = 0°, 5°, 10°, \cdots, 90°$. The analysis was performed with the integration parameter $N_r = 20$, since there was no improvement in accuracy above this value. It took approximately 2 s to compute 19 torque values on a 120 MHz Pentium computer (Fig. 3). Notice that the peak torque occurs at θ =45° and 135° when load magnet is, in effect, rotated half the angular span of a single a pole from its initial orientation. This data was checked by Knewston using the Maxwell 2-D Field Simulator which is an FEA based program from Ansoft Corp. [8]. This program uses the Local Jacobian method for computing the torque. The FEA model consisted of 2500 triangular elements, and it took approximately 2 min to compute each torque value using a 200 MHz Pentium computer (36 min for the entire analysis not including model development and setup).

An analysis was also performed to determine the decrease of the peak torque as a function of separation distance for a series of values $d = 60, 70, \cdots, 160$ mm. Again, the integration parameter N_r was set to 20, and it took less than 1 s to compute the 11 torque values. This data is compared to the corresponding FEA data in Fig. 4.

$$\vec{j}_M(p, r, \theta) = \begin{cases} -M^l_s\, \hat{z} & \begin{cases} R^d_1 \le r \le R^d_2 \\ \theta_2(p) = \theta + \frac{\pi}{N_{pole}} (1 + 2p) \end{cases} \\ M^l_s\, \hat{z} & \begin{cases} R^d_1 \le r \le R^d_2 \\ \theta_1(p) = \theta - \frac{\pi}{N_{pole}} (1 + 2p) \end{cases} \end{cases} \quad (22)$$

IV. Conclusion

A formula has been derived for computing the coupling between magnetic gears. This formula is readily programmed and is ideal for parametric analysis. It should be of considerable use in the design and optimization of novel gear devices.

Acknowledgment

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References

[1] E. P. Furlani, "Formulas for the force and torque of axial couplings," *IEEE Trans. Magn.*, vol. 29, no. 5, pp. 2295–2301, 1993.
[2] R. Wang, E. P. Furlani, and Z. J. Cendes, "Design and analysis of a permanent magnet axial coupling using 3D finite element field computations," *IEEE Trans. Magn.*, vol. 30, no. 4, pp. 2292–2295, 1994.
[3] E. P. Furlani, and T. S. Lewis, "A two dimensional model for the torque of radial couplings," *Int. J. Appl. Elect. Mech.*, vol. 6, pp. 187–196, 1995.
[4] E. P. Furlani, R. Wang, and H. Kusnadi, "A three-dimensional model for computing the torque of radial couplings," *IEEE Trans. Magn.*, vol. 31, no. 5, pp. 2522–2526, 1995.
[5] E. P. Furlani, "Analysis and optimization of synchronous magnetic couplings," *J. Appl. Phys.*, vol. 79, no 8, pp. 4692–4694, 1996.
[6] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. New York: Wiley, 1975.
[7] J. K. Lee and T. S. Lewis, "Epoxy-bonded Nd-Fe-B magnetic rollers: experimental and theoretical analysis," *J. Appl. Phys.*, vol. 70, no 10, pp. 6614–6617, 1991.
[8] MAXWELL, Ansoft Corp., Pittsburgh, PA., 1996