We analyze a dynamic queue-storage problem where the arrival and departure processes are those of the single-server Poisson (M/M/1) queue. The queue is stored in a linear array of cells numbered 1, 2, 3, ... , with at most one item (customer) per cell. The storage policy is first-fit, i.e., an item is placed at the time of its arrival into the lowest numbered unoccupied cell, where it remains until it is served and departs.

Let \( S(t) \) be the set of occupied cells at time \( t \), and define the wasted space as \( W(t) = \max S(t) - |S(t)| \), i.e. \( W(t) \) is the number of interior unoccupied cells. We analyze wasted space under the first-in-first-out (FIFO) and processor-sharing (PS) service disciplines. The results are expressed in terms of the ‘traffic intensity’ measure \( n = \lim_{t \to \infty} E[S(t)] \), i.e. the expected number in the system in statistical equilibrium. An asymptotic analysis of the steady state provides the following two tight bounds:

\[
EW^{\text{FIFO}}_n = O(\sqrt{n}),
\]

\[
EW^{\text{PS}}_n = O(\sqrt{n \log n}).
\]

These results are to be compared with the corresponding result, \( O(\sqrt{n \log \log n}) \), already known for the infinite server queue. In proving the new bounds, we also obtain estimates of the tails of the distributions of wasted space.

Dynamic storage allocation in computers is an important application of the above results. The bounds show that, on average, wasted space is asymptotically a negligible fraction of the total space spanned by the queue. This in turn means that in heavy traffic time-consuming compaction (garbage collection) schemes can have very little effect on storage efficiency.
(ii) An item is placed in a cell at the time of its arrival, and remains there until it is served and departs.

(iii) The arrival process is Poisson and service times are independent samples from a given exponential distribution.

In the computer application, the common item size can be taken to be any unit of information such as a byte or a fixed-length record or page.

Within this set-up, models are distinguished by their service discipline and storage policy. By the latter term, we mean the decision rule that selects the empty cells in which new arrivals are placed. Our analysis focuses on the policy most often proposed, viz. the first-fit rule, whereby an arrival is placed in the lowest numbered, unoccupied cell. We analyze two basic service disciplines: first-in-first-out (FIFO) and processor-sharing (PS).

The PS discipline models the round-robin rule of time-sharing systems (Kleinrock, 1975). According to the PS discipline, over any time interval \([t, t + T]\) during which the number in storage stays constant at \(m \geq 1\), each item in storage receives \(T/m\) units of service time. In other words, while there are \(m \geq 1\) items in storage, each item is receiving service at \((1/m)\)th the rate it would receive service were it alone in storage.

We are concerned primarily with the behavior of these systems in heavy traffic. For this reason, it is convenient to normalize the expected service time to 1, and to let \(1 - 1/n\) denote the arrival rate, with \(n > 1\). Heavy traffic then corresponds to large \(n\). For given \(n\), the processes of interest include \(Q_n(t)\), the number of occupied cells (queue-length) at time \(t\); \(H_n(t)\), the number of the highest occupied cell at time \(t\); and \(W_n(t) = H_n(t) - Q_n(t)\), the number of (interior) unoccupied cells with numbers less than \(H_n(t)\). We call \(W_n(t)\) the wasted space and define \(W_n(t) = H_n(t) = 0\) if \(Q_n(t) = 0\). When we omit the dependence on \(t\) from our process notation, we refer to a random variable with the stationary distribution, assuming that one exists.

The same Markov queue-length process \(Q_n(t)\) describes both the FIFO and PS systems. Classical results show that a unique stationary distribution exists for all \(n > 1\). The distribution \(q_i, i \geq 0\), and the tail probabilities are given by (Kleinrock, 1975)

\[
q_i = \frac{1}{n} \left(1 - \frac{1}{n}\right)^i, \quad i \geq 0,
\]

(1.1)

\[
\Pr\{Q_n \geq j\} = \sum_{i \geq j} q_i = (1 - 1/n)^j.
\]

Note that our measure of traffic intensity, \(n\), replaces the more commonly used ratio of arrival rate to service rate, \(1 - 1/n\). By (1.1),

\[
E[Q_n] = n - 1,
\]

(1.2)

so that asymptotics in \(n\) are asymptotics in the mean number in system.

Unfortunately, \(H_n(t)\) and \(W_n(t)\) are not Markov processes in the FIFO and PS systems, and finding explicit results for stationary distributions seems to be very
difficult. In the PS system a Markov process can be defined on the state space consisting of the class of all finite sets of integers denoting occupied cells. For some comments on this process, see Coffman et al. (1988). The FIFO system is even more difficult; the set of occupied cells must be augmented by a linear order by time of arrival so as to become a Markov process. Thus, we turn to asymptotic methods and prove the following bounds in Sections 3 and 4.

**Theorem 1.** For the FIFO system, there exists a $c > 0$ such that for all $n$ sufficiently large
\[
\Pr \{W_n \geq k\sqrt{n}\} = O(e^{-ck}).
\]
In addition,
\[
E[W_n] = \Theta(\sqrt{n}).
\]
i.e., there exist constants $c$, $c' > 0$ such that $c'\sqrt{n} \leq E[W_n] \leq c\sqrt{n}$ for all $n$ sufficiently large.

The multiplicative constant hidden in the big-oh notation of (1.3) is to be interpreted as independent of $n$ as well as $k$. Precisely, in addition to $c > 0$, this means that there exists a $C > 0$ such that $\Pr \{W_n \geq k\sqrt{n}\} \leq C e^{-ck}$ for all $n$ sufficiently large and for all $k \geq 0$. Since probabilities are at most 1, it suffices to prove the existence of such a $C$ for all $k$ and $n$ sufficiently large. The above meaning of the big-oh notation will also apply to similar bounds throughout the paper.

**Theorem 2.** For the PS system, there exists a $c > 0$ such that for all $n$ sufficiently large
\[
\Pr \{W_n \geq k\sqrt{n \log n}\} = O(e^{-ck'/3}).
\]
In addition,
\[
E[W_n] = \Theta(\sqrt{n \log n}).
\]

The following heuristic comments offer an explanation of the bounds (1.4) and (1.6). Let $T_c$ be the time it takes the system to clear itself of all its jobs. Simple computations show that for FIFO and PS we have respectively
\[
E(T_c) = \Theta(n), \quad E(T_c) = \Theta(n \log n),
\]
so that (1.4) and (1.6) state
\[
E(W) = \Theta(\sqrt{E(T_c)}).
\]
The intuition for (1.8) is that $Q_n(t)$ is similar to a random walk with a slight bias, and hence holes of cumulative size $\Theta(\sqrt{T})$ are likely to open up over $T$ steps. These holes represent wasted space until the outermost jobs are completed, which happens roughly at time $T_c$. Hence the wasted space will be $\Theta(\sqrt{E(T_c)})$. Formally verifying this simplistic intuition is a tricky endeavor, however, and is the task of this paper.

\footnote{To avoid a proliferation of constants, we use $c$ generically; i.e., unless stated otherwise, its value in one place need not be the same as in another.}
In preparation for the proofs in Sections 3 and 4, several preliminary results are presented in Section 2. Before getting into these results, we conclude this section with a brief discussion of related literature.

This paper extends the analysis of the single-server storage systems described in Coffman et al. (1988). For the PS system, it is shown that as \( n \to \infty \),
\[
\frac{1}{2} \sqrt{n} \approx E[W_n] = (\frac{1}{6}n^2 - 1)n. \tag{1.9}
\]
This result is to be compared with Theorem 2 above, which shows that asymptotically the proper dependence of \( E[W_n] \) on \( n \) is \( \Theta(\sqrt{n \log n}) \). The methods used in Coffman et al. (1988) are quite different from those used here. In that paper, tractable Markov processes \( W'_n(t) \) and \( W''_n(t) \) are defined so that \( W'_n(t) \) dominates \( W'_n(t) \) and is dominated by \( W''_n(t) \) throughout any sample path of the process. Analysis of \( W'_n(t) \) and \( W''_n(t) \) then leads to the bounds in (1.9).

In the infinite-server variation of our model, an item is assumed to begin its service time at the instant it arrives and is placed in a cell. This model was first studied by Kosten (1937) (see also the monograph by Newell (1984) on this problem). In a heavy-traffic analysis, the expected service time is again normalized to 1, but the arrival rate is taken to be \( n \). With these assumptions the average number in system is \( n \).

The infinite-server model has proved to be less difficult to analyze in that explicit results have been obtained for the distribution \( \Pr\{H_n > i\} \) (Coffman et al., 1985). By means of diffusion limits, Aldous (1986) has shown that\(^2\)
\[
E[W_n] \sim \sqrt{2n \log \log n} \quad \text{as } n \to \infty. \tag{1.10}
\]
(A much simpler proof of \( E[W_n] = \Theta(\sqrt{n \log \log n}) \) can be found in [6].) Thus, an asymptotically linear or nearly linear dependence on the square root of the average number in system describes the expected wasted space in the infinite-server as well as the FIFO and PS systems.

The infinite-server model has been extended in Coffman and Leighton (1989) to systems in which queued items can occupy more than one cell; the numbers of cells required by arriving items are independent samples from a given distribution and the cells occupied by an item must be contiguous. An asymptotic analysis of an efficient storage policy is worked out in Coffman and Leighton (1989). This type of extension to single-server models remains an interesting open problem.

2. Preliminary results

The proofs of Theorems 1 and 2 require a number of asymptotic properties of sums of independent, identically distributed (i.i.d.) bounded random variables. Our first lemma provides a very useful bound on the tails of the distributions. For completeness, we provide a compact proof tailored to our specific needs.

\(^2\) Further results of the same type have recently been found by Cinlar (1987).
Define the normalized sum \( S_m = S_m - m\mu \), where
\[
S_m = \sum_{i=1}^{m} X_i,
\]
and where the \( X_i \)'s are independent samples from a given distribution, with \( |X_i| \leq 1 \), \( 1 \leq i \leq m \), and \( \mu = E(X_i) \).

**Lemma 1** (Hoeffding, 1963). The following bounds hold for \( m \geq 1 \) and \( x > 0 \), with \( m \) an integer and \( x \) real:
\[
\begin{align*}
\Pr \{ S_m \geq x \sqrt{m} \} &\leq e^{-x^2/2}, \\
\Pr \{ S_m \leq -x \sqrt{m} \} &\leq e^{-x^2/2}.
\end{align*}
\]

**Proof.** We begin with the usual starting point for bounds of the above type. Let \( \lambda > 0 \). Since \( e^{\lambda (S_m - t)} \geq 1 \) when \( S_m - t \equiv 0 \), we have
\[
\Pr[S_m \geq t] \leq E[e^{\lambda (S_m - t)}],
\]
and hence
\[
\Pr[S_m \geq t] \leq e^{-\lambda t}E[e^{\lambda \sum(X_i - \mu)}] = e^{-\lambda t}[\phi(\lambda)]^m.
\]
where
\[
\phi(\lambda) = e^{-\lambda \mu}E(e^{\lambda X_i}).
\]

Now \( e^{\lambda x} \) is convex in \( x \), so that
\[
e^{\lambda X_i} \leq \frac{1}{2} e^{-\lambda (1 - X_1)} + \frac{1}{2} e^{\lambda (1 + X_1)}, \quad |X_1| \leq 1.
\]
Substituting in (2.4), we have
\[
\phi(\lambda) \leq e^{-\lambda \mu} \left[ \frac{1 - \mu}{2} e^{-\lambda} + \frac{1 + \mu}{2} e^{\lambda} \right] = e^{L},
\]
where
\[
L = L(\lambda, \mu) = -(1 + \mu)\lambda + \log \left[ \frac{1 - \mu}{2} + \frac{1 + \mu}{2} e^{2\lambda} \right].
\]
The first two derivatives of \( L \) with respect to \( \lambda \) are
\[
\frac{\partial L}{\partial \lambda} = -(1 + \mu) + \frac{1 + \mu}{\frac{1}{2}(1 - \mu) e^{-2\lambda} + \frac{1}{2}(1 + \mu)}
\]
and
\[
\frac{\partial^2 L}{\partial \lambda^2} = \frac{(1 + \mu)(1 - \mu) e^{-2\lambda}}{[\frac{1}{2}(1 + \mu + (1 - \mu) e^{-2\lambda})]^2} \leq 1,
\]
which we note is the geometric-arithmetic mean inequality. We observe that \( L(0, \mu) = (\partial L/\partial \lambda)(0, \mu) = 0 \). Hence, integrating twice the inequality (2.6) gives \( \phi(\lambda) \leq e^{\lambda^2/2} \), whereupon substitution in (2.4) gives \( \Pr[S_m \geq t] \leq e^{\lambda^2/2 - \lambda t} \). We prove (2.2) by putting \( \lambda = t/m \) and \( t = x \sqrt{m} \); (2.3) follows from (2.2) upon replacing \( X_i \) by \( -X_i \).
We also need probability bounds for the maximum and minimum of the partial sums $S_j$, $1 \leq j \leq m$. The following result combines Skorokhod's inequality (see Breiman (1968)) with the bounds in Lemma 1.

**Lemma 2.** For any $x \geq 0$, we have

$$\Pr\left(\max_{1 \leq i \leq m} \hat{S}_i \geq x \sqrt{m}\right) \leq 2e^{-x^2/8} \tag{2.7}$$

and

$$\Pr\left(\min_{1 \leq i \leq m} \hat{S}_i \leq -x \sqrt{m}\right) \leq 2e^{-x^2/8}. \tag{2.8}$$

**Proof.** Let $2t = x\sqrt{m}$ and define the event

$$C_j = \{\hat{S}_j \geq 2t \text{ and } \hat{S}_k < 2t \text{ for all } 1 \leq k < j\}.$$

Then

$$\left\{\max_{1 \leq i \leq m} \hat{S}_i \geq 2t\right\} = \bigcup_{j=1}^{m} C_j,$$

and since the $C_j$'s are disjoint,

$$\Pr\left(\max_{1 \leq i \leq m} \hat{S}_i \geq 2t\right) = \sum_{j=1}^{m} \Pr(C_j). \tag{2.9}$$

Now $C_j$ and $\{\hat{S}_m - \hat{S}_j \geq -t\}$ are independent events, and it is clear that

$$\bigcup_{j=1}^{m} \{C_j, \hat{S}_m - \hat{S}_j \geq -t\} \subset \{\hat{S}_m \geq t\}.$$

Hence,

$$\Pr(\hat{S}_m \geq t) \geq \sum_{j=1}^{m} \Pr(C_j) \Pr(\hat{S}_m - \hat{S}_j \geq -t) = \sum_{j=1}^{m} \Pr(C_j) \Pr(\hat{S}_m - \hat{S}_j \geq -t). \tag{2.10}$$

By Lemma 1,

$$\Pr(\hat{S}_m \geq t) \leq e^{-t^2/(2m)}$$

and

$$\Pr(\hat{S}_m - \hat{S}_j \geq -t) = 1 - \Pr(\hat{S}_m - \hat{S}_j < -t) \geq 1 - e^{-t^2/(2(m-j))} \geq 1 - e^{-t^2/(2m)}.$$

Substituting these bounds into (2.10), we obtain

$$e^{-t^2/(2m)} \geq [1 - e^{-t^2/(2m)}] \sum_{j=1}^{m} \Pr(C_j). \tag{2.11}$$
From (2.9) and (2.11) we get
\[
\Pr\{\max \hat{S}_i \geq x\sqrt{m}\} \leq \min\left(1, \frac{e^{-x^2/8}}{1 - e^{-x^2/8}}\right) \leq 2e^{-x^2/8},
\]
which is (2.7). Finally, (2.8) follows from (2.7) upon replacing \(X_i\) by \(-X_i\). \(\square\)

Lemmas 1 and 2 will often be used when \(X_i = \pm 1, 1 \leq i \leq m\). In this case \(\hat{S}_j\) may be interpreted as the deviation from the mean after \(j\) steps of an unrestricted (or free) random walk starting at the origin.

The following classical bound for Poisson distributed random variables can be found in Feller (1968). The proof, which we omit, is a standard application of Stirling’s formula.

**Lemma 3.** Let \(\Pr\{N = k\} = e^{-\lambda} \frac{\lambda^k}{k!}, k \geq 0\). Then there exists \(c > 0\) such that
\[
\Pr\{\frac{1}{2} \lambda < N < 2\lambda\} = 1 - O(e^{-c\lambda}).
\]

In the proofs of Theorems 1 and 2 we will analyze several properties of \(Q_n(t)\) over a fixed interval \([0, t]\). First, define
\[
U_n(t) = \sup_{0 \leq \tau \leq t} Q_n(\tau) - Q_n(0) \geq 0
\]
and
\[
D_n(t) = Q_n(0) - \inf_{0 \leq \tau \leq t} Q_n(\tau) \geq 0
\]
as the upward and downward excursions from \(Q_n(0)\) in \([0, t]\), respectively. Define the maximum downward excursion in \([0, t]\) over all starting points in \([0, t]\) as
\[
V_n(t) = \sup_{0 \leq t_1 \leq t_2 \leq t} [Q_n(t_1) - Q_n(t_2)].
\]
Trivially, we have for all \(n\) and \(t\)
\[
D_n(t) \leq V_n(t).
\]

We now show how the analysis of \(Q_n(t)\) can be reduced to the analysis of a corresponding, discrete-time random walk. Let \(N_n(t), t \geq 0\), denote the Poisson process with rate parameter \(2 - 1/n\), i.e.,
\[
\Pr\{N_n(t) = k\} = e^{-(2-1/n)t} \frac{(2-1/n)^k}{k!}, \quad k \geq 0.
\]
The discrete-time random walk \(Q_n^*(m), m \geq 0\), is defined as follows. We assume \(Q_n^*(0) = 0\). If \(Q_n^*(m-1) > 0\), then the \(m\)th step (at integer time \(m \geq 1\)) is +1 and −1 with probabilities \((1 - 1/n)/(2 - 1/n)\) and \(1/(2 - 1/n)\), respectively. If \(Q_n^*(m-1) = 0\), then the \(m\)th step is +1 and 0 with probabilities \((1 - 1/n)/(2 - 1/n)\) and \(1/(2 - 1/n)\), respectively. Observe that the limiting case \(n \to \infty\) yields the symmetric random walk with transition probabilities \(\frac{1}{2}, \frac{1}{2}\). In this case we denote \(Q_n^*(m)\) simply as \(Q_n^*(m)\). Define the excursions \(U_n^*(m)\) and \(D_n^*(m)\) of \(Q_n^*(m)\) in analogy with (2.12) and (2.13). Again, for \(n = \infty\) we drop the subscript from \(U_n^*(m)\) and \(D_n^*(m)\). We obtain the following result (see also Keilson (1979)).
Lemma 4. If $Q_n(0) = Q_n^+(0)$, then

$$Q_n(t) = Q_n^+(N_n(t)), \quad U_n(t) = U_n^+(N_n(t)), \quad D_n(t) = D_n^+(N_n(t)).$$ (2.17)

Proof. Suppose we randomize the times between the steps of $Q_n^+$ so that they form a sequence of i.i.d. exponential random variables with parameter $2 - 1/n$. We obtain a continuous-time jump process $Q_n^+(t)$. By definition, we have $Q_n^+(t) = Q_n^+(N_n(t))$. On the other hand, it is easy to verify that the infinitesimal generators of $Q_n(t)$ and $Q_n^+(t)$ are the same. The lemma follows. \qed

The next lemma provides a useful stochastic monotonicity result for random walks.

Let $S_m(p)$, $m \geq 0$, be a random walk on the non-negative integers with the transition probabilities $\pi_{i+1} = p$, $i \geq 0$, $\pi_{i-1} = q$, $i \geq 1$, and $\pi_{00} = q$, where $p, q \geq 0$ and $p + q = 1$.

Lemma 5. Let $x$ and $y$ be non-negative integers with $y > x$. Let $v(n, x, y, p)$ be the probability that the random walk $S_m(p)$, starting at $x$, visits $y$ within the first $n$ steps. Then $v$ is monotone increasing in $p$, i.e., if $0 \leq p_1 < p_2 \leq 1$ then

$$v(n, x, y, p_1) \leq v(n, x, y, p_2).$$ (2.18)

Remark. This result expresses the intuitive fact that the probability of a specific upward excursion increases with the upward drift.

Proof. We prove (2.18) by induction on $n$. The result is trivial for $n = 0$ and for $x = y$. Suppose it holds for some $n \geq 0$; we show that it persists for $n + 1$ and $x < y$. Suppressing $y$ from the notation, we can write

$$v(n + 1, x, p) = pv(n, x + 1, p) + qv(n, x - 1, p), \quad 1 \leq x < y,$$ (2.19)

$$v(n + 1, 0, p) = pv(n, x + 1, p) + qv(n, 0, p).$$

The desired monotonicity of $v$ with respect to $p$ will follow from (2.19) and its monotonicity with respect to $n$ and $x$. Monotonicity in $n$ is clear. For $0 \leq x_1 < x_2 \leq y$, let $w(k, x_1, x_2, p)$ denote the probability that the random walk $S_m(p)$, starting at $x_1$, first visits $x_2$ on the $k$th step. If a walk starts at $x_1$, $x_1 < x_2$, and visits $y$, $y > x_2$, then it must first pass through $x_2$. Hence, from the monotonicity in $n$,

$$v(n, x_1, p) = \sum_{k=0}^{n} w(k, x_1, x_2, p) v(n - k, x_2, p) \leq v(n, x_2, p),$$ (2.20)

thus establishing the monotonicity with respect to $x$, $0 \leq x < y$. From (2.19) and the monotonicity in $x$, we obtain for $1 \leq x \leq y$, $0 \leq p_1 < p_2 \leq 1$,

$$v(n + 1, x, p_1) = p_1 v(n, x + 1, p_1) + q_1 v(n, x - 1, p_1)$$

$$\leq p_2 v(n, x + 1, p_1) + q_2 v(n, x - 1, p_1)$$

$$\leq p_2 v(n, x + 1, p_2) + q_2 v(n, x + 1, p_2) = v(n + 1, x, p_2).$$

The same reasoning yields $v(0, x, p_1) \leq v(0, x, p_2).$ \qed
Lemma 6. Let $Q_n^*(0) = x$, and let $S_k = \sum_{i=1}^{k} X_i$, with $S_0 = 0$ and $\Pr\{X_i = +1\} = \Pr\{X_i = -1\} = \frac{1}{2}$, $i \geq 1$. Then

$$\Pr\{U_n^*(m) \geq y\} \geq \Pr\left\{ \max_{0 \leq k \leq m} S_k \geq y \right\} - \Pr\left\{ \min_{0 \leq k \leq m} S_k \leq -x \right\}. \quad (2.21)$$

Proof. In terms of the symmetric process $Q^*(k)$, with $Q^*(0) = 0$, we have by Lemma 5,

$$\Pr\{U_n^*(m) \geq y\} \leq \Pr\{U^*(m) \geq y\}. \quad (2.22)$$

Next, write the bound

$$\Pr\{U^*(m) \geq y\} \leq \Pr\{U^*(m) \geq y, D^*(m) < x\} + \Pr\{D^*(m) > x\}. \quad (2.23)$$

Since $Q^*(0) = x$, a sample path is in the set $\{U^*(m) \geq y, D^*(m) < x\}$ if and only if in the $m$ steps it never reaches the boundary at the origin and it visits position $y$ at least once. But the set of such sample paths in $Q^*(k)$ has the same probability as the set $\{\max_{0 \leq k \leq m} S_k \geq y, \min_{0 \leq k \leq m} S_k > -x\}$ in the free random walk $S_k$ starting at the origin.

Similarly, $\Pr\{D^*(m) \geq x\} = \Pr\{D^*(m) = x\}$ is the same as the probability, $\Pr\{\min_{0 \leq k \leq m} S_k \leq -x\}$, that the free random walk starting at the origin visits position $-x$ at least once in the first $m$ steps. Thus, from (2.22) and (2.23),

$$\Pr\{U_n^*(m) \geq y\} \leq \Pr\left\{ \max_{0 \leq k \leq m} S_k \geq y, \min_{0 \leq k \leq m} S_k > -x \right\} + \Pr\left\{ \min_{0 \leq k \leq m} S_k \leq -x \right\}. \quad (2.24)$$

Since

$$\left\{ \max_{0 \leq k \leq m} S_k \geq y, \min_{0 \leq k \leq m} S_k > -x \right\} \subseteq \left\{ \max_{0 \leq k \leq m} S_k \geq y \right\},$$

the bound in (2.21) follows directly. \qed

The final preliminary result is a central limit theorem for displacements in a free random walk, when the number of steps is a Poisson random variable. Define $R_n(t)$ as the free process corresponding to $Q_n(t)$, i.e., the process $Q_n(t)$ with the barrier at the origin removed. Let $R_n^*(m)$ be the discrete-time process corresponding to $R_n(t)$, just as $Q_n^*(m)$ corresponds to $Q_n(t)$, i.e., $R_n(t) = R_n^*(N_n(t))$. The absence of a subscript again denotes the symmetric process, with $n = \infty$. Thus, if $R^*(0) = 0$, then $R^*(m)$ is the free symmetric random walk starting at the origin.

Lemma 7. Let $t = t(n)$ be an increasing function of $n$ such that $t \to \infty$ and $t/n^2 \to 0$ as $n \to \infty$. Then

$$\Pr\left\{ \frac{R_n^*(N_n(t))}{\sqrt{2t}} \leq x \right\} \sim \Phi(x) \quad \text{as } n \to \infty,$$

where $\Phi(x)$ is the normal distribution with mean 0 and variance 1.
Proof. Let \( N_n^+(t) \) and \( N_n^-(t) \) be the number of positive and negative jumps, respectively, of \( R_n \) in \([0, t]\). Then \( N_n^+(t) \) and \( N_n^-(t) \) are independent Poisson random variables with means \((1 - 1/n)t\) and \( t\) respectively. Define the normalized random variables

\[
\hat{N}_n^+(t) = \frac{N_n^+(t) - (1 - 1/n)t}{\sqrt{(1 - 1/n)t}}, \quad \hat{N}_n^-(t) = \frac{N_n^-(t) - t}{\sqrt{t}}.
\]

Then by a classical limit theorem for Poisson random variables (see Feller (1968)), the distributions of both \( \hat{N}_n^+(t) \) and \( \hat{N}_n^-(t) \) converge to \( \Phi(x) \) as \( n \to \infty \). We have

\[
R_n^\pm(N_n(t)) = N_n^\pm(t) - N_n^\mp(t),
\]

and in terms of the normalized random variables in (2.24),

\[
\frac{R_n^\pm(N_n(t))}{\sqrt{t}} = \frac{\sqrt{(1 - 1/n)t}}{\sqrt{t}} \left( \frac{N_n^+(t)}{\sqrt{(1 - 1/n)t}} - \frac{N_n^-(t)}{\sqrt{t}} \right) = \sqrt{1 - 1/n} \left[ \hat{N}_n^+(t) + \sqrt{(1 - 1/n)t} \right] - \left[ \hat{N}_n^-(t) + \sqrt{t} \right] = \hat{N}_n^+(t) - \hat{N}_n^-(t) + \varepsilon,
\]

where

\[
\varepsilon = (\sqrt{1 - 1/n} - 1) \hat{N}_n^+(t) - \frac{\sqrt{t}}{n}.
\]

Since \( t = o(n^2) \), we have \( \varepsilon \approx 0 \) as \( n \to \infty \). The lemma then follows from the fact that, as \( n \to \infty \), \( \hat{N}_n^+(t) - \hat{N}_n^-(t) \) tends to a normally distributed random variable with mean 0 and variance 2. □

3. Proof of Theorem 1 for FIFO service

The proof consists of two parts. The first part proves the probability estimate and the second establishes \( E[ W_n] = \Theta(\sqrt{n}) \).

Part 1. We prove that there exists a constant \( c > 0 \) such that for all \( n \) sufficiently large

\[
\Pr\{ W_n > k \sqrt{n} \} = O(e^{-ck}).
\]

In estimating the stationary distribution of \( W_n(t) \) our approach is to analyze the events that can occur in a time interval of a suitably defined duration \( t \), assuming that the only known property of the storage state at the beginning of the interval is that the number in system has the stationary distribution. Since we are concerned only with the stationary process, we consider an interval \([0, t]\) without loss of generality.
The basic observation is that, at the end of the interval, \( W_n(t) \geq k\sqrt{n} \) obtains only if either

A: one or more items in storage at time 0 remain in storage at time \( t \), or

B: \( V_n(t) \geq k\sqrt{n} \); i.e., in some sub-interval of \([0, t]\) there is a drop in mass (number in system) of at least \( k\sqrt{n} \).

For if neither event A nor event B occur, then the item in position \( H_n(t) \) at time \( t \) must have arrived at a time \( 0 < t' \leq t \) when the first \( H_n(t) - 1 \) cells were occupied; the drop in mass during \([t', t]\) was less than \( k\sqrt{n} \), and hence

\[
W_n(t) = H_n(t) - Q_n(t) \leq Q(t') - Q_n(t) < k\sqrt{n}.
\]

Thus, \( W_n(t) \geq k\sqrt{n} \) implies that either A or B holds, and we have

\[
\Pr\{W_n(t) \geq k\sqrt{n}\} \leq \Pr\{A\} + \Pr\{B\}. \tag{3.2}
\]

The next step is to choose \( t \) sufficiently large that for some constant \( c > 0 \), \( \Pr\{A\} = O(e^{-ct}) \), but at the same time sufficiently small that \( \Pr\{B\} = O(e^{-ck}) \). The remainder of the proof shows that \( t = kn \) suffices for this purpose. We begin with the estimate of \( \Pr\{A\} \), which entails a standard calculation of virtual waiting times in the M/M/1 queue (see, e.g., Kleinrock (1976)).

Let \( T_i \) be the time required to serve \( i \geq 0 \) items consecutively. By the FIFO rule, items in the system at time 0 are served before any arrival in \([0, t]\). Then

\[
\Pr\{A\} = \sum_{i \geq 1} \Pr\{T_i > t\} q_i, \tag{3.3}
\]

where \( q_i \) is the stationary distribution in (1.1). A calculation yields

\[
\Pr\{A\} = \left(1 - \frac{1}{n}\right) e^{-t/n} < e^{-c}, \tag{3.4}
\]

for all \( n > 1 \), which is the desired bound.

Now consider event B and write

\[
\Pr\{B\} = \sum_{i = 0} \Pr\{V_n(t) \geq k\sqrt{n} | Q_n(0) = i\} q_i. \tag{3.5}
\]

For simplicity suppose \( \sqrt{n} \) is an integer (trivial modifications to the arguments below will take care of arbitrary \( \sqrt{n} \)).

If \( i \geq k\sqrt{n} \), then sample paths in the set \( \{V_n(t) < k\sqrt{n} | Q_n(0) = i\} \) never visit the barrier at the origin in \([0, t]\). Thus, as in the corresponding free process \( R_n(t) \), \( \Pr\{V_n(t) < k\sqrt{n} | Q_n(0) = i\} \) and hence \( \Pr\{V_n(t) \geq k\sqrt{n} | Q_n(0) = i\} \) is not a function of \( i \) for \( i \geq k\sqrt{n} \). Then we can write

\[
\Pr\{V_n(t) \geq k\sqrt{n} | Q_n(0) = i\} = \Pr\{V_n(t) \geq k\sqrt{n} | Q_n(0) = k\sqrt{n}\}, \quad i \geq k\sqrt{n}. \tag{3.6}
\]

If \( Q_n(0) < k\sqrt{n} \), then before the mass can drop by \( k\sqrt{n} \), it must first build up to \( k\sqrt{n} \). Thus, for \( Q_n(0) < k\sqrt{n} \), \( V_n(t) \geq k\sqrt{n} \) implies that there is a drop in mass of at
least $k\sqrt{n}$ in some interval $[t', t]$, $0 < t' < t$, with the mass at $t'$ equaling $k\sqrt{n}$. But $V_n(t)$ is clearly stochastically larger than $V_n(t-t')$, and hence
\[
\Pr(V_n(t) \geq k\sqrt{n} \mid Q_n(0) = i) \leq \Pr(V_n(t) \geq k\sqrt{n} \mid Q_n(0) = k\sqrt{n}), \quad t \leq k\sqrt{n}.
\]
(3.7)

Thus, from (3.5)-(3.7)
\[
\Pr[B] \leq \Pr(V_n(t) \geq k\sqrt{n} \mid Q_n(0) = k\sqrt{n}].
\]
(3.8)

But $Q_n(0) = k\sqrt{n}$ and $V_n(t) \geq k\sqrt{n}$ imply that either $U_n(t) \geq \frac{1}{2}k\sqrt{n}$ or $D_n(t) \geq \frac{1}{2}k\sqrt{n}$. Then,
\[
\Pr[B] \leq \Pr(D_n(t) \geq \frac{1}{2}k\sqrt{n} \mid Q_n(0) = k\sqrt{n}) + \Pr(U_n(t) > \frac{1}{2}k\sqrt{n} \mid Q_n(0) = k\sqrt{n}).
\]
(3.9)

Let $B_1$ and $B_2$ be the respective events on the right of (3.9). By Lemma 4, $D_n(t) = D_n^*(N_n(t))$, so in terms of the discrete-time random walk,
\[
\Pr(B_1) \leq \sum_{m=0}^{\infty} \Pr(D_n^*(m) \geq \frac{1}{2}k\sqrt{n} \mid Q_n^*(0) = k\sqrt{n}) \Pr(N_n(t) = m), \quad t = kn.
\]
(3.10)

Now $D_n^*(m)$ clearly increases stochastically with $m$, so that we obtain from (3.10),
\[
\Pr(B_1) \leq \Pr(D_n^*(4kn) \geq \frac{1}{2}k\sqrt{n} \mid Q_n^*(0) = k\sqrt{n}) \Pr(N_n(t) \leq 4kn)
\]
\[+ \Pr(N_n(t) > 4kn). \quad (3.11)
\]

By Lemma 3 with $h = (2-1/n)t < 2t = 2kn$, there exists a $c > 0$ such that for all $n \geq 1$,
\[
\Pr(N_n(t) > 4kn) = O(e^{-ck}). \quad (3.12)
\]

Introducing this bound and $\Pr(N_n(t) \leq 4kn) \leq 1$ in (3.11) gives
\[
\Pr(B_1) \leq \Pr(D_n^*(4kn) \geq \frac{1}{2}k\sqrt{n} \mid Q_n^*(0) = k\sqrt{n}) + O(e^{-ck}). \quad (3.13)
\]

As in the argument for (3.6), the probability of a downward excursion of $l$ or more from an initial position $Q_n^*(0) \geq l$ is the same in a random walk with a barrier at the origin as it is in the corresponding free random walk with the barrier (boundary conditions) removed. In the free random walk, downward excursions are independent of the initial position. Thus, letting $m = 4kn$, we can write
\[
\Pr\left(\min_{1 \leq i \leq m} S_i \leq -\frac{1}{2}k\sqrt{n}\right) - \Pr\{D_n^*(m) \geq \frac{1}{2}k\sqrt{n} \mid Q_n^*(0) = k\sqrt{n}\},
\]
where $S_j = \sum_{i=1}^{j} X_i$, $1 \leq j \leq 4kn$, is the free random walk starting at the origin, with
\[
\Pr(X_i = +1) = \frac{1 - 1/n}{2 - 1/n} \quad \text{and} \quad \Pr(X_i = -1) = \frac{1}{2 - 1/n}.
\]

To apply Lemma 2, we write
\[
\Pr\left(\min_{1 \leq i \leq m} S_i \leq -\frac{k}{2\sqrt{n}}\right) \leq \Pr\left(\min_{1 \leq i \leq m} \left(S_i + \frac{i}{2n-1} - \frac{m}{2n-1}\right) \leq -\frac{k}{2\sqrt{n}}\right).
\]
(3.14)
We note that \( \mu = E[X_i] = -1/(2n-1) \), so that \( S_i + i/(2n-1) = \hat{S}_i \), and rewrite (3.14) as
\[
\Pr\left\{ \min_{1 \leq i \leq m} S_i \leq -\frac{k}{2} \sqrt{n} \right\} \leq \Pr\left\{ \min_{1 \leq i \leq m} \hat{S}_i \leq -\left( \frac{k}{2} \sqrt{n} - \frac{1}{2n-1} m \right) \right\}.
\]
Now \( m = 4kn \), so if in Lemma 2 we put \( x = (\frac{1}{2} k \sqrt{n} - m/(2n-1)) \), then we obtain \( x = \frac{1}{2} \sqrt{k}[1 + O(1/\sqrt{n})] \). Thus, by Lemma 2 there exists a \( c > 0 \) such that for all \( n \) sufficiently large,
\[
\Pr[D_n^*(4kn) \geq \frac{1}{2} k \sqrt{n} | Q_n^*(0) = k \sqrt{n}] = O(e^{-ck}). \tag{3.15}
\]
Then (3.13) and (3.15) give
\[
\Pr[B_1] = O(e^{-ck}). \tag{3.16}
\]
To estimate \( \Pr[B_2] \), we first repeat the argument leading to (3.13) to obtain
\[
\Pr[B_2] \leq \Pr[U_n^*(4kn) \geq \frac{1}{2} k \sqrt{n} | Q_n^*(0) = k \sqrt{n}] + O(e^{-ck}). \tag{3.17}
\]
Next, let \( S_n = \sum_{i=1}^n X_i \), where \( \Pr[X_i = +1] = \Pr[X_i = -1] = \frac{1}{2} \). By Lemmas 2 and 6, we have
\[
\Pr[U_n^*(4kn) \geq \frac{1}{2} k \sqrt{n} | Q_n^*(0) = k \sqrt{n}]
\leq \Pr\left\{ \max_{0 \leq i \leq 4kn} S_i \geq \frac{1}{2} k \sqrt{n} \right\} + \Pr\left\{ \min_{0 \leq i \leq 4kn} S_i \leq -kn \right\}
\leq 2\Pr\left\{ \max_{0 \leq i \leq 4kn} S_i \geq \frac{1}{2} k \sqrt{n} \right\} = O(e^{-ck}). \tag{3.18}
\]
Then (3.17) and (3.18) yield
\[
\Pr[B_2] = O(e^{-ck}), \tag{3.19}
\]
and finally, (3.2), (3.4), (3.9), (3.16) and (3.19) yield (3.1).

**Part 2.** The goal here is a proof of \( E[W_n] = \Theta(\sqrt{n}) \). For the upper bound \( O(\sqrt{n}) \), define \( l(x) = \Pr[W_n \geq k], k-1 \leq x < k, 1 \leq k < \infty \), and \( u(x) = \Pr[W_n \geq \lambda k], \lambda k \leq x < \lambda(k+1), 0 \leq k < \infty \). Since \( \Pr[W_n \geq x] \) is a decreasing function, we have
\[
l(x) \leq \Pr[W_n \geq x] \leq u(x), \quad 0 \leq x < \infty.
\]
Hence,
\[
\sum_{k \geq 1} \Pr[W_n \geq k] = \int_0^\infty l(x) \, dx \leq \int_0^\infty u(x) \, dx = \lambda \sum_{k \geq 0} \Pr[W_n \geq \lambda k]. \tag{3.20}
\]
Letting \( \lambda = \sqrt{n} \), we have
\[
E[W_n] \leq \sqrt{n} \sum_{k \geq 0} \Pr[W_n \geq k \sqrt{n}]. \tag{3.21}
\]
The sum in (3.21) converges by virtue of (3.1), so \( E[W_n] = O(\sqrt{n}) \) follows.
More effort is required to prove the lower bound, i.e., there exists a $c > 0$ such that $E[W_n] \geq c \sqrt{n}$ for all $n$ sufficiently large. Note first that this bound will follow if we can show that there exists a $c > 0$ such that for all $n$ sufficiently large,

$$\Pr\{W_n > \sqrt{n}\} \geq c.$$  \hfill (3.22)

To prove (3.22), we again analyze the behavior of $Q_n$ in an interval of length $t$ in statistical equilibrium. As before, we assume that the interval is $[0, t]$ without loss of generality. In the argument below we take $t = n$ and prove that there exists an integer $k > 0$ and a constant $c > 0$ such that the joint probability of the following five events is at least $c$ for $n$ sufficiently large:

- $A_1 = \{Q_n(0) \geq 2n\}$,
- $A_2 = \{W_n(0) \leq k\sqrt{n}\}$,
- $A_3 = \{Q_n(\frac{3}{2}n) - Q_n(0) > k\sqrt{n}\}$,
- $A_4 = \{Q_n(\frac{3}{2}n) - Q_n(n) > \sqrt{n}\}$,
- $A_5 = \{\text{fewer than } 2n \text{ of the items in the system at time 0 depart by time } n\}$.

Before proving this fact, let us verify that $\Pr\{\cap_{j=1}^{5} A_j\} \geq c$ implies (3.22), and hence the lower bound on $E[W_n]$. Assume that $\cap_{j=1}^{5} A_j$ occurs. $A_2$ implies that the cells in position $i = Q_n(0) + k\sqrt{n} + 1$ or higher are unoccupied at time 0. $A_3$ implies that the cell in position $i$ contains an item at time $\frac{3}{2}n$ which arrived during $[0, \frac{3}{2}n]$. By $A_1$ and $A_5$, at least one of the items in the system at time 0 is still in the system at time $n$. Hence, by the FIFO rule the departures in $[0, n]$ must all be from cells with numbers no larger than $Q_n(0) + k\sqrt{n}$. By $A_4$, we must then have $W_n(n) > \sqrt{n}$, and we are done.

Now write

$$\Pr\left\{\bigcap_{j=1}^{5} A_j\right\} = \Pr\{A_3, A_4, A_5|A_1, A_2\} \Pr\{A_1, A_2\}. \hfill (3.23)$$

We want to show that both probabilities on the right of (3.23) are bounded away from zero for all $n$ sufficiently large. First, let $\bar{A}$ denote the complement of event $A$ and write

$$\Pr\{A_1\} = \Pr\{A_1, A_2\} + \Pr\{A_1, \bar{A}_2\},$$

so that

$$\Pr\{A_1, A_2\} = \Pr\{A_1\} - \Pr\{A_1, \bar{A}_2\} \geq \Pr\{A_1\} - \Pr\{\bar{A}_2\}. \hfill (3.24)$$

According to the stationary distribution in (1.1) we have

$$\Pr\{A_1\} = \left(1 - \frac{1}{n}\right)^{2n} \sim \frac{1}{e^2} \quad \text{as } n \to \infty. \hfill (3.25)$$

By (3.1), $\Pr\{\bar{A}_2\} = \Pr\{W_n > k\sqrt{n}\} = O(e^{-cn})$ for some $c > 0$ and all $n$ sufficiently large. Therefore, there exists a $k > 0$ and a constant $0 < c < 1/e^2$, such that by (3.24)

$$\Pr\{A_1, A_2\} > c \hfill (3.26)$$

for all $n$ sufficiently large.
We turn next to the probability \( \Pr\{A_3, A_4, A_5 | A_1, A_2\} \) and observe that, given the number of initially occupied cells, the joint event \( \{A_3, A_4, A_5\} \) is independent of their locations. We conclude that

\[
\Pr\{A_3, A_4, A_5 | A_1, A_2\} = \Pr\{A_3, A_4, A_5 | A_1\}. \quad (3.27)
\]

Now let \( B_j \) denote the event corresponding to \( A_j \) in the free process \( R_n(t), t \geq 0 \), i.e., \( Q_n(t), t \geq 0 \), with the barrier at 0 removed and \( R_n(0) \equiv Q_n(0) \). Then

\[
B_1 = \{R_n(0) \geq 2n\},
\]

\[
B_3 = \{R_n(\frac{1}{2}n) - R_n(0) > k\sqrt{n}\},
\]

\[
B_4 = \{R_n(\frac{1}{2}n) - R_n(n) > \sqrt{n}\},
\]

\[
B_5 = \{\text{fewer than } 2n \text{ downward jumps in } [0, n]\}.
\]

Since \( A_5 \) occurs only if \( Q_n(\tau) > 0, 0 \leq \tau \leq n \), it is easy to see that

\[
\Pr\{A_3, A_4, A_5 | A_1\} = \Pr\{B_3, B_4, B_5 | B_1\}; \quad (3.28)
\]

and since the joint event \( B_3, B_4, B_5 \) is clearly independent of the initial state in the free process, we have

\[
\Pr\{B_3, B_4, B_5 | B_1\} = \Pr\{B_3, B_4, B_5\}. \quad (3.29)
\]

Arguing as for (3.24) we get

\[
\Pr\{B_3, B_4, B_5\} \equiv \Pr\{B_3, B_4\} - \Pr\{\bar{B}_5\}. \quad (3.30)
\]

The number of downward jumps during \([0, t]\) is a Poisson process with parameter 1. Hence, by Lemma 3 we have for some \( c > 0 \) and for all \( n > 0 \),

\[
\Pr\{\bar{B}_5\} = O(e^{-cn}). \quad (3.31)
\]

Clearly, since \( B_3 \) and \( B_4 \) bound displacements in \( R_n(t) \) over disjoint intervals of time, \( B_3 \) and \( B_4 \) are independent. To apply Lemma 7, we observe that

\[
R_n^\#(N_n(n/2)) \ominus R_n(\frac{1}{2}n) - R_n(0) \ominus -[R_n(\frac{1}{2}n) - R_n(n)].
\]

We conclude from Lemma 7 that as \( n \to \infty \),

\[
\Pr\{B_3, B_4\} = \Pr\{B_3\} \Pr\{B_4\} \sim [1 - \Phi(k)][1 - \Phi(1)]. \quad (3.32)
\]

By (3.28)–(3.32) there exists a \( c > 0 \) such that for all \( n \) sufficiently large,

\[
\Pr\{A_3, A_4, A_5 | A_1\} = \Pr\{B_3, B_4, B_5\} > c. \quad (3.33)
\]

By (3.23), (3.26), (3.27) and (3.33) there exists a \( c > 0 \) such that for all \( n \) sufficiently large, \( \Pr\{\bigcap_{j=1}^5 A_j\} > c \); (3.16) then holds and the lower bound is proved. \( \square \)

4. Proof of Theorem 2 for processor sharing

In Part 1 we prove that there exists a \( c > 0 \) such that for all \( n \) sufficiently large

\[
\Pr\{W_n > k\sqrt{n \log n}\} = O(e^{-c^{1/2}}),
\]

and in Part 2 we prove that \( E[W_n] = \Theta(\sqrt{n \log n}) \).
Part 1. As in Theorem 1, we analyze the queue-length process over an interval \([0, t]\), but this time we take \(t = kn \log n\). In analogy with the proof of Theorem 1, we observe that \(W_n(t) \geq k\sqrt{n \log n}\) only if either

A: one or more items in storage at time 0 are still in storage at time \(t\), or

B: \(V_n(t) \geq k\sqrt{n \log n}\).

The goal is to show that for all \(n\) sufficiently large the probability of each of these events is \(O(e^{-ck^{1/2}})\) for some \(c > 0\), and hence that

\[
Pr\{W_n(t) \geq k\sqrt{n \log n}\} \leq Pr\{A\} + Pr\{B\} = O(e^{-ck^{1/2}}) \tag{4.1}
\]

for all \(n\) sufficiently large. In processor sharing the items are not served sequentially and formula (3.3) is no longer valid. As a consequence, it becomes more difficult to bound \(Pr\{A\}\). We consider event \(B\) first, for its analysis is virtually identical to that in the proof of Theorem 1.

Choose \(t = kn \log n\) and repeat the development from (3.5) to (3.14), with \(n\) replaced by \(n \log n\) in the bounds. We obtain

\[
Pr\{B\} \leq 2 Pr\left\{\min_{1 \leq i \leq m} \hat{S}_i \leq \left(\frac{1}{2} k\sqrt{n \log n} - \frac{m}{2n-1}\right)\right\} + O(e^{-ck}), \tag{4.2}
\]

where \(m = \lceil 4kn \log n \rceil\). Put

\[
x\sqrt{m} = \frac{1}{2} k\sqrt{n \log n} - \frac{m}{2n-1}
\]

so that

\[
x = \frac{1}{4} k \left[1 + O\left(\sqrt{\frac{\log n}{n}}\right)\right].
\]

By Lemma 2 and (4.2) there exists a \(c > 0\) such that for all \(n\) sufficiently large,

\[
Pr\{B\} = O(e^{-ck}). \tag{4.3}
\]

We turn next to an estimate of \(Pr\{A\}\). Define the events

\[
C_1 = \left\{\sup_{0 \leq \tau \leq t} Q_n(\tau) < 2nk^{2/3}\right\} \quad \text{and} \quad C_2 = \{Q_n(0) \leq k^{2/3}n\}.
\]

We have

\[
Pr\{A\} \leq Pr\{A \mid C_1\} + Pr\{\overline{C}_1\} \leq Pr\{A \mid C_1, C_2\} + Pr\{\overline{C}_1 \mid C_2\} + Pr\{\overline{C}_2\}. \tag{4.4}
\]

To bound the last probability on the right of (4.4), we apply (1.1) and obtain

\[
Pr\{\overline{C}_2\} = Pr\{Q_n(0) > k^{2/3}n\} \leq \left(1 - \frac{1}{n}\right) k^{2/3}n \leq e^{-k^{2/3}}. \tag{4.5}
\]

For an analysis of the event \(\{A \mid C_1, C_2\}\), consider a single item that remains in storage throughout \([0, t]\). Clearly, by definition of PS service, such an item receives at least \(t/\sup_{0 \leq \tau \leq t} Q_n(\tau)\) time units of service. Hence, given \(C_1 = \{\sup_{0 \leq \tau \leq t} Q_n(\tau) < 2nk^{2/3}\}\), we see that for \(t = kn \log n\),

\[
e^{-t/(2nk^{2/3})} = e^{-(k^{1/3}/2) \log n}
\]
is an upper bound to the probability that an item in storage at time 0 is still in
storage at time \( t = nk \log n \). Thus, given \( C_2 = \{ Q_n(0) \leq k^{2/3} n \} \) as well as \( C_1 \), we have
for the at most \( k^{2/3} n \) items in storage at time 0,
\[
\Pr(A \mid C_1, C_2) \leq 2k^{2/3} n \ e^{-(k^{2/3}) \log n} = 2k^{2/3} n \ e^{-(k^{2/3}/2 - 1) \log n}.
\]
Hence, there is a \( c > 0 \) such that for all \( n \) sufficiently large,
\[
\Pr(A \mid C_1, C_2) = O(e^{-ck^{1/3}}).
\] (4.6)
Thus, to prove Part 1 of Theorem 2, it remains to show that \( \Pr(C_1 \mid C_2) = O(e^{-ck^{1/3}}) \). We first transform our problem from continuous to discrete time. By Lemma 4 we have
\[
\Pr(C_1 \mid C_2) = \Pr\left\{ \sup_{0 < r < t} Q_n^*(r) \leq k^{2/3} n \mid Q_n(0) \leq k^{2/3} n, N_n(t) \leq 4t \right\} + \Pr\{N_n(t) > 4t \mid Q_n(0) \leq k^{2/3} n\}. \] (4.7)
Let \( P_1 \) and \( P_2 \) denote the first and second probabilities on the right of (4.7). The Poisson process \( N_n(t) \) is independent of \( Q_n(0) \), so we may apply Lemma 3 with \( \lambda = (2 - 1/n) t \) to obtain
\[
P_2 = \Pr\{N_n(t) > 4t \mid Q_n(0) = k^{2/3} n\} = O(e^{-ck}). \] (4.8)
We show next that \( P_1 = O(e^{-ck}) \) also holds.
Since \( \sup_{0 < r < t} Q_n^*(r) \) increases stochastically in \( s \), we obtain
\[
P_1 \leq \Pr\left\{ \sup_{0 < r < m} Q_n^*(r) \geq 2k^{2/3} n \mid Q_n(0) \leq k^{2/3} n \right\} \] (4.9)
where \( m = 4t \). A random walk \( Q_n^*(r) \) for which \( Q_n^*(0) \leq k^{2/3} n \) and \( \sup_{0 < r < m} Q_n^*(r) > 2k^{2/3} n \) must pass through \( k^{2/3} n \) and then perform an upward excursion of at least \( k^{2/3} n \) in at most \( m \) steps. Since \( U_n(r) \) is stochastically increasing in \( r \), we conclude from (4.9) that
\[
P_1 \leq \Pr\{ U_n^*(m) \geq k^{2/3} n \mid Q_n(0) = k^{2/3} n\}. \] (4.10)
Now if \( S_k \) denotes the free symmetric random walk starting at the origin, then by Lemma 6,
\[
\Pr\{ U_n^*(m) \geq k^{2/3} n \mid Q_n(0) = k^{2/3} n\} \leq \Pr\left\{ \max_{0 \leq k < m} S_k \geq k^{2/3} n \right\} + \Pr\left\{ \min_{0 \leq k < m} S_k \leq -k^{2/3} n \right\}
\leq 2 \Pr\left\{ \max_{0 \leq k < m} S_k \geq k^{2/3} n \right\}. \] (4.11)
For \( m = 4t = 4kn \log n \), we write \( k^{2/3} n = \sqrt{k^{1/3} n/(4 \log n)} \sqrt{m} \), so by (4.11) and Lemma 2, there is a \( c > 0 \) such that
\[
\Pr\{ U_n^*(m) \geq k^{2/3} n \mid Q_n(0) = k^{2/3} n\} = O(e^{-ck^{1/3}}). \] (4.12)
The desired result $\Pr\{ C_1 | C_2 \} = O(e^{-\epsilon \sqrt{n}})$ then follows from (4.7), (4.8), (4.10), and (4.12).

**Part 2.** We prove next that $E[W_n] = O(\sqrt{n \log n})$. The upper bound $O(\sqrt{n \log n})$ is easy, since by (3.20) with $\lambda = \sqrt{n \log n}$,

$$E[W_n] \leq \sqrt{n \log n} \sum_{k=0}^{\infty} \Pr\{ W_n \geq k\sqrt{n \log n} \}.$$ 

The sum converges by (4.1), so the upper bound follows.

For the lower bound, we proceed as in Theorem 1; it is sufficient to prove that there exists a $c > 0$ such that over the interval $[0, t]$ in statistical equilibrium, if $t = \frac{1}{2} n \log n$, then for all $n$ sufficiently large,

$$\Pr\{ W_n(t) \geq \sqrt{n \log n} \} > c. \quad (4.13)$$

(The coefficient $\frac{1}{2}$ is determined by the choice of $\sqrt{n}$ in event $A_2$ below. Choices other than $\frac{1}{2}$ and $\sqrt{n}$ are possible, as will be clear from the analysis below.)

To prove (4.13) we show that the joint probability of the following events is bounded away from 0 for all $n$ sufficiently large.

- $A_1 = \{ Q_0(0) \geq 2n + \sqrt{n} \}$,
- $A_2 = \{ Q_0(0) - Q_n(t) \geq 2\sqrt{n \log n} \}$,
- $A_3 = \{ D_n(t) \leq n \}$,
- $A_4 = \{ \text{there are } > \sqrt{n} \text{ occupied cells at time 0, and at least 1 of the items in the } \sqrt{n} \text{ highest numbered of these cells has a remaining service time } > t/n \}$.

It is easy to verify that

$$A_1, A_2, A_3, A_4 \subset \{ W_n(t) \geq \sqrt{n \log n} \}, \quad n \geq 3. \quad (4.14)$$

Let cell $i+1$ be occupied by an item fitting the description of $A_4$. By $A_1$ and $A_3$, $Q_n(\tau) > n$, $0 \leq \tau \leq t$. By the PS discipline, the item in cell $i+1$ receives at most $t/n$ units of service during $[0, t]$, and hence is still in storage at time $t$. Thus, the highest numbered occupied cell at time $t$ satisfies $H_n(t) \geq i+1$. Except for at most $\sqrt{n}$ items, all items in storage at time 0 are in the first $i$ cells, so that $H_n(t) > i \equiv Q_n(0) - \sqrt{n}$. At time $t$, the number of items in the first $i$ cells is at most $Q_0(t) \leq Q_n(0) - 2\sqrt{n \log n}$, by $A_2$. Hence,

$$W_n(t) = H_n(t) - Q_n(t) \geq Q_n(0) - \sqrt{n} - [Q_n(0) - 2\sqrt{n \log n}] > \sqrt{n \log n}, \quad n \geq 3.$$ 

By (4.14) we need only show that $\Pr\{ A_1, A_2, A_3, A_4 \}$ is bounded away from 0 for all $n$ sufficiently large. To prove this, first write

$$\Pr\{ A_1, A_2, A_3, A_4 \} = \Pr\{ A_1, A_2, A_3 \} - \Pr\{ A_1, A_2, A_3, A_4 \}$$

$$\geq \Pr\{ A_1, A_2, A_3 \} - \Pr\{ A_1, A_4 \}. \quad (4.15)$$
The remaining service times of items in storage at time 0 are independently and exponentially distributed with parameter 1. Therefore, since
\[ \Pr\{A_1, \tilde{A}_4\} = \Pr\{\tilde{A}_4|A_1\} \Pr\{A_1\} = \Pr\{\tilde{A}_4|A_1\}, \]
we have
\[ \Pr\{A_1, \tilde{A}_4\} \leq (1 - e^{-\log n/3})^{n/2} = \left(1 - \frac{1}{n^{1/3}}\right)^{n^{1/2}} \sim e^{-n^{1/6}} \text{ as } n \to \infty. \quad (4.16) \]

By (4.15) and (4.16), it is sufficient to show that there exists a \( c > 0 \) such that
\[ \Pr\{A_1, A_2, A_3\} > c \]
for all \( n \) large enough. To show this, let \( B_j \) denote the event corresponding to \( A_j \) in the free process \( R_n(t) \), where \( R_n(0) = Q_n(0) \). By \( A_1 \) and \( A_3 \), \( Q_n(\tau) > n + \epsilon n \) for \( 0 \leq \tau \leq t \), so that
\[ \Pr\{A_1, A_2, A_3\} = \Pr\{B_1, B_2, B_3\}. \quad (4.17) \]

Since \( B_1 \) and \( B_2 \) are independent events, we have
\[ \Pr\{B_1, B_2, B_3\} = \Pr\{B_1, B_2\} - \Pr\{B_3\} = \Pr\{B_1\} \Pr\{B_2\} - \Pr\{B_3\}. \quad (4.18) \]

By (1.11),
\[ \Pr\{B_1\} = \Pr\{Q_n(0) \geq 2n + \sqrt{n}\} \sim \left(1 - \frac{1}{n}\right)^{2n + \sqrt{n}} \sim e^{-2}, \]
and since \( R_n(t) - R_n(0) = R^*_n(N_n(t)) \), we obtain from Lemma 7,
\[ \Pr\{B_2\} = \Pr\{R^*_n(N_n(t)) \leq -2\sqrt{3t}\} \sim 1 - \phi(\sqrt{6}) \text{ as } n \to \infty. \quad (4.19) \]

To bound \( \Pr\{B_3\} \) we condition on \( N_n(t) \leq m = \frac{2}{3} n \log n \) and use Lemmas 2 and 3. We find that
\[ \lim_{n \to \infty} \Pr\{B_3\} = 0. \quad (4.20) \]

Assembling (4.17)-(4.20) we conclude that there exists a \( c > 0 \) such that
\[ \Pr\{A_1, A_2, A_3\} > c \]
for all \( n \) sufficiently large. \( \square \)

**Acknowledgment**

The authors gratefully acknowledge helpful discussions with J.C. Lagarias, C.L. Mallows, and L.A. Shepp.

**References**


