ON THE EXISTENCE OF DISJOINT SPANNING PATHS IN FAULTY HYPERCUBES

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Assume that \( n \) is a positive integer with \( n \geq 4 \) and \( F \) is a subset of the edges of the hypercube \( Q_n \) with \( |F| \leq n - 4 \). Let \( u, x \) be two distinct white vertices of \( Q_n \) and \( v, y \) be two distinct black vertices of \( Q_n \), where black and white refer to the two parts of the bipartition of \( Q_n \). Let \( l_1 \) and \( l_2 \) be odd integers, where \( l_1 \geq d_{Q_n}(u,v) \), \( l_2 \geq d_{Q_n}(x,y) \), and \( l_1 + l_2 = 2^n - 2 \). Moreover, let \( l_3 \) and \( l_4 \) be even integers, where \( l_3 \geq d_{Q_n}(u,x) \), \( l_4 \geq d_{Q_n}(v,y) \), and \( l_3 + l_4 = 2^n - 2 \). In this paper, we prove that there are two disjoint paths \( P_1 \) and \( P_2 \) such that (1) \( P_1 \) is a path joining \( u \) to \( v \) with length \( l(P_1) = l_1 \), (2) \( P_2 \) is a path joining \( x \) to \( y \) with length \( l(P_2) = l_2 \), and (3) \( P_1 \cup P_2 \) spans \( Q_n - F \). Moreover, there are two disjoint paths \( P_3 \) and \( P_4 \) such that (1) \( P_3 \) is a path joining \( u \) to \( x \) with length \( l(P_3) = l_3 \), (2) \( P_4 \) is a path joining \( v \) to \( y \) with length \( l(P_4) = l_4 \), and (3) \( P_3 \cup P_4 \) spans \( Q_n - F \).
1. Introduction

In this paper, a network is represented as a loopless undirected graph. For the graph definition and notation, we follow Bondy and Murty\(^1\) and Hsu et al.\(^5\)

The \(G = (V, E)\) is a graph if \(V\) is a finite set and \(E\) is a subset of \(\{(a, b) \mid (a, b)\text{ is an unordered pair of } V\}\). We say that \(V\) is the vertex set and \(E\) is the edge set. Two vertices \(u\) and \(v\) are adjacent if \((u, v) \in E\). A graph \(G = (V_0 \cup V_1, E)\) is bipartite if \(V(G)\) is the union of two disjoint sets \(V_0\) and \(V_1\) such that every edge joins a vertex of \(V_0\) and a vertex of \(V_1\). A path is a sequence of adjacent vertices, written as \(\langle v_0, v_1, \ldots, v_m \rangle\), in which all the vertices \(v_0, v_1, \ldots, v_m\) are distinct except possibly \(v_0 = v_m\). A path can also be written as \(\langle v_0, P, v_m \rangle\), where \(P = \langle v_0, v_1, \ldots, v_m \rangle\), and we use \(P^{-1}\) to denote the inverse path \(\langle v_m, v_m-1, \ldots, v_0 \rangle\). The length of a path \(P\), denoted by \(l(P)\), is the number of edges in \(P\). Let \(u\) and \(v\) be two vertices of \(G\). The distance between \(u\) and \(v\), denoted by \(d_G(u, v)\), is the length of the shortest path of \(G\) joining \(u\) and \(v\). The diameter of \(G\) is \(\max \{d_G(u, v) \mid u, v \in V(G)\}\). A cycle is a path with at least three vertices such that the first vertex is the same as the last one. A Hamiltonian cycle is a cycle of length \(|V(G)|\), and a Hamiltonian path is a path of length \(|V(G)| - 1\). A graph is Hamiltonian if it has a Hamiltonian cycle. It is easy to check that a Hamiltonian bipartite graph \(G = (V_0 \cup V_1, E)\) satisfies \(|V_0| = |V_1| \geq 2\).

Let \(u = u_{n-1}u_{n-2} \ldots u_1u_0\) be an \(n\)-bit binary string. We use \((u)_i\) to denote \(u_i\) for \(0 \leq i \leq n - 1\). The Hamming weight \(w(u)\) of \(u\) is \(|\{i \mid (u)_i = 1 \text{ for } 0 \leq i \leq n - 1\}|\). Let \(u\) and \(v\) be two \(n\)-bit binary strings. The Hamming distance \(h(u, v)\) between \(u\) and \(v\) is the number of different bits in the corresponding strings. The \(n\)-dimensional hypercube, denoted by \(Q_n\), consists of all \(n\)-bit binary strings as its vertices, and two vertices \(u\) and \(v\) are adjacent if and only if \(h(u, v) = 1\). The hypercube is one of the most popular interconnection networks for parallel computer/communication system, see Leighton.\(^9\) This is partly due to its attractive properties such as regularity, recursive structure, vertex and edge symmetry, maximum connectivity, as well as effective routing and broadcasting algorithm.

It is known that \(d_{Q_n}(u, v) = h(u, v)\). Moreover, \(Q_n\) is a bipartite graph with bipartition \(V_0 = \{u \mid w(u)\text{ is even}\}\) and \(V_1 = \{u \mid w(u)\text{ is odd}\}\). A vertex \(u\) of \(Q_n\) is white if \(w(u)\) is even, otherwise \(u\) is black. For \(0 \leq k \leq n - 1\), we use \(u^k\) to denote the vertex \(v\) with \((v)_k = 1 - (u)_k\) and \((v)_i = (u)_i\) if \(i \neq k\). An edge is a \(k\)-dimensional edge if it joins a vertex \(u\) to \(u^k\). We use \(E_k\) to denote the set of all \(k\)-dimensional edges of \(Q_n\). For \(i = 0, 1\), let \(Q_{n-1}^i\) denote the subgraph of \(Q_n\) induced by \(\{u \mid (u)_0 = i\}\). Obviously, \(Q_{n-1}^i\) is isomorphic to \(Q_{n-1}\).

Assume that \(n\) is any positive integer with \(n \geq 2\). Let \(u\) and \(x\) (respectively, \(v\) and \(y\)) be two distinct white (respectively, black) vertices of \(Q_n\). It has been proven

\[(2) P_3\text{ is a path joining } v \text{ to } y \text{ with } l(P_3) = l_3, \text{ and (3) } P_3 \cup P_4 \text{ spans } Q_n - F \text{ except the following cases: (a) } l_3 = 2 \text{ with } d_{Q_n - F}(u, x) = 2 \text{ and } d_{Q_n - F - \{v, y\}}(u, x) > 2, \text{ and (b) } l_4 = 2 \text{ with } d_{Q_n - F}(v, y) = 2 \text{ and } d_{Q_n - F - \{u, x\}}(v, y) > 2.\]
that there are two disjoint paths $P_1$ and $P_2$ such that (1) $P_1$ is a path joining $u$ to $v$, (2) $P_2$ is a path joining $x$ to $y$, and (3) $P_1 \cup P_2$ spans $Q_n$. This property has been used in many applications of hypercubes, see Hsu et al.\cite{8} and Lin et al.\cite{9} Recently, Lee et al.\cite{10} further studied the possible paths of joining $u$ to $x$ and $v$ to $y$, and they obtained the following result:

**Theorem 1.1.** (Lee et al.\cite{10}) Assume that $n \geq 4$. Let $u$ and $x$ (respectively, $v$ and $y$) be two distinct white (respectively, black) vertices of $Q_n$. Let $l_1$ and $l_2$ be odd integers, where $l_1 \geq h(u, v)$, $l_2 \geq h(x, y)$, and $l_1 + l_2 = 2^n - 2$. Moreover, let $l_3$ and $l_4$ be even integers, where $l_3 \geq h(u, x)$, $l_4 \geq h(v, y)$, and $l_3 + l_4 = 2^n - 2$ except for the case where $\{l_3, l_4\} = \{2, 2^n - 4\}$ and $\{u, x, v, y\}$ induces a cycle of length 4. Then there are two disjoint paths $P_1$ and $P_2$ such that (1) $P_1$ is a path joining $u$ to $v$ with $l(P_1) = l_1$, (2) $P_2$ is a path joining $x$ to $y$ with $l(P_2) = l_2$, and (3) $P_1 \cup P_2$ spans $Q_n$. Furthermore, there are two disjoint paths $P_3$ and $P_4$ such that (1) $P_3$ is a path joining $u$ to $x$ with $l(P_3) = l_3$, (2) $P_4$ is a path joining $v$ to $y$ with $l(P_4) = l_4$, and (3) $P_3 \cup P_4$ spans $Q_n$.

In this paper, we prove that the above theorem can be extended even with edge faults. Let $u$ and $x$ (respectively, $v$ and $y$) be two distinct white (respectively, black) vertices of $Q_n$, and let $F$ be a subset of $E(Q_n)$. Then different $F$'s may result in different distances between two vertices in $Q_n - F$. Assume that $n \geq 4$ and $|F| \leq n - 4$. Let $l_1$ and $l_2$ be odd integers, where $l_1 \geq d_{Q_n - F}(u, v)$, $l_2 \geq d_{Q_n - F}(x, y)$, and $l_1 + l_2 = 2^n - 2$. Moreover, let $l_3$ and $l_4$ be even integers, where $l_3 \geq d_{Q_n - F}(u, x)$, $l_4 \geq d_{Q_n - F}(v, y)$, and $l_3 + l_4 = 2^n - 2$. We will prove that there are two disjoint paths $P_1$ and $P_2$ such that (1) $P_1$ is a path joining $u$ to $v$ with $l(P_1) = l_1$, (2) $P_2$ is a path joining $x$ to $y$ with $l(P_2) = l_2$, and (3) $P_1 \cup P_2$ spans $Q_n - F$. Furthermore, there are two disjoint paths $P_3$ and $P_4$ such that (1) $P_3$ is a path joining $u$ to $x$ with $l(P_3) = l_3$, (2) $P_4$ is a path joining $v$ to $y$ with $l(P_4) = l_4$, and (3) $P_3 \cup P_4$ spans $Q_n - F$ except for the following cases: (a) $l_3 = 2$ and $d_{Q_n - F}(u, x) = 2$ and $d_{Q_n - F - \{v, x\}}(u, x) > 2$, and (b) $l_4 = 2$ and $d_{Q_n - F}(v, y) = 2$ and $d_{Q_n - F - \{u, y\}}(v, y) > 2$.

In the following section we review some properties of hypercubes. In Section 3 we prove the property mentioned above for faulty hypercubes, and we prove that the bound $(n - 4)$ is tight. In the final section we discuss the applications of this paper.

2. Some properties of hypercubes

A bipartite graph $G = (V_0 \cup V_1, E)$ is hamiltonian laceable if there exists a hamiltonian path joining any two vertices from different partite sets, i.e., one in $V_0$ and one in $V_1$. A hamiltonian laceable graph $G$ is $k$-edge fault-tolerant hamiltonian laceable if $G - F$ remains hamiltonian laceable for every $F \subseteq E(G)$ with $|F| \leq k$.

**Lemma 2.1.** (Tsai et al.\cite{11}) $Q_n$ is $(n - 2)$-edge fault-tolerant hamiltonian laceable if $n \geq 2$. 

A hamiltonian laceable graph \( G = (V_0 \cup V_1, E) \) is hyper-hamiltonian laceable if for any vertex \( v \in V_i, i = 0, 1 \), there is a hamiltonian path of \( G \setminus \{v\} \) between any two vertices of \( V_{1-i} \). A hyper-hamiltonian laceable graph \( G \) is \( k \)-edge fault-tolerant hyper-hamiltonian laceable if \( G - F \) remains hyper-hamiltonian laceable for every \( F \subseteq E(G) \) with \( |F| \leq k \).

**Lemma 2.2.** (Tsai et al.\textsuperscript{15}) \( Q_n \) is \((n - 3)\)-edge fault-tolerant hyper hamiltonian laceable if \( n \geq 3 \).

We can discuss the hamiltonian laceability for hypercubes that contain faulty vertices. However, this problem seems very difficult. It is conjectured that \( Q_n - F \) is hamiltonian if \( F \) consists of a faulty vertex set \( F_V \) and a faulty edge set \( F_e \) with \( |F_V \cap V_0| = |F_V \cap V_1| = r \) and \( r + |F_e| \leq n - 2 \), see Sun et al.\textsuperscript{14}. Sun et al.\textsuperscript{14} restricted the study by assuming that the faulty vertices are only on disjoint adjacent pairs.

**Lemma 2.3.** (Sun et al.\textsuperscript{14}) Let \( F \) be a subset of \( V(Q_n) \cup E(Q_n) \) such that \( F \) can be decomposed into two parts \( F_{av} \) and \( F_e \) where \( F_{av} \) is a union of fault-adjacent paths joining \( v \) and \( \hat{v} \) and \( F_e \) consists of \( f_e \) edges. Then \( Q_n - F \) is hamiltonian laceable if \( |F| = f_{av} + f_e \leq n - 2 \).

**Lemma 2.4.** (Sun et al.\textsuperscript{14}) Let \( F \) be a subset of \( V(Q_n) \cup E(Q_n) \) such that \( F \) can be decomposed into two parts \( F_{av} \) and \( F_e \) where \( F_{av} \) is a union of \( f_{av} \) disjoint adjacent pairs of \( Q_n \) and \( F_e \) consists of \( f_e \) edges. Then \( Q_n - F \) is hyper hamiltonian laceable if \( |F| = f_{av} + f_e \leq n - 3 \).

Assume that \( x \) and \( y \) are two different vertices in the \( n \)-dimensional hypercube. Saad and Shultz\textsuperscript{13} constructed \( n \) internally disjoint paths joining \( x \) and \( y \) by the following manner: Let \( \{\alpha_j\}_{i=0}^{h(x,y) - 1} \) be the increasing sequence of indices such that \( (x)_{\alpha_i} \neq (y)_{\alpha_i} \), and let \( \{\alpha_j\}_{j=h(x,y)}^{n-1} \) be the increasing sequence of indices such that \( (x)_{\alpha_j} = (y)_{\alpha_j} \). For \( 0 \leq i \leq h(x,y) - 1 \), we set \( P_i = (z_i,0 = x, z_i,1 = z_i^{0+i}, z_i,2 = z_i^{0+i+1}, \ldots, z_i,h(x,y) = z_i^{h(x,y)-1} = y) \) with the addition of superscripts performed under modulo \( h(x,y) \). For \( h(x,y) \leq j \leq n - 1 \), we set \( P_j = (x, x^{\alpha_j}, z_0^{\alpha_j}, z_0^{\alpha_j}, \ldots, z_0^{\alpha_j}, y^{\alpha_j}, y) \). For example, let \( n = 5, u = 00000, v = 01110 \). Then \( P_1 = (00000, 00100, 00110, 01110), P_2 = (00000, 00100, 01100, 01110), P_3 = (00000, 01000, 01010, 01110), P_4 = (00000, 00001, 00111, 01111, 01111), \) and \( P_5 = (00000, 10000, 10010, 10110, 11110, 01110) \).

**Theorem 2.1.** (Saad and Shultz\textsuperscript{13}) Let \( x \) and \( y \) be two different vertices of \( Q_n \). Then there exist \( n \) internally disjoint paths \( P_0, P_1, \ldots, P_{n-1} \) of \( Q_n \) joining \( x \) to \( y \), where \( l(P_i) = h(x,y) \) if \( 0 \leq i \leq h(x,y) - 1 \) and \( l(P_i) = h(x,y) + 2 \) if \( h(x,y) \leq i \leq n - 1 \). Furthermore, each \( P_i, 0 \leq i \leq n - 1, \) uses exactly one \( j \)-dimensional edge if \( (x)_j \neq (y)_j \).

**Lemma 2.5.** Suppose that \( n \geq 4 \). Let \( x \) and \( y \) be two different vertices of \( Q_n \), and let \( F \) be a subset of \( E(Q_n) \) with \( |F| \leq n - 4 \). Then \( d_{Q_n-F}(x,y) \) is either \( h(x,y) \)
or \( h(x, y) + 2 \). Furthermore, \( d_{Q_n - F}(x, y) = h(x, y) \) if \( h(x, y) \geq n - 3 \). Thus, the diameter of \( Q_n - F \) is \( n \).

**Proof.** It is trivial that \( d_{Q_n - F}(x, y) \geq h(x, y) \). Moreover, \( d_{Q_n - F}(x, y) = h(x, y) \) if there exists a path of length \( h(x, y) \) joining \( x \) to \( y \) in \( Q_n - F \), and \( d_{Q_n - F}(x, y) > h(x, y) \) otherwise. Since \( Q_n \) is a bipartite graph, \( d_{Q_n - F}(x, y) \) is either \( h(x, y) \) or \( h(x, y) + 2 \) following from Theorem 2.1. Suppose \( h(x, y) \geq n - 3 \). Since \( |F| \leq n - 4 \), by Theorem 2.1, there exists a path of length \( h(x, y) \) joining \( x \) to \( y \) in \( Q_n - F \), and \( d_{Q_n - F}(x, y) = h(x, y) \). Hence, the diameter of \( Q_n - F \) is \( n \).

**Lemma 2.6.** Suppose that \( n \geq 5 \). Let \( x, y, u, \) and \( v \) be four different vertices of \( Q_n \) and \( (x)_0 \neq (y)_0 \). Moreover, let \( F \) be a subset of \( E(Q_n) \) with \( |F| \leq n - 4 \) and \( F \cap E_0 \neq \emptyset \). Then there exists a path \( T \) in \( Q_n - F \) joining \( x \) to \( y \) such that \( l(T) = h(x, y) + 2 \) and \( \{u, v\} \cap V(T) = \emptyset \). Furthermore, there exists a shortest path \( P \) of \( Q_n - F \) joining \( x \) to \( y \) such that \( \{u, v\} \cap V(T) = \emptyset \) and \( |P| = 1 \).

**Proof.** By Theorem 2.1, there are \( n \) internally disjoint paths \( P_0, P_1, \ldots, P_{n-1} \) in \( Q_n \) joining \( x \) to \( y \). We consider the following cases.

**Case 1:** \( h(x, y) = 1 \). By Theorem 2.1, \( l(P_i) = 1 \) and \( l(P_i) = 2 \) for \( 1 \leq i \leq n - 1 \). Since \( |F| \leq n - 4 \), at least three \( P_i \) with \( 1 \leq i \leq n - 1 \) is in \( Q_n - F \). Thus, there exists a path \( T \) in \( Q_n - F \) joining \( x \) to \( y \) such that \( l(T) = 3 \) and \( \{u, v\} \cap V(T) = \emptyset \). Obviously, \( T \) contains exactly one 0-dimensional edge.

**Case 2:** \( h(x, y) = 2 \). Without loss of generality, assume that \( y = (x^0)^1 \). By Theorem 2.1, \( l(P_i) = 2 \) if \( i = 0, 1 \) and \( l(P_i) = 4 \) if otherwise. Since \( |F| = n - 4 \), at least one \( P_i \) is in \( Q_n - F - \{u, v\} \). Suppose that \( i > 1 \). Obviously, \( P_i \) is of length 4 joining \( x \) to \( y \) not passing \( u \) nor \( v \) that contains exactly one 0-dimensional edge. Thus, we assume that \( P_i \) is not in \( Q_n - F - \{u, v\} \). By the pigeon-hole principle, \( P_1 \) and \( P_2 \) are in \( Q_n - F \) and there are exactly two paths \( P_k \) and \( P_l \) in \( Q_n - F \) with \( 2 \leq k \neq l < n \), \( u \) is in \( P_k \), and \( v \) is in \( P_l \). Thus, \( u \in \{x^k, (x^k)^0, y^k\} \) and \( v \in \{x^l, (x^l)^0, y^l\} \).

Suppose that \( u = x^k \) or \( v = x^l \). Without loss of generality, we assume that \( u = x^k \). Obviously, \( (x, x^0, (x^k)^0, y^k, y) \) is a path of length 4 joining \( x \) to \( y \) not passing \( u \) nor \( v \). Suppose that \( u = y^k \) or \( v = y^l \). Without loss of generality, we assume that \( u = y^k \). Obviously, \( (x, x^k, (x^k)^0, x^0, y) \) is a path of length 4 joining \( x \) to \( y \) not passing \( u \) nor \( v \). Finally, we consider \( u = (x^k)^0 \) and \( v = (x^l)^0 \). Obviously, \( (x, x^k, (x^k)^1, x^1, y) \) is a path of length 4 joining \( x \) to \( y \) not passing \( u \) nor \( v \). By Lemma 2.5, there exists a shortest path of \( Q_n - F \) joining \( x \) to \( y \) not passing \( u \) nor \( v \) that contains exactly one 0-dimensional edge.

**Case 3:** \( h(x, y) = 3 \). Without loss of generality, assume that \( y = ((x^0)^1)^2 \). By Theorem 2.1, \( l(P_i) = 3 \) if \( i = 0, 1, 2 \) and \( l(P_i) = 5 \) otherwise. Since \( |F| \leq n - 4 \), at least one of \( P_i \) is in \( Q_n - F - \{u, v\} \). Suppose that \( r \geq 3 \). Obviously, \( P_r \) is a path of length 5 joining \( x \) to \( y \) not passing \( u \) nor \( v \) that contains exactly one 0-dimensional edge. Thus, we assume that \( r \leq 2 \). Obviously, \( P_r \) can be written as \( (x, x^t, y^j, y), \) where \( (x^t)^0 = y^j \) and \( \{i, j, k\} = \{0, 1, 2\} \). Let \( A = \{(x, x^t, (x^t)^0, x^t) \mid t \in \{0, 1, 2, \ldots, n - 1\} - \{i, k, 0\}\} \) and \( B = \{(y^j, (y^j)^t, y^t, y) \mid t \in \{0, 1, 2, \ldots, n - 1\} - \{i, k, 0\}\} \).
1} \{j, k, 0\}. Then \(A \cup B\) forms a set of \(2n - 6\) edge-disjoint paths of length 3. Since \(n \geq 5\), \((2n - 6) - |F| \geq 3\). There exists one path \(T'\) in \(A \cup B\) not passing \(u\) nor \(v\). Suppose that \(T'\) is of the form \((x, x^i, (x^i)^j, x^i)\). Then \((x, x^i, (x^i)^j, x^i, y^j, y)\) is a path \(T\) in \(Q_n - F\) such that \(l(T) = 5\) and \(\{u, v\} \cap V(T) = \emptyset\). Suppose that \(T'\) is of the form \((y^j, (y^j)^i, y^j, y)\). Then \((x, x^i, y^j, (y^j)^i, y^j, y)\) is a path \(T\) in \(Q_n - F\) such that \(l(T) = 5\) and \(\{u, v\} \cap V(T) = \emptyset\). Thus, there exists a path \(T\) in \(Q_n - F\) joining \(x\) to \(y\) such that \(l(T) = 5\) and \(\{u, v\} \cap V(T) = \emptyset\) that contains exactly one 0-dimensional edge.

**Case 4:** \(h(x, y) > 3\). Without loss of generality, we can assume that \(y = (((((x^0)^1)^2) \ldots)^h(x, y))^{-1}\). By Theorem 2.1, \(l(P_t) = h(x, y)\) if \(i < h(x, y)\) and \(l(P_t) = h(x, y) + 2\) otherwise. Since \(|F| \leq n - 4\), at least one of \(P_t\) is in \(Q_n - F - \{u, v\}\). Suppose that \(r \geq h(x, y)\). Obviously, \(P_t\) forms a path of length \(h(x, y) + 2\) joining \(x\) to \(y\) not passing \(u\) nor \(v\) that contains exactly one 0-dimensional edge. Thus, we assume that \(r < h(x, y)\). Then, \(P_t\) can be written as \((x, x^i, (x^i)^j, t, (y^p)^q, y^p, y)\) where \([i, j, p, q]\) are four distinct element of \(\{0, 1, \ldots, h(x, y)\}\). Let \(A = \{x, x^i, (x^i)^j, x^i\} | t \in \{0, 1, 2, \ldots, n - 1\} - \{i, j, 0\}\) and \(B = \{(y^p, (y^p)^q, y^p, y) | t \in \{0, 1, 2, \ldots, n - 1\} - \{p, q, 0\}\}\). Then \(A \cup B\) forms a set of \(2n - 6\) edge-disjoint paths of length 3, and there exists one path \(T'\) in \(A \cup B\) not passing \(u\) nor \(v\). Suppose that \(T'\) is of the form \((x, x^i, (x^i)^j, x^i)\). Then \((x, x^i, (x^i)^j, x^i, (x^i)^j, t, (y^p)^q, y^p, y)\) is a path \(T\) in \(Q_n - F\) such that \(l(T) = h(x, y) + 2\) and \(\{u, v\} \cap V(T) = \emptyset\). Suppose that \(T'\) is of the form \((y^p, (y^p)^q, y^p, y)\). Then \((x, x^i, (x^i)^j, t, (y^p)^q, y^p, (y^p)^q, y^p, y)\) is a path \(T\) in \(Q_n - F\) such that \(l(T) = h(x, y) + 2\) and \(\{u, v\} \cap V(T) = \emptyset\). Thus, there exists a path \(T\) in \(Q_n - F\) joining \(x\) to \(y\) such that \(l(T) = h(x, y) + 2\) and \(\{u, v\} \cap V(T) = \emptyset\) that contains exactly one 0-dimensional edge.

From the above proof, there exists a shortest path \(P\) of \(Q_n - F\) joining \(x\) to \(y\) such that \(\{u, v\} \cap V(P) = \emptyset\) and \(|E(P) \cap E_0| = 1\).

For a vertex \(w\), let the set of its neighbors in graph \(G\) be denoted by \(Nbd_G(w)\).

For some neighbor vertex \(w^i\) of \(w\) in \(Q_n^0\), \(1 \leq i \leq n - 1\), we define its reduced edge set \(RE_w(w^i)\) as \(\{(w^i)^0, (w^i)^0\} \cup \{(w^i)^0, (w^i)^0) | 1 \leq j \leq n - 1 \text{with } i \neq j\}\). Then we have the following obvious lemma.

**Lemma 2.7.** \(RE_w(w^i) \cap RE_w(w^k) = \emptyset\) if \(i \neq k\). Moreover, \(\{(w^i)^0, (w^i)^0\} \cup \{(w^i)^0, (w^i)^0) | 1 \leq j \leq n - 1 \text{with } i \neq j\}\) is a path in \(Q_n - F\), and \(\{(w^i)^0, (w^i)^0\} \cup \{(w^i)^0, (w^i)^0) | 1 \leq j \leq n - 1 \text{with } i \neq j\}\) is a path in \(Q_n - F\). Moreover, let \(l_1\) and \(l_2\) be odd integers with \(l_1 \geq d_{Q_n - F}(u, v), l_2 \geq d_{Q_n - F}(x, y)\), and \(l_1 + l_2 = 2^n - 2\). Moreover, let \(l_3\) and \(l_4\) be even integers with \(l_3 \geq d_{Q_n - F}(u, x), l_4 \geq d_{Q_n - F}(v, y)\), and \(l_3 + l_4 = 2^n - 2\). There are two disjoint paths \(P_1\) and \(P_2\) such that \((1)\ P_1\) is a path joining \(u\) to \(v\)
with \( l(P_1) = l_1 \), (2) \( P_2 \) is a path joining \( x \) to \( y \) with \( l(P_2) = l_2 \), and (3) \( P_1 \cup P_2 \) spans \( Q_n - F \). Moreover, there are two disjoint paths \( P_3 \) and \( P_4 \) such that (1) \( P_3 \) is a path joining \( u \) to \( x \) with \( l(P_3) = l_3 \), (2) \( P_4 \) is a path joining \( v \) to \( y \) with \( l(P_4) = l_4 \), and (3) \( P_3 \cup P_4 \) spans \( Q_n - F \) except the following cases: (a) \( l_3 = 2 \) with \( d_{Q_n-F}(u,x) = 2 \) and \( d_{Q_n-F}(v,x) > 2 \), and (b) \( l_4 = 2 \) with \( d_{Q_n-F}(v,y) = 2 \) and \( d_{Q_n-F}(u,y) > 2 \). Furthermore, the bound \((n - 4)\) is tight.

**Proof.** We first prove the bound \((n - 4)\) is tight. Let \( u \) be any white vertex of \( Q_n \) and let \( v = u^0 \). We set \( F = \{ (u, u^i) \mid 1 \leq i \leq n - 3 \} \), \( x = v^{n-1} \) and \( y = u^{n-2} \). Obviously, \( u \) and \( x \) are white vertices, \( v \) and \( y \) are black vertices, and \( d_{Q_n-F}(u,v) = 1 \). It is easy to check that there is no path in \( Q_n - F \) of length 3 joining \( u \) to \( v \). Thus, the bound \((n - 4)\) is tight.

We prove the theorem by induction. With Theorem 1.1, the theorem holds for \( n \geq 4 \) and \(|F| = 0\). Assume the theorem holds for any \( Q_k \) with \( 4 \leq k < n \). Thus, we assume that \(|F| > 0\). Since \( Q_n \) is edge symmetric, we may assume that \( F \cap E_0 \neq \emptyset \). Let \( F_i = E(Q^i_{n-1}) \cap F \) for \( i = 0, 1 \). Thus, \(|F_i| \leq n - 5\) for \( i = 0, 1 \). Without loss of generality, we may assume that \( l_1 \geq l_2 \) and \( u \in V(Q^0_{n-1}) \). We prove the existence of \( P_1 \) and \( P_2 \). Since the proof of existence of \( P_3 \) and \( P_4 \) is very similar, it is deferred to Appendix B. We will consider eight cases depending on the location of \( v, x, \) and \( y \), and prove the first four cases here. The proof of the remaining cases are similar—though more tedious—to these, so we prove them in Appendix A.

**Case 1:** \( v \in V(Q^0_{n-1}) \) and \( \{x,y\} \subset V(Q^1_{n-1}) \).

![Fig. 1. Illustration of Case 1.](image)

Suppose that \( l_2 < 2^{n-1} - 1 \). By Lemma 2.1, there exists a hamiltonian path \( R \) of \( Q^0_{n-1} - F_0 \) joining \( u \) and \( v \). Note that the length of \( R \) is \( 2^{n-1} - 1 \) and \( \lceil (2^{n-1} - 1)/2 \rceil > (n-4) + 2 \geq |F| + 2 \) for \( n \geq 5 \). We can write \( R \) as \( \langle u, R_1, p, q, R_2, v \rangle \) for some vertices \( p \) and \( q \) with \( \{p^0, q^0\} \cap \{x, y\} = \emptyset \), \( (p, p^0) \notin F \), \( (q, q^0) \notin F \), and \( (p^0, q^0) \notin F \). Obviously, \( d_{Q^1_{n-1}-F_1}(p^0, q^0) = 1 \), thus \( p^0 \) and \( q^0 \) are of different color. By induction, there exist two disjoint paths \( S_1 \) and \( S_2 \) such that (1) \( S_1 \) is a path joining \( p^0 \) to \( q^0 \) with \( l(S_1) = l_1 = 2^{n-1} - 1 \), (2) \( S_2 \) is a path joining \( x \) to \( y \) with \( l(S_2) = l_2 \), and (3) \( S_1 \cup S_2 \) spans \( Q^1_{n-1} - F_1 \). We set \( P_1 \) as \( \langle u, R_1, p, p^0, S_1, q^0, q, R_2, v \rangle \) and set \( P_2 \) as \( \langle u, R_1, p, p^0, S_1, q^0, q, R_2, v \rangle \). We set \( P_1 \) and \( P_2 \) are the required paths. See Figure 1(a) for illustration.
Suppose that \( l_2 = 2^{n-1} - 1 \). By Lemma 2.1, there exists a hamiltonian path \( P_1 \) of \( Q^0_{n-1} - F_0 \) joining \( u \) to \( v \) and there exists a hamiltonian path \( P_2 \) of \( Q^1_{n-1} - F_1 \) joining \( x \) to \( y \). Obviously, \( P_1 \) and \( P_2 \) are the required paths. See Figure 1(b) for illustration.

**Case 2:** \( \{x, y\} \subset V(Q^0_{n-1}) \) and \( v \in V(Q^1_{n-1}) \). Since \( |F| \leq n - 4 \), there exists a vertex \( p \in \text{Nbd}_{Q^0_{n-1} - F_0}(u) - \{y\} \) such that \( (p, p^0) \notin F \).

Suppose that \( l_2 < 2^{n-1} - 1 \). By induction, there exist two disjoint paths \( S_1 \) and \( S_2 \) such that \((1) S_1 \) is a path joining \( u \) to \( p \) with \( l(S_1) = 2^{n-1} - l_2 - 2 \), \((2) S_2 \) is a path joining \( x \) to \( y \) with \( l(S_2) = l_2 \), and \((3) S_1 \cup S_2 \) spans \( Q^0_{n-1} - F_0 \). By Lemma 2.1, there exists a hamiltonian path \( R \) of \( Q^1_{n-1} - F_1 \) joining \( p^0 \) to \( v \). We set \( P_1 \) as \( \langle u, S_1, p, p^0, R, v \rangle \) and set \( P_2 \) as \( S_2 \). Obviously, \( P_1 \) and \( P_2 \) are the required paths. See Figure 2(a) for illustration.

\[
Q^0_{n-1}, \quad Q^1_{n-1}, \quad Q^0_{n-1}, \quad Q^1_{n-1}.
\]

(a) \( Q^0_{n-1} \), (b) \( Q^1_{n-1} \)

**Fig. 2. Illustration of Case 2.**

Suppose that \( l_2 = 2^{n-1} - 1 \). By induction, there exist two disjoint paths \( S_1 \) and \( S_2 \) such that \((1) S_1 \) is a path joining \( u \) to \( p \) with \( l(S_1) = 2^{n-1} - 2 \), \((2) S_2 \) is a path joining \( x \) to \( y \) with \( l(S_2) = 2^{n-1} - 3 \), and \((3) S_1 \cup S_2 \) spans \( Q^0_{n-1} - F_0 \). Since \( [(2^{n-1} - 3)/2] > (n - 4) + 1 \geq |F| + 1 \) for \( n \geq 5 \), we can write \( S_2 \) as \( \langle x, S^1_2, r, s, S^2_2, y \rangle \) for some vertices \( r \) and \( s \) such that \( v \notin \{r^0, s^0\}, (r, r^0) \notin F, (s, s^0) \notin F, \) and \( (r^0, s^0) \notin F \). By induction, there exist two disjoint paths \( R_1 \) and \( R_2 \) such that \((1) R_1 \) is a path joining \( p^0 \) to \( v \) with \( l(R_1) = 2^{n-1} - 3 \), \((2) R_2 \) is a path joining \( r^0 \) to \( s^0 \) with \( l(R_2) = 1 \), and \((3) R_1 \cup R_2 \) spans \( Q^1_{n-1} - F_1 \). We set \( P_1 \) as \( \langle u, S_1, p, p^0, R_1, v \rangle \) and set \( P_2 \) as \( \langle x, S^1_2, r, r^0, s, s^0, S^2_2, y \rangle \). Obviously, \( P_1 \) and \( P_2 \) are the required paths. See Figure 2(b) for illustration.

**Case 3:** \( \{v, x, y\} \subset V(Q^0_{n-1}) \).

Suppose that \( l_2 \leq 2^{n-2} - 1 \). By induction, there exist two disjoint paths \( S_1 \) and \( S_2 \) such that \((1) S_1 \) is a path joining \( u \) to \( v \) with \( l(S_1) = 2^{n-1} - l_2 - 2 \), \((2) S_2 \) is a path joining \( x \) to \( y \) with \( l(S_2) = l_2 \), and \((3) S_1 \cup S_2 \) spans \( Q^0_{n-1} - F_0 \). Since \( [(2^{n-2} - 1)/2] > (n - 4) + 1 \) for \( n \geq 5 \), we can write \( S_2 \) as \( \langle u, S^1_2, r, s, S^2_2, v \rangle \) for some vertices \( r \) and \( s \) with \( (r, r^0) \notin F \) and \( (s, s^0) \notin F \). By Lemma 2.1, there exists a hamiltonian path \( R \) of \( Q^1_{n-1} - F_1 \) joining \( r^0 \) to \( s^0 \). We set \( P_1 \) as \( \langle u, S^1_2, r, r^0, R, s, s^0, S^2_2, v \rangle \) and set \( P_2 \) as \( S_2 \). Obviously, \( P_1 \) and \( P_2 \) are the required paths. See Figure 3(a) for illustration.
Suppose that \( l_2 \geq 2^{n-2} + 1 \). By induction, there exist two disjoint paths \( S_1 \) and \( S_2 \) such that (1) \( S_1 \) is a path joining \( u \) to \( v \) with \( l(S_1) = 2^{n-2} - 1 \), (2) \( S_2 \) is a path joining \( x \) to \( y \) with \( l(S_2) = 2^{n-2} - 1 \), and (3) \( S_1 \cup S_2 \) spans \( Q^0_{n-1} - F_0 \). Again, we can write \( S_1 \) as \( \langle u, S_1^1, r, s, S_2^0, v \rangle \) for some vertices \( r \) and \( s \) with \( (r, r^0) \notin F \) and \( (s, s^0) \notin F \). Similarly, we can write \( S_2 \) as \( \langle x, S_2^1, p, q, S_2^2, y \rangle \) for some vertices \( p \) and \( q \) with \( (p, p^0) \notin F \), \( (q, q^0) \notin F \), and \( (p^0, q^0) \notin F \). By induction, there exist two disjoint paths \( R_1 \) and \( R_2 \) such that (1) \( R_1 \) is a path joining \( r^0 \) to \( s^0 \) with \( l(R_1) = 2^{n-1} - l_2 + 2^{n-2} - 2 \), (2) \( R_2 \) is a path joining \( p^0 \) to \( q^0 \) with \( l(R_2) = l_2 - 2^{n-2} \), and (3) \( R_1 \cup R_2 \) spans \( Q^1_{n-1} - F_1 \). We set \( P_1 \) as \( \langle u, S_1^1, r, r^0, R_1, s^0, s, S_2^0, v \rangle \) and set \( P_2 \) as \( \langle x, S_2^1, p, p^0, R_2, q^0, q, S_2^2, y \rangle \). Obviously, \( P_1 \) and \( P_2 \) are the required paths. See Figure 3(b) for illustration.

**Case 4:** \( \{v, x, y\} \subset V(Q^1_{n-1}) \). Since \( |F| \leq n - 4 \), in \( Nbd_{Q^1_{n-1} - F_1}(v) - \{x\} \) we can choose a vertex \( p \) such that \( (p, p^0) \notin F \). Obviously, \( p \) is a white vertex.

Suppose that \( l_2 < 2^{n-1} - 1 \). By induction, there exist two disjoint paths \( S_1 \) and \( S_2 \) such that (1) \( S_1 \) is a path joining \( p \) to \( v \) with \( l(S_1) = l_1 - 2^{n-1} \), (2) \( S_2 \) is a path joining \( x \) to \( y \) with \( l(S_2) = l_2 \), and (3) \( S_1 \cup S_2 \) spans \( Q^1_{n-1} - F_1 \). By Lemma 2.1, there exists a hamiltonian path \( R \) of \( Q^0_{n-1} - F_0 \) joining \( u \) and \( p^0 \). We set \( P_1 \) as \( \langle u, R, p^0, p, S_1, v \rangle \) and we set \( P_2 \) as \( S_2 \). Obviously, \( P_1 \) and \( P_2 \) are the required paths. See Figure 4(a) for illustration.

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**Fig. 3.** Illustration of Case 3.

**Fig. 4.** Illustration of Case 4.
Suppose that \( l_2 = 2^{n-1} - 1 \). By induction, there exist two disjoint paths \( S_1 \) and \( S_2 \) such that (1) \( S_1 \) is a path joining \( p \) to \( v \) with \( l(S_1) = 1 \), (2) \( S_2 \) is a path joining \( x \) to \( y \) with \( l(S_2) = 2^{n-1} - 3 \), and (3) \( S_1 \cup S_2 \) spans \( Q_{n-1} \). Again, we can write \( S_2 \) as \( \langle x, S_1^2, r, s, S_2^1, y \rangle \) for some vertices \( r \) and \( s \) with \( u \notin \{r^0, s^0\}, (r, r^0) \notin F, (s, s^0) \notin F \), and \( (r^0, s^0) \notin F \). By induction, there exist two disjoint paths \( R_1 \) and \( R_2 \) such that (1) \( R_1 \) is a path joining \( u \) to \( p^0 \) with \( l(R_1) = 2^{n-1} - 3 \), (2) \( R_2 \) is a path joining \( r^0 \) to \( s^0 \) with \( l(R_2) = 1 \), and (3) \( R_1 \cup R_2 \) spans \( Q_{n-1} \). We set \( P_1 \) as \( \langle u, R_1, p^0, p, v \rangle \) and set \( P_2 \) as \( \langle x, S_1^2, r^0, S_2^1, s, y \rangle \). Obviously, \( P_1 \) and \( P_2 \) are the required paths. See Figure 4(b) for illustration.

The theorem is proved.

\[ \square \]

4. Applications

Lai and Hsu\(^7\) began the study of two equal disjoint path cover for an interconnection network. Let \( G \) be a graph with four vertices \( u, v, x, \) and \( y \). We say \( G \) is \((u, v, x, y)\)-two-equal-disjoint path coverable if there are two spanning disjoint paths \( P \) and \( Q \) such that \( P \) is a path joining \( u \) to \( v \), \( Q \) is a path joining \( x \) to \( y \), and \( l(P) = l(Q) \). With Theorem 3.1, \( Q_n - F \) for any \( F \subseteq E(Q_n) \) with \( |F| \leq n - 4 \) is \((u, v, x, y)\)-two-equal-disjoint path coverable if \( (1) \) \( u \) and \( v \) are white vertices whereas \( x \) and \( y \) are black vertices or \( (2) \) \( u \) and \( x \) are white vertices whereas \( v \) and \( y \) are black vertices.

The path embedding problem, which deals with all possible length of the paths between given two vertices in a given graph, is investigated in a lot of interconnection networks, see Chang et al., Fan et al.,\(^3\)\(^4\) and Ma and Xu.\(^12\) In particular, Li et al.\(^10\) proved that \( Q_n \) is bipanconnected. A bipartite graph \( G \) is bipanconnected if there exists a path \( P_l(x, y) \) of length \( l \) joining any two different vertices \( x \) and \( y \) with \( d_G(x, y) \leq l \leq |V(G)| - 1 \) such that \( 2 \mid (l - d_G(x, y)) \). Later, we prove that \( Q_n - F \) remains bipanconnected for any \( F \subseteq E(Q_n) \) with \( |F| \leq n - 4 \). Yet, we expect such path \( P_l(x, y) \) can be further extended by including the vertices not in \( P_l(x, y) \) into a hamiltonian path from \( x \) to a fixed vertex \( z \) or a hamiltonian cycle.

**Theorem 4.1.** Assume that \( n \) is any positive integer with \( n \geq 2 \) and \( F \) is a subset of \( E(Q_n) \) with \( |F| \leq n - 4 \). Let \( x \) and \( z \) be two vertices from different partite sets of \( Q_n \) and \( y \) be a vertex of \( Q_n \) that is not in \( \{x, z\} \). For any integer \( l \) with \( d_{Q_n-F}(x, y) \leq l \leq 2^n - 1 - d_{Q_n-F}(y, z) \) and \( 2 \mid (l - d_{Q_n-F}(x, y)) \), there exists a hamiltonian path \( R(x, y, z, l) \) of \( Q_n - F \) from \( x \) to \( z \) such that \( d_{R(x,y,z,l)}(x, y) = l \).

**Proof.** By brute force, we can check the theorem holds for \( n = 2, 3 \). Now, we consider \( n \geq 4 \). Without loss of generality, we assume that \( x \) is a white vertex and \( z \) is a black vertex.

Suppose that \( y \) is a black vertex. Obviously, \( d_{Q_n-F}(y, z) \geq 2 \). There exists a neighbor \( w \) of \( y \) such that \( w \neq x \) and \( d_{Q_n-F}(w, z) = d_{Q_n-F}(y, z) - 1 \). Obviously, \( w \) is a white vertex. By Theorem 3.1, there exist two disjoint paths \( R_1 \) and \( R_2 \) such that (1) \( R_1 \) is a path joining \( x \) to \( y \) with \( l(R_1) = l \), (2) \( R_2 \) is a path joining \( w \) to \( z \) with \( l(R_2) = 2^n - l - 2 \), and (3) \( R_1 \cup R_2 \) spans \( Q_n - F \). We set \( R \) as \( \langle x, R_1, y, w, R_2, z \rangle \).
Obviously, $R$ is the required hamiltonian path.

Suppose that $y$ is a white vertex. Obviously, $d_{Q_n-F}(x, y) \geq 2$. There exists a neighbor $w$ of $y$ such that $w \neq z$ and $d_{Q_n-F}(w, x) = d_{Q_n-F}(y, x) - 1$. Obviously, $w$ is a black vertex. By Theorem 3.1, there exist two disjoint paths $R_1$ and $R_2$ such that (1) $R_1$ is a path joining $x$ to $w$ with $l(R_1) = l - 1$, (2) $R_2$ is a path joining $y$ to $z$ with $l(R_2) = 2^n - l - 1$, and (3) $R_1 \cup R_2$ spans $Q_n - F$. We set $R$ as $(x, R_1, w, y, R_2, z)$. Obviously, $R$ is the required hamiltonian path.

Corollary 4.1. Assume that $n$ is a positive integer with $n \geq 2$ and $F$ is a subset of $E(Q_n)$ with $|F| \leq n - 4$. Let $x$ and $y$ be any two different vertices of $Q_n$. For any integer $l$ with $d_{Q_n-F}(x, y) \leq l \leq 2^n - 1$ there exists a hamiltonian cycle $S(x, y; l)$ of $Q_n - F$ such that $d_{S(x,y;l)}(x, y) = l$ and $2 | (l - d_{Q_n-F}(x, y))$.

Proof. Let $z$ be a neighbor of $x$ such that $z \neq y$. By Theorem 4.1, there exits a hamiltonian path $R$ of $Q_n - F$ joining $x$ to $z$ such that $d_{R(x,z;1)}(x, y) = 1$. We set $S$ as $(x, R, z, x)$. Obviously, $S$ forms the required hamiltonian cycle.

Corollary 4.2. Assume that $n$ is any positive integer with $n \geq 2$ and $F$ is a subset of $E(Q_n)$ with $|F| \leq n - 4$. Then $Q_n - F$ is bipanconnected.

Proof. Obviously, $Q_n - F$ is bipanconnected for $n = 1, 2, 3$. Now, we consider $n \geq 4$. Without loss of generality, we assume that $x$ is a white vertex.

Suppose that $y$ is a black vertex. Thus, $d_{Q_n-F}(x, y)$ is odd. Let $l$ be any odd integer with $d_{Q_n-F}(x, y) \leq l \leq 2^n - 1$. Suppose that $l = 2^n - 1$. By Lemma 2.1, there exists a hamiltonian path of $Q_n - F$ joining $x$ to $y$. Obviously, it is of length $2^n - 1$. Suppose that $l < 2^n - 1$. Since $n \geq 4$, we can choose a pair of adjacent vertices $u$ and $v$ such that $u$ is a white vertex with $u \neq x$ and $v$ be a black vertex with $v \neq y$. Obviously, $d_{Q_n-F}(u, v) = 1$. By Theorem 3.1, there exist two disjoint paths $P_1$ and $P_2$ such that (1) $P_1$ is a path joining $u$ to $v$ with $l(P_1) = 2^n - 2 - l$, (2) $P_2$ is a path joining $x$ to $y$ with $l(P_2) = l$, and (3) $P_1 \cup P_2$ spans $Q_n - F$. Obviously, $P_2$ is a path of length $l$ joining $x$ to $y$.

Suppose that $y$ is a white vertex. Thus, $d_{Q_n-F}(x, y)$ is even. Let $l$ be any even integer with $d_{Q_n-F}(x, y) \leq l < 2^n - 1$. Since $n \geq 4$, we can choose two different neighbors $u$ and $v$ of $y$ such that $d_{Q_n-F}(x, u) = d_{Q_n-F}(x, y) - 1$. By Theorem 3.1, there exist two disjoint paths $P_1$ and $P_2$ such that (1) $P_1$ is a path joining $x$ to $u$ with $l(P_1) = l - 1$, (2) $P_2$ is a path joining $y$ to $v$ with $l(P_2) = 2^n - l - 1$, and (3) $P_1 \cup P_2$ spans $Q_n - F$. Obviously, $(x, P_1, u, y)$ is a path of length $l$ joining $x$ to $y$.

Thus, $Q_n - F$ is bipanconnected.

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Appendix A: Proof of Cases 5–8 in Theorem 3.1

Case 5: $y \in V(Q_{n-1}^0)$ and $\{x, v\} \subset V(Q_{n-1}^1)$. By Lemma 2.6, there exists a shortest path $T$ of $Q_n - F$ joining $x$ to $y$ such that $\{u, v\} \cap V(T) = \emptyset$. Moreover, $l(T)$ is either $h(x, y)$ or $h(x, y) + 2$ and $|E(T) \cap E_0| = 1$. Thus, $T$ can be written as $(y, T_1, z, z^0, T_2, x)$ such that $(z, z^0)$ is the only 0-dimensional edge. (Note that $y = z$ if $l(T_1) = 0$ and $z^0 = x$ if $l(T_2) = 0$.)

(i) $l(T_1) \neq 0$ and $l(T_2) = 0$. Obviously, $z$ is a black vertex. Since $|F| \leq n - 4$, there exists a vertex $p$ in $V(Q_{n-1}^0) - \{y^0\}$ such that $(p, p^0) \notin F$ and $d_{Q_{n-1}^0 - F - \{y, z\}}(p, u) = 2$.

Suppose that $l_2 < 2^{n-1} - 1$. By induction, there exist two disjoint paths $S_1$ and $S_2$ such that (1) $S_1$ is a path joining $u$ to $p$ with $l(S_1) = 2^{n-1} - l_2 - 1$, (2) $S_2$ is a path joining $z$ to $y$ with $l(S_2) = l_2 - 1$, and (3) $S_1 \cup S_2$ spans $Q_{n-1}^0 - F_0$. By Lemma 2.2, there exists a hamiltonian path $R$ of $(Q_{n-1}^1 - F_1) - \{x\}$ joining $p^0$ to $v$. We set $P_1$ as $(u, S_1, p, p^0, R, v)$ and set $P_2$ as $(x = z^0, z, S_2, y)$. Obviously, $P_1$ and $P_2$ are the required paths. See Figure 5(a) for illustration.

Suppose that $l_2 = 2^{n-1} - 1$. By induction, there exist two disjoint paths $S_1$ and $S_2$ such that (1) $S_1$ is a path joining $u$ to $p$ with $l(S_1) = 2$, (2) $S_2$ is a path joining $z$ to $y$ with $l(S_2) = 2^{n-1} - 4$, and (3) $S_1 \cup S_2$ spans $Q_{n-1}^0 - F_0$. We can write $S_2$ as $(z, S^1_2, r, s, S^2_2, y)$ for some vertices $r$ and $s$ with $v \notin \{r^0, s^0\}$, $(r, r^0) \notin F$, $(s, s^0) \notin F$, and $(r^0, s^0) \notin F$. By Lemma 2.4, there exists a hamiltonian path $R$ of $(Q_{n-1}^1 - F_1) - \{x\}$ joining $p^0$ to $v$. We set $P_1$ as $(u, S_1, p, p^0, R, v)$ and set $P_2$ as $(x = z^0, z, S^1_2, r, r^0, s^0, s, S^2_2, y)$. Obviously, $P_1$ and $P_2$ are the required paths. See Figure 5(b) for illustration.
(ii) \(l(T_1) = 0\) and \(l(T_2) \neq 0\). Since \(|F| \leq n - 4\), there exists a vertex \(p\) in \(V(Q^1_{n-1}) - \{u^0\}\) such that \((p,p^0) \not\in F\) and \(d_{Q^1_{n-1} - F_1 - \{x,x^0\}}(p,v) = 2\).

Suppose that \(l_2 < 2^{n-1} - 1\). By induction, there exist two disjoint paths \(R_1\) and \(R_2\) such that (1) \(R_1\) is a path joining \(p\) to \(v\) with \(l(R_1) = 2^{n-1} - l_2 - 1\), (2) \(R_2\) is a path joining \(R\) to \(z^0\) with \(l(R_2) = l_2 - 1\), and (3) \(R_1 \cup R_2\) spans \(Q^1_{n-1} - F_1\). We set \(P_1\) as \(\langle u, S, p^0, p, R_1, v \rangle\) and set \(P_2\) as \(\langle x, R_2, z^0, z = y \rangle\). Obviously, \(P_1\) and \(P_2\) are the required paths. See Figure 5(c) for illustration.

Suppose that \(l_2 = 2^{n-1} - 1\). By Lemma 2.2, there exists a hamiltonian path \(S\) of \((Q^1_{n-1} - F_0) - \{y\}\) joining \(u\) to \(p^0\). By induction, there exist two disjoint paths \(R_1\) and \(R_2\) such that (1) \(R_1\) is a path joining \(p\) to \(v\) with \(l(R_1) = 2\), (2) \(R_2\) is a path joining \(x\) to \(z^0\) with \(l(R_2) = 2^{n-1} - 4\), and (3) \(R_1 \cup R_2\) spans \(Q^1_{n-1} - F_1\). We can write \(R_2\) as \(\langle x, R_1^0, r, s, R_2^0, z^0 \rangle\) for some vertices \(r\) and \(s\) and set \(F = \{r^0, s^0\}\). By Lemma 2.4, there exists a hamiltonian path \(S\) of \((Q^1_{n-1} - F_0) - \{r^0, s^0\}\) joining \(u\) to \(p^0\). We set \(P_1\) as \(\langle u, S, p^0, p, R_1, v \rangle\) and set \(P_2\) as \(\langle x, R_2, r^0, s^0, s, R_1^0, z^0, z = y \rangle\). Obviously, \(P_1\) and \(P_2\) are the required paths. See Figure 5(d) for illustration.

(iii) \(l(T_1) = l(T_2) = 0\). Thus, \(y = x^0\).

(iv) \(l(T_1) \neq 0\) and \(l(T_2) \neq 0\).

Assume \(z\) is a white vertex. Since \(|F| \leq n - 4\), there exists a vertex \(p\) in \(Nbd_{Q^1_{n-1} - F_1}(v) - \{x, y^0\}\) such that \((p,p^0) \not\in F\). Obviously, \(p\) is a white vertex. By induction, there exist two disjoint paths \(S_1\) and \(S_2\) such that (1) \(S_1\) is a path joining \(u\) to \(p^0\) with \(l(S_1) = 2^{n-1} - 3\), (2) \(S_2\) is a path joining \(z\) to \(y\) with \(l(S_2) = 1\), and (3) \(S_1 \cup S_2\) spans \(Q^1_{n-1} - F_0\). Similarly, there exist two disjoint paths \(R_1\) and \(R_2\) such that (1) \(R_1\) is a path joining \(p\) to \(v\) with \(l(R_1) = 2^{n-1} - l_2\), (2) \(R_2\) is a path joining \(x\) to \(z^0\) with \(l(R_2) = l_2 - 2\), and (3) \(R_1 \cup R_2\) spans \(Q^1_{n-1} - F_1\). We set \(P_1\) as \(\langle u, S_1, p^0, p, R_1, v \rangle\) and set \(P_2\) as \(\langle x, R_2, z^0, z, S_2, y \rangle\). Obviously, \(P_1\) and \(P_2\) are the required paths. See Figure 5(f) for illustration.
Assume that \( z \) is a black vertex. Since \(|F| \leq n - 4\), there exists a vertex \( p \) in \( V(Q_{n-1}^1) - \{u^0\}\) such that \((p, p^0) \notin F\) and \(d_{Q_{n-1}^1-F_1-(x,z^0)}(p,v) = 2\). Thus, \( p \) is a black vertex. By induction, there exist two disjoint paths \( S_1 \) and \( S_2 \) such that (1) \( S_1 \) is a path joining \( u \) to \( p^0 \) with \( l(S_1) = 2^{n-1} - l(T_1) - 2 \), (2) \( S_2 \) is a path joining \( z \) to \( y \) with \( l(S_2) = l(T_1) \), and (3) \( S_1 \cup S_2 \) spans \( Q_{n-1}^0 - F_0 \). Similarly, there exist two disjoint paths \( R_1 \) and \( R_2 \) such that (1) \( R_1 \) is a path joining \( p \) to \( v \) with \( l(R_1) = 2^{n-1} - l_2 - l(T_1) - 2 \), (2) \( R_2 \) is a path joining \( x \) to \( z^0 \) with \( l(R_2) = l_2 - l(T_1) - 1 \), and (3) \( R_1 \cup R_2 \) spans \( Q_{n-1}^1 - F_1 \). We set \( P_1 \) as \((u, S_1, p^0, p, R_1, v)\) and set \( P_2 \) as \((x, R_2, z^0, z, S_2, y)\). Obviously, \( P_1 \) and \( P_2 \) are the required paths. See Figure 5(h) for illustration.

**Case 6:** \( \{v, y\} \subset V(Q_{n-1}^0) \) and \( x \in V(Q_{n-1}^1) \).

Suppose \( l_2 \geq h(x,y) + 4 \). Since \(|F| \leq n - 4\), there exists a vertex \( z \) in \( Nbd_{Q_{n-1}^1-F_0}(y) - \{u\} \) such that \((z, z^0) \notin F\). Obviously, \(d_{Q_{n-1}^1-F_1}(x, z^0) \leq h(x, z^0) + 2 \leq l_2 - 2\). By induction, there exist two disjoint paths \( S_1 \) and \( S_2 \) such that (1) \( S_1 \) is a path joining \( u \) to \( v \) with \( l(S_1) = 2^{n-1} - 3 \), (2) \( S_2 \) is a path joining \( z \) to \( y \) with \( l(S_2) = 1 \), and (3) \( S_1 \cup S_2 \) spans \( Q_{n-1}^0 - F_0 \). As before, we can write \( S_1 \) as \((u, S_1^1, r, s, S_1^2, v)\) for some vertices \( r \) and \( s \) such that \( x \notin \{r^0, s^0\}, (r, r^0) \notin F, (s, s^0) \notin F \). By induction, there exist two disjoint paths \( R_1 \) and \( R_2 \) such that (1) \( R_1 \) is a path joining \( r^0 \) to \( s^0 \) with \( l(R_1) = l_1 - 2^{n-1} + 2 \), (2) \( R_2 \) is a path joining \( x \) to \( z^0 \) with \( l(R_2) = l_2 - 2 \), and (3) \( R_1 \cup R_2 \) spans \( Q_{n-1}^1 - F_1 \). We set \( P_1 \) as \((u, S_1^1, r, r^0, R_1, x, s, S_1^2, v)\) and set \( P_2 \) as \((x, R_2, z^0, z, y)\). Obviously, \( P_1 \) and \( P_2 \) are the required paths. See Figure 6(a) for illustration.

Fig. 6. Illustration of Case 6.

Now we consider \( l_2 \leq h(x,y) + 2 \). Thus, \( l_2 \leq n + 2 \). By Lemma 2.6, there exists a shortest path \( T \) of \( Q_n - F \) joining \( x \) to \( y \) such that \( \{u, v\} \cap V(T) = \emptyset \). Moreover, \(|E(T) \cap E_0| = 1\) and \( l(T) \in \{h(x,y), h(x,y) + 2\}\). Thus \( T \) can be written
as \((y, T_1, z, z^0, T_2, x)\) such that \((z, z^0) \in E_0\). (Note that \(y = z\) if \(l(T_1) = 0\), and \(z^0 = x\) if \(l(T_2) = 0\).) Among all shortest paths of \(Q_n - F\) joining \(x\) to \(y\) such that \(\{u, v\} \cap V(T) = \emptyset\), we choose \(T\) as the one with \(z\) being a white vertex if possible. If not, we choose \(T\) such that \(l(T_2) > 0\) if possible.

(i) \(l(T_1) = 0\). Suppose \(l_2 = 1\). Let \(F_{av} = \{\{x, y\}\}\). Obviously, \(|F| + |F_{av}| \leq n - 3\).

By Lemma 2.3, there exists a Hamiltonian path \(P_1\) of \(Q_n - F - F_{av}\) joining \(u\) and \(v\).

We set \(P_2\) as \(\langle x, y \rangle\). Obviously, \(P_1\) and \(P_2\) are the required paths. See Figure 6(b) for illustration.

Suppose now \(l_2 = 2\). Let \((x, w_1, w_2, y)\) be a shortest path in \(Q_n - F\) joining \(x\) to \(y\) such that \(\{u, v\} \cap V(T) = \emptyset\). Let \(F_{av} = \{\{x, w_1\}, \{w_2, y\}\}\). Obviously, \(|F| + |F_{av}| \leq n - 2\). By Lemma 2.3, there exists a Hamiltonian path \(P_1\) of \(Q_n - F - F_{av}\) joining \(u\) and \(v\) and we set \(P_2\) as \(\langle x, w_1, w_2, y \rangle\). Obviously, \(P_1\) and \(P_2\) are the required paths.

Suppose that \(l_2 \geq 5\). Since there are \(2^{n-2}\) white vertices in \(Q_{n-1}^0\) and \(|F| \leq n - 4\), there exists a white vertex \(w\) in \(V(Q_{n-1}^0) - \{u\}\) such that \((w, w^0) \notin F\). Let \(F' = F_0 \cup \{(v, t) \mid t \in Nbd_{Q_{n-1}^0}(v)\}\) such that \((t, t^0) \in F\). Obviously, \(|F'| \leq n - 4\). By Lemma 2.2, there exists a Hamiltonian path \(S\) of \((Q_{n-1}^0 - F') - \{y\}\) joining \(u\) to \(w\).

We can write \(S\) as \(\langle u, S_1, p, v, S_2, w \rangle\). Obviously, \(p\) is a white vertex. By induction, there exist two disjoint paths \(R_1\) and \(R_2\) such that (1) \(R_1\) is a path joining \(w^0\) to \(w\) with \(l(R_1) = 2^{n-1} - l_2 - 1\), (2) \(R_2\) is a path joining \(x\) to \(z^0\) with \(l(R_2) = l_2 - 1\), and (3) \(R_1 \cup R_2\) spans \(Q_{n-1}^1 - F\) (we can’t get the exceptional case since \(h(x, z^0) \geq 4\)). We set \(P_1\) as \(\langle u, S_1, p, p^0, R_1, w^0, w, (S_2)^{-1}, v \rangle\) and set \(P_2\) as \(\langle x, R_2, z^0, y, R_2 \rangle\). Obviously, \(P_1\) and \(P_2\) are the required paths. See Figure 6(c) for illustration.

(ii) \(l(T_1) \neq 0\) and \(l(T_2) = 0\).

Assume that \(z\) is a white vertex. By induction, there exist two disjoint paths \(S_1\) and \(S_2\) such that (1) \(S_1\) is a path joining \(u\) to \(v\) with \(l(S_1) = 2^{n-1} - l(T_1) - 2\), (2) \(S_2\) is a path joining \(z\) to \(y\) with \(l(S_2) = l(T_1)\), and (3) \(S_1 \cup S_2\) spans \(Q_{n-1}^0 - F_0\). Again, we can write \(S_1\) as \(\langle u, S_1^1, r, s, S_1^2, v \rangle\) for some vertices \(r\) and \(s\) such that \(x \notin \{r^0, s^0\}\), \((r, r^0) \notin F\) and \((s, s^0) \notin F\). By induction, there exist two disjoint paths \(R_1\) and \(R_2\) such that (1) \(R_1\) is a path joining \(r^0\) to \(s^0\) with \(l(R_1) = 2^{n-1} - l_2 + l(T_1) - 1\), (2) \(R_2\) is a path joining \(x\) to \(z^0\) with \(l(R_2) = l_2 - l(T_1) - 1\), and (3) \(R_1 \cup R_2\) spans \(Q_{n-1}^1 - F_1\).

We set \(P_1\) as \(\langle u, S_1^1, r, r^0, R_1, s^0, s, S_1^2, v \rangle\) and set \(P_2\) as \(\langle x, R_2, z^0, z, S_2, y \rangle\). Obviously, \(P_1\) and \(P_2\) are the required paths. See Figure 6(d) for illustration.

Assume that \(z\) is a black vertex, hence \(l(T_1)\) is an even integer. Moreover, we can write \(T_1\) as \(\langle y = w_0, w_1, \ldots, w_{l(T_1) - 1}, z, T_2, x \rangle\). Let \(F_0' = F_0 \cup \{(v, t) \mid t \in Nbd_{Q_{n-1}^0}(v)\}\) such that \((t, t^0) \in F\). Obviously, \(|F_0'| \leq |F| \leq n - 4\).

Suppose that \(\frac{l(T_1)}{2} + |F_0'| \leq n - 4\). We set \(F_{av} = \{\{w_{2i-1}, w_{2i}\} \mid 1 \leq i \leq \frac{l(T_1)}{2}\}\). Since \(|F| \leq n - 4\), there exists a white vertex \(s\) in \(V(Q_{n-1}^0) - (\{u\} \cup \{w_i \mid 1 \leq i \leq l(T_1) - 1\})\) such that \((s, s^0) \notin F\). By Lemma 2.4, there exists a Hamiltonian path \(S\) of \((Q_{n-1}^0 - F') - F_{av} - \{y\}\) joining \(u\) to \(s\). We can write \(S\) as \(\langle u, S_1, r, v, S_2, s, R_1, \rangle\) since \((r, v) \notin F_0', (r, r^0) \notin F\). By induction, there exist two disjoint paths \(R_1\) and \(R_2\) such
that (1) $R_1$ is a path joining $r^0$ to $s^0$ with $l(R_1) = 2^{n-1} - l_2 + l(T_1) - 1$, (2) $R_2$ is a path joining $x$ to $z^0$ with $l(R_2) = l_2 - l(T_1) - 1$, and (3) $R_1 \cup R_2$ spans $Q_{n-1}^1 - F$. We set $P_1$ as $\langle u, S_1, r, r^0, R_1, s^0, s, (S_2)^{-1}, v \rangle$ and set $P_2$ as $\langle x, R_2, z^0, z, (T_1)^{-1}, y \rangle$. Obviously, $P_1$ and $P_2$ are the required paths. See Figure 6(e) for illustration.

Suppose that $l(T_1) + |F_0^*| > n - 4$. We claim that $\{(w_{l(T_1)-1}, w_{l(T_1)-1}^0), (w_{l(T_1)-1}^0, z_0)\} \cap F \neq \emptyset$. If not, then $\langle y = w_0, w_1, \ldots, w_{l(T_1)-1}, w_{l(T_1)-1}^0, z_0, Q_2, x \rangle$ is a shortest path of $Q_n - F$ joining $y$ to $x$ such that the only 0-dimensional edge joins a white vertex $w_{l(T_1)-1}$ to a black vertex $w_{l(T_1)-1}^0$ in $Q_{n-1}^1$, contradicting the choice of $z$.

Obviously, $z_1 = (z_0)^k$. We claim that $\{(z, z^k), (z^k, z_1)\} \cap F \neq \emptyset$. If not, then similarly $\langle y, T_1, z, (z)^k, z_1, T_2, x \rangle$ is a shortest path of $Q_n - F$ joining $y$ to $x$ such that the only 0-dimensional edge joining a white vertex $(z)^k$ in $Q_{n-1}^0$ to a black vertex $z_1$ in $Q_{n-1}^1$, contradicting the choice of $z$.

Let $F_0^* = F_0 \cup \{(z, z') | z' \neq v, RE_z(z') \cap F \neq \emptyset\}$. Obviously, $|F_0^*| \leq n - 5$ and $(z, z^k) \in F_0^*$. By induction, there exist two disjoint paths $S_1$ and $S_2$ such that (1) $S_1$ is a path joining $u$ to $v$ with $l(S_1) = 2^{n-1} - l(T_1) - 1$, (2) $S_2$ is a path joining $w_{l(T_1)-1}$ to $y$ with $l(S_2) = l(T_1) - 1$, and (3) $S_1 \cup S_2$ spans $Q_{n-1}^1 - F_0^*$. Since $z$ can’t be on $S_2$, the path $S_1$ can be written as $\langle u, S_1^1, z^1, z, S_1^2, v \rangle$ such that $k \notin \{i, j\}$.

In $Q_{n-1}^1 - F_1$ we have the path $Z = \langle z_0, z_1, \ldots, z_{l-1}(T_1-1), x \rangle$ where $l_2 - l(T_1) - 1$ is an even integer. Note that $l(T_1) + (2^{n-1} - l(T_1) - 1) = 2^{n-1} - l_2 - 1 \leq n + 1$ and $|F_0^*| + |F_1| \leq n - 4$. Since $l(T_1) + |F_0^*| > n - 4$, $l_2 - l(T_1) - 1 < \frac{n+1}{2}$.

We set $F_{av}$ as $\{(z_0, (z^i)^0) \cup \{(z^i)^0, ((z^i)^0)^j)\} \cup \{(z_{2^i-1}, z_{2^i}) | 1 \leq i \leq \frac{l_2 - l(T_1) - 1}{2}\}$. Obviously, $|F_1| + |F_{av}| \leq |\frac{n+1}{2}| - 2 \leq (n - 3)$ for $n \geq 8$. Consider first the case when $n \geq 8$. Obviously, there exists an edge $(r, s)$ in $S_1^1$ or $S_2^1$ such that $\{r^0, s^0\} \cap \{(z^i)^0, ((z^i)^0)^j)\} \cup \{(z_i | 0 \leq i \leq l_2 - l(T_1) - 1)\} = \emptyset$. Without loss of generality, $(r, s)$ is in $S_1^1$ and $S_2^1$ can be written as $\langle z^i, S_1^2, r, s, S_2^2, v \rangle$. By Lemma 2.3, there exists a hamiltonian path $R$ of $Q_{n-1}^1 - F_1 - F_{av}$ joining $r^0$ to $s^0$. We set $P_1$ as $\langle u, S_1^1, z^i, (z^i)^0, ((z^i)^0)^j, (z^i)^0, z^j, S_1^2, r, r^0, R, s^0, s, S_2^2, v \rangle$ and set $P_2$ as $\langle x, (Z)^{-1}, z^0, z, T_1, y \rangle$. Obviously, $P_1$ and $P_2$ are the required paths. See Figure 6(f) for illustration.

Finally, consider when $n < 8$. If $l_2 = 3$, we can do what we did in subcase (i), so we can assume $l_2 \geq 5$. We will show that we can always choose $T$ such that $z$ is a white vertex. Let $n = 5$, thus $|F| = 1$. Then $h(y, x) = 3$ or $h(y, x) = 5$. In either case, consider the four paths of length 2 starting from $y$, going to a neighbor in $Q_{n-1}^1$, then using the 0-dimensional edge to $Q_{n-1}^1$. These are all edge-disjoint, so only one of these paths can have an edge of $F$, and only one of them can contain $u$, while none can contain $v$, since it is a black vertex in $Q_{n-1}^1$. Let $\langle y, z, z^0 \rangle$ be one of the remaining paths. Since there is no fault in $Q_{n-1}^1$, and $u$ and $v$ are both in $Q_{n-1}^0$, we can easily extend this path to a shortest path $T$ of $Q_n - F$ joining $y$ to $x$ such that $\{u, v\} \cap V(T) = \emptyset$. The cases $n = 6$ and $n = 7$ can be handled similarly.

(iii) $l(T_2) = 0$. 

We can write $T_1$ as $\langle y = w_0, w_1, \ldots, w_{l(T_1)-1}, w_{l(T_1)} = z \rangle$, where $l(T_1)$ is an even integer. Let $i$ be the smallest index such that $\{w_i, (w_i)^0\}$ is not a fault (i is at most $l(T_1)$). By Lemma 2.6, we get that if $l(T_1) - i \geq h((w_i)^0, x) + 2$, then there is a path $P_2$ of length $l(T_1) - i$ joining $(w_i)^0$ to $x$ in $Q_{n-1}^i - F$, contradicting the choice of $T$. Thus $l(T_1) - i = h((w_i)^0, x)$, and again by Lemma 2.6 there are $l(T_1) - i$ edge-disjoint paths in $Q_{n-1}^i$ joining $(w_i)^0$ to $x$. By the choice of $T$ each of these paths must have a faulty edge, so $|F_i| \geq l(T_1) - i$. This implies that $F_0 \leq n - 4 - i - l(T_1) - i = n - 4 - l(T_1).

Now we can choose a white vertex $s \neq u$ in $Q_{n-1}^i$ such that $s$ is not a vertex on $T_1$ and $\{s, s^0\} \not\in F$. Let $F_0' = F_0 \cup \{(v, t) \mid t \in \text{Nbd}_{Q_{n-1}^i}(v) \text{ such that } t \text{ is not on } T_1 \text{ and } (t, t^0) \in F\}$. Clearly $|F_0'| \leq |F| - (l(T_1) - i) - i = |F| - l(T_1)$. We set $F_{av}$ as $\{\{w_{2j-1}, w_{2j}\} \mid 1 \leq j \leq l(T_1)/2\}$. Obviously, $|F_0'| + |F_{av}| \leq |F| - l(T_1)/2 \leq n - 5,$ so by Lemma 2.4 there exists a hamiltonian path $S_1$ of $Q_{n-1}^i - F_0' - F_{av} - y$ joining $u$ to $s$. Obviously, $S_1$ can be written as $\langle u, S_1^1, t, v, S_1^2, s \rangle$ where $t = u$ is possible. Also by Lemma 2.4 there exists a hamiltonian path $S_2$ of $Q_{n-1}^i - F - x$ joining $t^0$ to $s^0$. We set $P_1$ as $\langle u, S_1^1, t, t^0, S_2^0, s, (S_2^1)^{-1}, v \rangle$ and set $P_2$ as $T$. Obviously, $P_1$ and $P_2$ are the required paths. See Figure 6(g) for illustration.

**Case 7:** $\langle v, x \rangle \subset V(Q_{n-1}^0)$ and $y \in V(Q_{n-1}^1)$. This case is the same as Case 6 because we can interchange white vertices with black vertices, interchange $x$ with $y$, and interchange $u$ with $v$.

**Case 8:** $x \in V(Q_{n-1}^0)$ and $\{y, v\} \subset V(Q_{n-1}^1)$. In this case, we may assume that $|F_0| \geq |F_1|$ because we can interchange white vertices with black vertices, interchange $x$ with $y$, and interchange $u$ with $v$. Thus, $|F_1| \leq \lfloor \frac{m}{2} \rfloor$.

We first consider the case $l_2 \geq h(x, y) + 2$ and the case $l_2 = h(x, y) \geq n - 2$. We first construct a desired path $T$ joining $x$ to $y$ as follows:

Suppose that $l_2 \geq h(x, y) + 2$. We first consider the case that $h(x, y) \neq 1$. By Theorem 2.1, there exist $(n - 1)$ internally disjoint paths $T'_1, T'_2, \ldots, T'_{n-1}$ of $Q_{n-1}^i$ joining $x^0$ to $y$ such that $l(T'_i) \in \{h(x^0, y), h(x^0, y) + 2\}$. We can write $T'_i$ as $\langle x^0, z^0, T''_i, y \rangle$ and set $T_i$ as $\langle x, z, z^0, T''_i, y \rangle$ for $1 \leq i \leq n - 1$. Obviously, $T_1, T_2, \ldots, T_{n-1}$ forms a set of $(n - 1)$ internally disjoint paths joining $x$ to $y$ such that $l(T_i) \in \{h(x, y), h(x, y) + 2\}$. Now, we consider the case that $h(x, y) = 1$. Let $T_i = \langle x, x^0, y, \ldots, y^i \rangle$ for $1 \leq i \leq n - 1$. Obviously, $T_1, T_2, \ldots, T_{n-1}$ forms a set of $(n - 1)$ internally disjoint paths joining $x$ to $y$ such that $l(T_i) = 3$. Since $|F| \leq n - 4$, at least one of $T_i$ with $1 \leq i \leq n - 1$, say $T$, is in $Q_{n-1}^i - F$ satisfying $\{u, v\} \cap V(T) = \emptyset$. Obviously, we can rewrite $T$ as $\langle x, z, z^0, T''_i, y \rangle$. (Note that $T''_i$ is empty if $z^0 = y$.)

Suppose that $h(x, y) = l_2 \geq n - 2$. Thus, $h(x^0, y) \geq n - 3$. By Theorem 2.1, there exists $(n - 3)$ internal disjoint paths $T'_1, T'_2, \ldots, T'_{n-3}$ of $Q_{n-1}^i$ joining $x^0$ to $y$ such that $l(T'_i) = h(x^0, y)$ for $1 \leq i \leq n - 3$. Obviously, we can write $T'_i$ as $\langle x^0, z^0, T''_i, y \rangle$. Then we set $T_i = \langle x, z, x^0, T''_i, y \rangle$. Obviously, $T_1, T_2, \ldots, T_{n-3}$ forms a set of $(n - 3)$ internally disjoint paths of $Q_n$ joining $x$ to $y$ such that $l(T_i) = h(x, y)$ for $1 \leq i \leq n - 3$. Since $|F| \leq n - 4$, at least one of these paths, say $T$, is in $Q_{n-1}^i - F$. We write this path as $\langle x, z, z^0, R, y \rangle$. Obviously, $d_{Q_n-F}(z^0, y) = h(x, y) = l_2 = 2 \geq n - 4 \geq 1$. Suppose that $h(z^0, y) = 1$. Obviously, $T$ is a path of $Q_n - F$ joining $x$ to $y$ such that
\{u, v\} \cap V(T) = \emptyset and \(l(T) = l_2\). Suppose that \(h(z^0, y) > 1\). By Theorem 2.1, there exists \((n - 4)\) internally disjoint paths of \(Q_{n-1}^1\) joining \(z^0\) to \(y\) such that each path is of length \(h(z^0, y)\). Since \(|F_1| \leq n - 4\), there exists a path \(T''\) of \(Q_{n-1}^1 - F_1\) joining \(z^0\) to \(y\) such that \(v \notin V(T'')\) and \(l(T'') = h(z^0, y)\). We reset \(T\) as \((x, z, z^0, T'', y)\). Obviously, \(T\) is a path of \(Q_n - F\) joining \(x\) to \(y\) such that \(\{u, v\} \cap V(T) = \emptyset\) and \(l(T) = l_2\).

Since \(|F| \leq n - 4\), there exists a vertex \(p^0\) in \(Nbd_{Q_{n-1}^1}(v) - \{z^0\}\) such that \((p, p^0) \notin F\). By induction, there exist two disjoint paths \(S_1\) and \(S_2\) such that \((1) S_1\) is a path joining \(u\) to \(p\) with \(l(S_1) = 2^{n-1} - 3\), \((2) S_2\) is a path joining \(x\) to \(z\) with \(l(S_2) = 1\), and \((3) S_1 \cup S_2\) spans \(Q_{n-1}^1 - F_0\). Again, there exist two disjoint paths \(R_1\) and \(R_2\) such that \((1) R_1\) is a path joining \(p^0\) to \(v\) with \(l(R_1) = 2^{n-1} - l_2\), \((2) R_2\) is a path joining \(z^0\) to \(y\) with \(l(R_2) = l_2 - 2\), and \((3) R_1 \cup R_2\) spans \(Q_{n-1}^1 - F_1\). We set \(P_1\) as \((u, S_1, p, p^0, R_1, v)\) and set \(P_2\) as \((x, S_2, z, z^0, R_2, y)\). Obviously, \(P_1\) and \(P_2\) are the required paths. See Figure 7(a) for illustration.

![Figure 7](image_url)

Fig. 7. Illustration of Case 8.

Next, we consider \(l_2 = h(x, y)\) with \(l_2 \leq n - 3\). In this case, there exists a shortest path \(T\) of length of \(l_2\) in \(Q_n - F\) joining \(x\) to \(y\). Thus, \(T\) can be written as \((x, T_1, z, z^0, T_2, y)\) such that \((z, z^0)\) is the only 0-dimensional edge. (Note that \(x = z\) if \(l(T_1) = 0\) and \(z^0 = y\) if \(l(T_2) = 0\).)

Suppose that there exists a shortest path \(T\) of \(Q_n - F\) joining \(x\) to \(y\) such that the only 0-dimensional edge \((z, z^0)\) joins a black vertex \(z\) to a white vertex \(z^0\). Since \(|F| \leq n - 4\), there exists a vertex \(p^0\) in \(Nbd_{Q_{n-1}^1}(v) - \{z^0\}\) such that \((p, p^0) \notin F\). By
induction, there exist two disjoint paths $S_1$ and $S_2$ such that (1) $S_1$ is a path joining $u$ to $p$ with $l(S_1) = 2^{n-1} - l(T_1) - 2$, (2) $S_2$ is a path joining $x$ to $z$ with $l(S_2) = l(T_1)$, and (3) $S_1 ∪ S_2$ spans $Q^0_{n-1} - F_0$. Again, there exist two disjoint paths $R_1$ and $R_2$ such that (1) $R_1$ is a path joining $p^0$ to $v$ with $l(R_1) = 2^{n-1} - l_2 + l(T_1) - 1$, (2) $R_2$ is a path joining $z^0$ to $y$ with $l(R_2) = l_2 - l(T_1) - 1$, and (3) $R_1 ∪ R_2$ spans $Q^1_{n-1} - F_1$. We set $P_1$ as $(u, S_1, p, p^0, R_1, v)$ and set $P_2$ as $(x, S_2, z, z^0, R_2, y)$. Obviously, $P_1$ and $P_2$ are the required paths. Again, see Figure 7(b) for illustration.

Now, we consider the case that there is no shortest path of $Q_n - F$ joining $x$ to $y$ such that the only 0-dimensional edge $(z, z^0)$ joins a black vertex $z$ in $Q^0_{n-1}$ to a white vertex $z^0$ in $Q^1_{n-1}$. Let $T$ be a shortest path of $Q_n - F$ joining $x$ to $y$. Obviously, both $l(T_1)$ and $l(T_2)$ are even.

(i) $l(T_1) = 0$. Thus, $T_2$ can be written as $\langle z^0 = w_0, w_1, \ldots, w_{2r} = y \rangle$ for some positive integer $r \leq \frac{n-4}{2}$. Obviously, $w_1 = (w_0)^k$ for some index $k$. Since $|F| \leq n - 4$, there exists a black vertex $p$ in $Q^0_{n-1}$ such that $h(p, x) \neq 1$, $p^0 \notin V(T_2)$, and $(p, p^0) \notin F$. Let $F^*_0 = F_0 \cup \{(x, x') | RE_{z}(x') \cap F \neq \emptyset \} \cup \{(x, x^k) \}$. By Lemma 2.7, $|F^*_0| \leq n - 3$. By Lemma 2.1, there exists a hamiltonian path $S$ of $Q^0_{n-1} - F^*_0$ joining $u$ to $p$. Obviously, $S$ can be written as $(u, S_1, x, x', x, \ldots, x', S_1, p)$ such that $k \notin \{i, j\}$. Thus, $w_2 \neq ((x)^0)^j$. We set $F_{av}$ as $\{(z^0, (x)^0) \cup \{(x)^0, ((x)^0)^j) \} \cup \{(w_1, w_2) \}$.

(2) $l(T_1) = 2$. Thus, $T_1$ can be written as $\langle x, q, z \rangle$ and $T_2$ can be written as $\langle z^0 = w_0, w_1, \ldots, w_{2r} = y \rangle$ for some positive integer $r \leq \frac{n-6}{2}$. Obviously, $w_1 = (w_0)^k$ for some index $k$. Let $p$ be a vertex in $Nbd_{Q^0_{n-1}}(x) - \{q\}$ such that $(p, p^0) \notin F$. Suppose that $\{(q, q^0), (q^0, w_0) \} \cap F = \emptyset$. Obviously, $\langle x, q, q^0, w_0, w_1, \ldots, w_{2r} = y \rangle$ is a shortest path of $Q_n - F$ joining $u$ to $y$ such that the only 0-dimensional edge joining a black vertex $q$ in $Q^0_{n-1}$ to a white vertex $q^0$ in $Q^1_{n-1}$, so we get a contradiction. Thus, $\{(q, q^0), (q^0, w_0) \} \cap F \neq \emptyset$. Suppose that $\{(z, z^k), (z^k, w_1) \} \cap F = \emptyset$. Obviously, $\langle x, q, z, (z)^k, w_1, \ldots, w_{2r} = y \rangle$ is a shortest path of $Q_n - F$ joining $x$ to $y$ such that the only 0-dimensional edge joining a black vertex $(z)^k$ in $Q^0_{n-1}$ to a white vertex $w_1$ in $Q^1_{n-1}$, so we get a contradiction. Thus, $\{(z, z^k), (z^k, w_1) \} \cap F \neq \emptyset$. Let $F^*_0 = F_0 \cup \{(z, z^k) | \emptyset \neq q, RE_{z}(z^k) \cap F \neq \emptyset \}$. Obviously, $|F^*_0| \leq n - 5$ and $(z, z^k) \notin F_0^*$. By induction, there exist two disjoint paths $S_1$ and $S_2$ such that (1) $S_1$ is a path joining $u$ to $p$ with $l(S_1) = 2^{n-1} - 3$, (2) $S_2$ is a path joining $x$ to $q$ with $l(S_2) = 1$, and (3) $S_1 ∪ S_2$ spans $Q^0_{n-1} - F^*_0$. Obviously, $S_1$ can be written as $(u, S_1^1, z', z, z^0, S_1^2, p)$ such that $k \notin \{i, j\}$. Thus, $w_2 \neq ((x)^0)^j$. We set $F_{av}$ as $\{(x)^0, (x)^0) \cup \{(x)^0, ((x)^0)^j) \} \cup \{(w_1, w_2) \}$.
(iii) \(l(T_1) \geq 4\). Thus, \(T_1\) can be written as \(\langle x, s, T'_1, q, z \rangle\) and \(T_2\) can be written as \(\langle z^0 = w_0, w_1, \ldots, w_{2r} = y \rangle\) for some positive integer \(r \leq \lceil \frac{n-8}{2} \rceil\). Obviously, \(w_1 = (w_0)^k\) for some index \(k\). Suppose that \(\{(q,q^0), (q^0, w_1)\} \cap F = \emptyset\). Obviously, \(\langle x, s, T'_1, q, q^0, w_1, \ldots, w_{2r} = y \rangle\) is a shortest path of \(Q_n - F\) joining \(x\) to 
\(y\) such that the only 0-dimensional edge joining a black vertex \(q\) in \(Q_{n-1}^0\) to a white vertex \(q^0\) in \(Q_{n-1}^1\), so we get a contradiction. Thus, \(\{(q,q^0), (q^0, w_1)\} \cap F = \emptyset\). Suppose that \(\{(z,z^k), (z^k, w_1)\} \cap F = \emptyset\). Obviously, \(\langle x, q, z, (z)^k, w_1, \ldots, w_{2r} = y \rangle\) is a shortest path of \(Q_n - F\) joining \(x\) to 
\(y\) such that the only 0-dimensional edge joining a black vertex \((z)^k\) in \(Q_{n-1}^0\) to a white vertex \(z\) in \(Q_{n-1}^1\), so we get a contradiction. Thus, \(\{(z,z^k), (z^k, w_1)\} \cap F = \emptyset\). Let \(F^*_0 = F_0 \cup \{(z, z') | z' \neq q, R_{Ez}(z') \cap F = \emptyset\}\). Obviously, \(|F^*_0| \leq n - 5\) and \((z, z^k) \in F^*_0\). Let \(p\) be a vertex in \(\text{Nbd}_{Q_{n-1}^0}(x) - \{q\}\) such that \((p, p^0) \notin F\). By induction, there exist two disjoint paths \(S_1\) and \(S_2\) such that (1) \(S_1\) is a path joining \(x\) to \(p\) with \(l(S_1) = 2^{n-1} - l(T_1)\), (2) \(S_2\) is a path joining \(x\) to \(q\) with \(l(S_2) = l(T_1) - 2\), and (3) \(S_1 \cup S_2\) spans \(Q_{n-1}^0 - F^*_0\). Obviously, \(S_1\) can be written as \(\langle u, S'_1, z^i, z, z^j, S''_1, p, p \rangle\) such that \(k \notin \{i, j\}\). We set \(F_{av}\) as \(\{(z^0, (z^i)^0) \cup \{(z^0, (z^i)^0) \cup \{(z^0, (z^j)^0) \cup \{(w_{2i-1}, w_{2j}) \mid 1 \leq i \leq r\}\}\}\). Obviously, \(|F_1| + |F_{av}| \leq \lceil \frac{n-5}{2} \rceil + \lceil \frac{n-8}{2} \rceil + 2 \leq (n-3)\). By Lemma 2.3, there exists a hamiltonian path \(R\) of \(Q_{n-1}^1 - F_1 - F_{av}\) joining \(p^0\) to \(v\). We set \(P_1\) as \(\langle u, S'_1, z^i, (z^0)^i, (z^0)^j, z^j, S''_1, p, p^0, R, v \rangle\) and set \(P_2\) as \(\langle x, S_2, q, z, z^0, T_2, y \rangle\). Obviously, \(P_1\) and \(P_2\) are the required paths. See Figure 7(e) for illustration.
Appendix B: Proof of the existence of $P_3$ and $P_4$ in Theorem 3.1

Now we prove the existence of $P_3$ and $P_4$. Without loss of generality, we can assume that $l_3 \geq l_4$ and $u \in V(Q_{n-1}^0)$. We need to consider the following eight cases.

**Case 9:** $v \in V(Q_{n-1}^0)$ and $\{x, y\} \subset V(Q_{n-1}^1).

Suppose that $l_4 \geq h(v, y) + 4$. Since $|F| \leq n - 4$, there exists a vertex $p^0$ in $V(Q_{n-1}^1) - \{v^0\}$ such that $(p, p^0) \notin F$ and $d_{Q_{n-1}^1 - \{y\}}(x, p^0) = 2$. Obviously, $p$ is a black vertex. Again, there exists a vertex $z$ in $\text{Nbd}_{Q_{n-1}^1 - F_0}(v) - \{u\}$ such that $(z, z^0) \notin F$. Obviously, $z$ is a white vertex and $d_{Q_{n-1}^1 - F_1}(y, z^0) \leq h(y, z^0)+2 \leq l_4-2$.

By induction, there exist two disjoint paths $S_1$ and $S_2$ such that (1) $S_1$ is a path joining $u$ to $p$ with $l(S_1) = 2^{n-1} - 3$, (2) $S_2$ is a path joining $v$ to $z$ with $l(S_2) = 1$, and (3) $S_1 \cup S_2$ spans $Q_{n-1}^0 - F_0$. Again, there exist two disjoint paths $R_1$ and $R_2$ such that (1) $R_1$ is a path joining $p^0$ to $x$ with $l(R_1) = 2^{n-1} - l_4$, (2) $R_2$ is a path joining $z^0$ to $y$ with $l(R_2) = l_4 - 2$, and (3) $R_1 \cup R_2$ spans $Q_{n-1}^1 - F_1$. We set $P_3$ as $(u, S_1, p, p^0, R_1, x)$ and set $P_4$ as $(v, S_2, z, z^0, R_2, y)$. Obviously, $P_3$ and $P_4$ are the required paths. See Figure 8(a) for illustration.

![Fig. 8. Illustration of Case 9.](image)

Now, we consider $l_4 \leq h(v, y) + 2$. Thus, $l_4 \leq n + 2$. By Lemma 2.6, there exists a shortest path $T$ of $Q_n - F$ joining $v$ to $y$ such that $\{u, x\} \cap V(T) = \emptyset$. Moreover, $|E(T) \cap E_0| = 1$ and $l(T) \in \{h(v, y), h(v, y) + 2\}$. Thus, $T$ can be written as $(v, T_1, z, z^0, T_2, y)$ such that $(z, z^0) \in E_0$. (Note that $v = z$ if $l(T_1) = 0$ and $z^0 = y$ if $l(T_2) = 0$.)

(i) $l(T_1)$ is odd. Since $|F| \leq n - 4$, there exists a vertex $p^0$ in $V(Q_{n-1}^1) - \{v^0\}$ such that $(p, p^0) \notin F$ and $d_{Q_{n-1}^1 - \{y\}}(x, p^0) = 2$. Obviously, $p$ is a black vertex. By induction, there exist two disjoint paths $S_1$ and $S_2$ such that (1) $S_1$ is a path joining $u$ to $p$ with $l(S_1) = 2^{n-1} - l(T_1) - 2$, (2) $S_2$ is a path joining $v$ to $z$ with $l(S_2) = l(T_1)$, and (3) $S_1 \cup S_2$ spans $Q_{n-1}^0 - F_0$. Again, there exist two disjoint paths $R_1$ and $R_2$ such that (1) $R_1$ is a path joining $p^0$ to $x$ with $l(R_1) = 2^{n-1} - l_4 + l(T_1) - 1$, (2) $R_2$ is a path joining $z^0$ to $y$ with $l(R_2) = l_4 - l(T_1) - 1$, and (3) $R_1 \cup R_2$ spans $Q_{n-1}^1 - F_1$. We set $P_3$ as $(u, S_1, p, p^0, R_1, x)$ and set $P_4$ as $(v, S_2, z, z^0, R_2, y)$. Obviously, $P_3$ and $P_4$ are the required paths. See Figure 8(b) for illustration.
\[(ii) l(T_1) \text{ is even. Since } |F| \leq n - 4, \text{ there exists a white vertex } p \text{ in } V(Q_{n-1}^0) - \{u\} \text{ such that } (p p^0) \notin F.\]

Suppose that \(l(T_1) = 0\). By Lemma 2.2, there exist a hamiltonian path \(S\) of \(Q_{n-1}^0 - F_0 - \{v\}\) joining \(u\) to \(p\). By induction, there exist two disjoint paths \(R_1\) and \(R_2\) such that (1) \(R_1\) is a path joining \(p^0\) to \(x\) with \(l(R_1) = 2^{n-1} - 1 - 1\), (2) \(R_2\) is a path joining \(z^0\) to \(y\) with \(l(R_2) = l_4 - 1\), and (3) \(R_1 \cup R_2\) spans \(Q_{n-1}^0 - F_1\). We set \(P_3 = (u, S, p, p^0, R_1, x)\) and set \(P_4 = (v = z^0, R_2, y)\). Obviously, \(P_3\) and \(P_4\) are the required paths. See Figure 8(c) for illustration.

Suppose that \(l(T_1) \neq 0\). By induction, there exist two disjoint paths \(S_1\) and \(S_2\) such that (1) \(S_1\) is a path joining \(u\) to \(p\) with \(l(S_1) = 2^{n-1} - l(T_1) - 2\), (2) \(S_2\) is a path joining \(v\) to \(z\) with \(l(S_2) = l(T_1)\), and (3) \(S_1 \cup S_2\) spans \(Q_{n-1}^0 - F_1\). Again, there exist two disjoint paths \(R_1\) and \(R_2\) such that (1) \(R_1\) is a path joining \(p^0\) to \(x\) with \(l(R_1) = 2^{n-1} - l_4 + l(T_1) - 1\), (2) \(R_2\) is a path joining \(z^0\) to \(y\) with \(l(R_2) = l_4 - l(T_1) - 1\), and (3) \(R_1 \cup R_2\) spans \(Q_{n-1}^0 - F_1\). We set \(P_3 = (u, S_1, p, p^0, R_1, x)\) and set \(P_4 = (v, S_2, z, z^0, R_2, y)\). Obviously, \(P_3\) and \(P_4\) are the required paths. See Figure 8(d) for illustration.

**Case 10:** \(\{x, y\} \subset V(Q_{n-1}^0)\) and \(v \in V(Q_{n-1}^1)\).

Suppose that \(l_4 \geq h(v, y) + 4\). Since \(|F| \leq n - 4\), there exists a vertex \(z\) in \(V(Q_{n-1}^0)\) such that \((z, z^0) \notin F, d_{Q_{n-1}^0 - (u, x)}(y, z) = 2\) and \(d_{Q_{n-1}^1 - F_1}(v, z^0) \leq h(v, z^0) + 2 \leq l_4 - 2\). Obviously, \(z\) is a black vertex. By induction, there exist two disjoint paths \(S_1\) and \(S_2\) such that (1) \(S_1\) is a path joining \(u\) to \(x\) with \(l(S_1) = 2^{n-1} - 4\), (2) \(S_2\) is a path joining \(z\) to \(y\) with \(l(S_2) = 2\), and (3) \(S_1 \cup S_2\) spans \(Q_{n-1}^0 - F_0\). Obviously, we can write \(S_1\) as \((u, S_1^0, r, s, S_1^0, x)\) for some vertices \(r\) and \(s\) with \((r, r^0) \notin F, (s, s^0) \notin F\). By induction, there exist two disjoint paths \(R_1\) and \(R_2\) such that (1) \(R_1\) is a path joining \(r^0\) to \(s^0\) with \(l(R_1) = 2^{n-1} - l_4 + 1\), (2) \(R_2\) is a path joining \(v\) to \(z^0\) with \(l(R_2) = l_4 - 3\), and (3) \(R_1 \cup R_2\) spans \(Q_{n-1}^0 - F_1\). We set \(P_3 = (u, S_1^0, r, r^0, R_1, s, S_2^0, x)\) and set \(P_4 = (v, R_2, z^0, S_2, y)\). Obviously, \(P_3\) and \(P_4\) are the required paths. See Figure 9(a) for illustration.

Now, we consider \(l_4 \leq h(v, y) + 2\). Thus, \(l_4 \leq n + 2\). By Lemma 2.6, there exists a shortest path \(T\) of \(Q_n - F\) joining \(v\) to \(y\) such that \(\{u, x\} \cap V(T) = \emptyset\). Moreover, \(|E(T) \cap E_0| = 1\) and \(l(T) \in \{h(v, y), h(v, y) + 2\}\). Thus, \(T\) can be written as \((v, T_1, z^0, z, T_2, y)\) such that \((z, z^0) \in E_0\). (Note that \(v = z^0\) if \(l(T_1) = 0\) and
\[ \textbf{Case 11:} \{v, x, y\} \subseteq V(Q_{n-1}^0) \]

Suppose that \( l_4 \leq 2^{n-2} - 2 \). By induction, there exist two disjoint paths \( S_1 \) and \( S_2 \) such that (1) \( S_1 \) is a path joining \( u \) to \( x \) with \( l(S_1) = 2^{n-1} - l_4 - 2 \), (2) \( S_2 \) is a path joining \( v \) to \( y \) with \( l(S_2) = l_4 \), and (3) \( S_1 \cup S_2 \) spans \( Q_{n-1}^0 - F_0 \). Again, we can write \( S_1 \) as \( \langle u, s_1, r, r_0, r, s, s_0, x \rangle \) for some vertices \( r \) and \( s \) with \( (r, r_0) \notin F \) and \( (s, s_0) \notin F \). By induction, there exist two disjoint paths \( R_1 \) and \( R_2 \) such that (1) \( R_1 \) is a path joining \( r_0 \) to \( s_0 \) with \( l(R_1) = 2^{n-1} - l_4 + l(T_2) - 1 \), (2) \( R_2 \) is a path joining \( v \) to \( z^0 \) with \( l(R_2) = l_4 - l(T_2) - 1 \), and (3) \( R_1 \cup R_2 \) spans \( Q_{n-1}^1 - F_1 \). We set \( P_3 \) as \( \langle v, R_2, z^0, z, S_2^3, y \rangle \) and \( P_4 \) as \( \langle v, R_2, z^0, z, S_2^3, y \rangle \). Obviously, \( P_3 \) and \( P_4 \) are the required paths. See Figure 9(d) for illustration.
Suppose that \( l_4 \geq 2^{n-2} \). By induction, there exist two disjoint paths \( S_1 \) and \( S_2 \) such that (1) \( S_1 \) is a path joining \( u \) to \( x \) with \( l(S_1) = 2^{n-2} \), (2) \( S_2 \) is a path joining \( v \) to \( y \) with \( l(S_2) = 2^{n-2} - 2 \), and (3) \( S_1 \cup S_2 \) spans \( Q_{n-1}^0 - F_0 \). Again, we can write \( S_1 \) as \( \langle u, S_1^1, r, s, S_1^2, x \rangle \) for some vertices \( r \) and \( s \) with \( (r, r^0) \notin F \) and \((s, s^0) \notin F \). Similarly, we can write \( S_2 \) as \( \langle v, S_2^1, p, q, S_2^2, y \rangle \) for some vertices \( p \) and \( q \) with \((p, p^0) \notin F \), \((q, q^0) \notin F \), and \((p^0, q^0) \notin F \). By induction, there exist two disjoint paths \( R_1 \) and \( R_2 \) such that (1) \( R_1 \) is a path joining \( r^0 \) to \( s^0 \) with \( l(R_1) = 2^{n-1} - l_4 + 2^{n-2} - 3 \), (2) \( R_2 \) is a path joining \( p^0 \) to \( q^0 \) with \( l(R_2) = l_4 - 2^{n-2} + 1 \), and (3) \( R_1 \cup R_2 \) spans \( Q_{n-1}^1 - F_1 \). We set \( P_1 \) as \( \langle u, S_1^1, r, r^0, R_1, s^0, s, S_1^2, x \rangle \) and set \( P_2 \) as \( \langle v, S_2^1, p, p^0, R_2, q^0, q, S_2^2, y \rangle \). Obviously, \( P_1 \) and \( P_2 \) are the required paths. See Figure 10(b) for illustration.

**Case 12**: \( \{v, x, y\} \subset V(Q_{n-1}^1) \). Since \( |F| \leq n - 4 \), there exists a vertex \( p \in V(Q_{n-1}^1) \) such that \((p, p^0) \notin F \) and \( d_{Q_{n-1}^1 - F_1 - \{v, y\}}(x, p) = 2 \).

Suppose that \( l_2 < 2^{n-1} - 2 \). By induction, there exist two disjoint paths \( S_1 \) and \( S_2 \) such that (1) \( S_1 \) is a path joining \( p \) to \( x \) with \( l(S_1) = 2^{n-1} - l_4 - 2 \), (2) \( S_2 \) is a path joining \( v \) to \( y \) with \( l(S_2) = l_4 \), and (3) \( S_1 \cup S_2 \) spans \( Q_{n-1}^1 - F_1 \). By Lemma 2.1, there exists a hamiltonian path \( R \) of \( Q_{n-1}^0 - F_0 \) joining \( u \) to \( p^0 \). We set \( P_3 \) as \( \langle u, R, p^0, p, S_1, x \rangle \) and set \( P_4 \) as \( S_2 \). Obviously, \( P_3 \) and \( P_4 \) are the required paths. See Figure 11(a) for illustration.

Suppose that \( l_2 = 2^{n-1} - 2 \). By induction, there exist two disjoint paths \( S_1 \) and \( S_2 \) such that (1) \( S_1 \) is a path joining \( p \) to \( x \) with \( l(S_1) = 2^{n-1} - 4 \), (2) \( S_2 \) is a path joining \( v \) to \( y \) with \( l(S_2) = 2^{n-1} - 4 \), and (3) \( S_1 \cup S_2 \) spans \( Q_{n-1}^1 - F_1 \). We can write \( S_2 \)
as \(\langle v, S_1^2, r, s, S_2^2, y \rangle\) for some vertices \(r\) and \(s\) such that \(u \notin \{r^0, s^0\}, (r, r^0) \notin F, (s, s^0) \notin F\). By induction, there exist two disjoint paths \(R_1\) and \(R_2\) such that (1) \(R_1\) is a path joining \(u\) to \(p^0\) with \(l(R_1) = 2^{n-1} - 3\), (2) \(R_2\) is a path joining \(r^0\) to \(s^0\) with \(l(R_2) = 1\), and (3) \(R_1 \cup R_2\) spans \(Q_{n-1}^0 - F_0\). We set \(P_3\) as \(\langle u, R_1, p^0, p, S_1, x \rangle\) and set \(P_4\) as \(\langle v, S_1^2, r, r^0, s^0, s, S_2^2, y \rangle\). Obviously, \(P_3\) and \(P_4\) are the required paths. See Figure 11(b) for illustration.

**Case 13:** \(y \in V(Q_{n-1}^0)\) and \(\{v, y\} \subset V(Q_{n-1}^1)\). This case is symmetric with Case 9.

**Case 14:** \(\{v, y\} \subset V(Q_{n-1}^0)\) and \(x \in V(Q_{n-1}^1)\). This case is symmetric with Case 12.

**Case 15:** \(\{v, x\} \subset V(Q_{n-1}^0)\) and \(y \in V(Q_{n-1}^1)\). This case is symmetric with Case 10.

**Case 16:** \(x \in V(Q_{n-1}^0)\) and \(\{y, v\} \subset V(Q_{n-1}^1)\). Since \(|F| \leq n - 4\), there exist two vertices \(r\) and \(s\) in \(Q_{n-1}^0\) such that \((x, s) \in E(Q_{n-1}^0 - F_0)\), \((r, r^0) \notin F, (s, s^0) \notin F\) and \(d_{Q_{n-1}^0 - F_1}(r^0) = 2\). Obviously, \(r\) and \(s\) are black vertices in \(Q_{n-1}^0\).

Suppose that \(l_4 < 2^{n-1} - 2\). By induction, there exist two disjoint paths \(R_1\) and \(R_2\) such that (1) \(R_1\) is a path joining \(u\) to \(r\), (2) \(R_2\) is a path joining \(s\) to \(x\), and (3) \(R_1 \cup R_2\) spans \(Q_{n-1}^0 - F_0\). Again by induction, there exist two disjoint paths \(S_1\) and \(S_2\) such that (1) \(S_1\) is a path joining \(r^0\) to \(s^0\) with \(l(S_1) = 2^{n-1} - l_4 - 2\), (2) \(S_2\) is a path joining \(v\) to \(y\) with \(l(S_2) = l_4\), and (3) \(S_1 \cup S_2\) spans \(Q_{n-1}^0 - F_1\). We set \(P_3\) as \(\langle u, R_1, r, r^0, S_1, s^0, s, R_2, x \rangle\) and set \(P_4\) as \(S_2\). Obviously, \(P_3\) and \(P_4\) are the required paths. See Figure 12(a) for illustration.

Fig. 12. Illustration of Case 16.

Suppose that \(l_2 = 2^{n-1} - 2\). Let \(t^0\) be a common neighbor of \(r^0\) and \(s^0\) such that \(v \notin \{v, y\}\). Obviously, \(t^0\) is a black vertex in \(Q_{n-1}^1\). We set \(F_{av}^1\) as \(\{r^0, t^0\}\).

Obviously, \(|F_1_l| + |F_{av}^1| \leq (n-5) + 1 \leq n - 4\). By Lemma 2.4, there exists a hamiltonian path \(S\) of \(Q_{n-1}^0 - F_1 - F_{av}^1\). We can write \(S\) as \(\langle v, S_1^1, p^0, q^0, S_2^2, y \rangle\) for some vertices \(p^0\) and \(q^0\) such that \(\{p, q\} \cap \{u, x\} = \emptyset, (p, p^0) \notin F, (q, q^0) \notin F, (p, q) \notin F\). We set \(F_{av}^0\) as \(\{\{x, s\}, \{p, q\}\}\). Obviously, \(|F_0_l| + |F_{av}^0| \leq (n-5) + 2 \leq n - 3\). By Lemma 2.3, there exists a hamiltonian path \(R\) of \(Q_{n-1}^0 - F_0 - F_{av}^0\) joining \(u\) to \(r\). We set \(P_3\) as \(\langle u, R, r, r^0, t^0, s^0, s, x \rangle\) and set \(P_4\) as \(\langle v, S_1^1, p^0, p, q, q^0, S_2^2, y \rangle\). Obviously, \(P_3\) and \(P_4\) are the required paths. See Figure 12(b) for illustration.
References