Accurate Numerical Solution of Initial Value Problems for the Time Dependent Convection-Diffusion Equation

R. Company, L. Jódar and E. Ponsoda
Instituto de Matemática Multidisciplinar
Universidad Politécnica de Valencia, Spain
<rcomp@,ljodar@,eponsoda@mat.upv.es

(Received August 2001; accepted October 2001)

Abstract—By using the Fourier transform method and composite Simpson’s integration formula, a symbolically computable approximate solution of an initial value problem for the time dependent convection-diffusion equation is constructed. Given an admissible error $\epsilon$ and a point $(x,t)$, the approximation constructed with error less than $\epsilon$ at $(x,t)$ can be symbolically obtained using Mathematica 4.0. © 2003 Elsevier Science Ltd. All rights reserved.

Keywords—Convection-diffusion equation, A priori error bound, Initial value problem.

1. INTRODUCTION

Linear flow of heat where thermal properties are time dependent but independent of position are modeled by the time dependent convection-diffusion equation [1]. The analysis of American call options also addresses these types of equations [2, p. 111]. In this paper, we consider the problem

\begin{align}
\frac{\partial u(x,t)}{\partial t} + a(t)\frac{\partial u(x,t)}{\partial x} &= b(t)u_{xx}(x,t), & -\infty < x < \infty, \quad t > 0, \\
u(x,0) &= f(x), & -\infty < x < \infty,
\end{align}

(1.1)

where $a(t)$, $b(t)$ are positive continuous functions and $f(x)$ is a function with properties to be determined. Difference methods are widely used in the literature, see [3] for instance, but they are not appropriate to construct approximations with a priori error bounds. In this paper, we apply first the Fourier transform method, as in [4], and further numerical integration using composite Simpson’s rule. Throughout this paper, $L^1$ denotes the space of all absolutely Lebesgue integrable functions on the real line. Here $\mathcal{F}$ denotes the exponential Fourier transform, and recall that if $f$ is continuous and piecewise smooth and $f'$ lies in $L^1$, then [5, p. 214]

\[ \mathcal{F}[f'(x)](w) = iw\mathcal{F}[f(x)](w) = iwF(w). \]  

(1.3)
On the other hand, if \( zf(z) \) is Li, then \( F(w) \) is differentiable and
\[
\mathcal{F}(zf(z))(w) = i \frac{d}{dw} F(w). \tag{1.4}
\]

Let \( \Phi(x) \) be the error function defined by the relation
\[
\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \tag{1.5}
\]
and note that by [6, p. 306] for \( q > 0 \), one gets
\[
\int_0^u e^{-q^2 x^2} dx = \frac{\sqrt{\pi}}{2q} \Phi(\sqrt{q} u), \quad \int_0^\infty e^{-q^2 x^2} dx = \frac{\sqrt{\pi}}{2q}. \tag{1.6}
\]

2. ACCURATE NUMERICAL SOLUTION

Let us assume that problem (1.1), (1.2) admits a solution \( u(x,t) \) such that \( u(\cdot,t), u_x(\cdot,t), u_{xx}(\cdot,t), u_t(\cdot,t) \) are absolutely integrable functions of the variable \( x \), for a fixed value of \( t > 0 \).

Let us denote
\[
\mathcal{F}(u(\cdot,t))(w) = U(t)(w), \tag{2.1}
\]
assume that \( f(x) \) is in \( L^1 \), and denote
\[
F(w) = \mathcal{F}(f(x))(w). \tag{2.2}
\]

By applying Fourier transform to problem (1.1), (1.2) and taking into account property (1.3) of this transform, it follows that
\[
\frac{d}{dt} U(t)(w) = - (iwa(t) + w^2b(t)) U(t)(w); \quad U(0)(w) = F(w), \quad t > 0. \tag{2.3}
\]
Solving (2.3) and applying the Fourier inversion formula, one gets
\[
U(t)(w) = F(w)e^{-w^2 \int_0^t b(s) ds} e^{-iw \int_0^t a(s) ds}, \tag{2.4}
\]
\[
u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F(w)e^{-w^2 \int_0^t b(s) ds} e^{iw(x-\int_0^t a(s) ds)} dw. \tag{2.5}
\]
It is easy to check that \( u(x,t) \) given by (2.5) satisfies (1.1) and (1.2). Let \( M > 0 \) such that
\[
|F(w)| < M, \quad -\infty < w < \infty, \tag{2.6}
\]
note that
\[
|F(w)e^{-w^2 \int_0^t b(s) ds} e^{iw(x-\int_0^t a(s) ds)}| \leq M e^{-w^2 \int_0^t b(s) ds}, \tag{2.7}
\]
and by (1.5), (1.6), it follows that
\[
\left| \int_{|w| > R} U(t)(w)e^{iwx} dw \right| \leq M \int_{|w| > R} e^{-w^2 \int_0^t b(s) ds} dw
\]
\[
= M \left[ \frac{\pi}{\int_0^t b(s) ds} \left( 1 - \Phi \left( R \sqrt{\int_0^t b(s) ds} \right) \right) \right]. \tag{2.8}
\]
Taking $R$ large enough so that
\[ 1 \Phi \left( R \sqrt{\int_0^t b(s) \, ds} \right) < \frac{\epsilon}{M}, \]  \hspace{1cm} (2.9) \]
and letting
\[ I(x, t, R) = \int_{-R}^R F(w) e^{-w^2 \int_0^t b(s) \, ds} e^{iw(x-f_0^s a(s) \, ds)} \, dw, \]  \hspace{1cm} (2.10) \]
then by (2.5)-(2.10), it follows that
\[ |u(x, t) - I(x, t, R)| < \frac{\epsilon}{2}. \]  \hspace{1cm} (2.11) \]
Let $n$ be an even integer and
\[ h = \frac{2R}{n}; \hspace{1cm} w_j = -R + jh, \quad 0 \leq j \leq n, \]  \hspace{1cm} (2.12) \]
\[ \bar{f}(x, t, w, h) = F(w) e^{-w^2 \int_0^t b(s) \, ds} e^{iw(x-f_0^s a(s) \, ds)}; \]  \hspace{1cm} (2.13) \]
\[ f_j = \bar{f}(x, t, w_j, h), \quad 0 \leq j \leq n. \]  \hspace{1cm} (2.14) \]
Simpson’s formula approximating the integral (2.10) takes the form
\[ S(x, t, R, h) = \frac{h}{3} [f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n], \]  \hspace{1cm} (2.15) \]
where $f_j$ is defined by (2.14). By [7, p. 258], the error when $S(x, t, R, h)$ approximates $I(x, t, R)$ takes the form
\[ |S(x, t, R, h) - I(x, t, R)| \leq \frac{h^4 R}{90} \max_{|w| \leq R} \left| \frac{d^4}{dw^4} \bar{f}(x, t, w, h) \right|. \]  \hspace{1cm} (2.16) \]
Assume that
\[ x^k f(x) \text{ lies in } L^1 \text{ for } 0 \leq k \leq 4. \]  \hspace{1cm} (2.17) \]
Then by (1.4), it follows that
\[ F^{(j)}(w) = (-i)^j \int_{-\infty}^{\infty} x^j f(x) e^{-ixw} \, dw. \]  \hspace{1cm} (2.18) \]
Let us introduce the expressions
\[ \beta(t) = \int_0^t b(s) \, ds, \hspace{1cm} \alpha(x, t) = x - \int_0^t a(s) \, ds. \]  \hspace{1cm} (2.19) \]
Using Leibnitz’s formula for the fourth derivative of the product of three functions, one gets
\[ \frac{d^4}{dw^4} \left( e^{-w^2 \beta(t)} F(w) e^{i\omega(x,t)} \right) = \sum_{k=0}^4 \sum_{j=0}^{4-k} \binom{4}{k} \binom{4-k}{j} \frac{d^k}{dw^k} \left( e^{-w^2 \beta(t)} \right) \frac{d^{4-k-j}}{dw^{4-k-j}} \left( e^{i\omega(x,t)} \right). \]  \hspace{1cm} (2.20) \]
Computing the involved derivatives, taking absolute values in (2.20), and denoting
\[ A_j = \int_{-\infty}^{\infty} |x^j f(x)| \, dx, \quad 0 \leq j \leq 4, \]  \hspace{1cm} (2.21) \]
it follows that
\[
\max_{|w| \leq R} \left\{ \left| \frac{d^4}{dw^4} \left( e^{-w^2\beta(t)} F(w) e^{iw\alpha(x,t)} \right) \right| \right\} \leq P_4(x, t, R),
\]  
where
\[
P_4(x, t, R) = A_0(\alpha(x, t))^4 + 4A_1|\alpha(x, t)|^3 + 6A_2|\alpha(x, t)|^2 + 4A_3|\alpha(x, t)| + A_4  
+ 8\beta(t)R \left( A_0|\alpha(x, t)|^3 + 3A_1|\alpha(x, t)|^2 + 3A_2|\alpha(x, t)| + A_3 \right)  
+ 12\beta(t)R(1 + 2R\beta(t)) \left( A_0|\alpha(x, t)|^2 + 2A_1|\alpha(x, t)| + A_2 \right)  
+ 8\beta(t) \left[ 2\beta(t) + 4R(\beta(t))^2 + 8R^2(\beta(t))^3 \right] (A_0|\alpha(x, t)| + A_1)  
+ 2\beta(t)A_0 \left[ 16R^3(\beta(t))^3 + 8R^2(\beta(t))^2 + 4R(1 + 4(\beta(t))^3) + 4(\beta(t))^2 \right].
\]

By (2.11), (2.16), and (2.22), taking \( h \) small enough so that
\[
h < \left[ \frac{90P_4(x, t, R)}{R} \right]^{1/4} = h_0,
\]
one gets
\[
|S(x, t, R, h) - I(x, t, R)| < \frac{\epsilon}{2}; \quad |u(x, t) - S(x, t, R, h)| < \epsilon.
\]

Summarizing the following result provides a constructive procedure to obtain an approximate solution of \( u(x, t) \) with a prefixed accuracy.

**Theorem 2.1.** Let \( a(t), b(t) \) be continuous positive functions and let \( f(x) \) be a function satisfying (2.17). Let \( M \) be defined by (2.6) where \( F(w) \) is the Fourier transform of \( f(x) \). Given \( \epsilon > 0 \) and \( (x, t) \) with \(-\infty < x < \infty, t > 0\), take \( R > 0 \) large enough so that (2.9) holds. Then take \( h > 0 \) satisfying (2.24) and an even positive integer such that \( nh = 2R \). Let \( f_j \) be defined by (2.12)–(2.15) for \( 0 \leq j \leq n \) and \( S(x, t, R, h) \) by (2.15). If \( u(x, t) \) is the exact solution of problem (1.1),(1.2), then \( S(x, t, R, h) \) approximates \( w(x, t) \) with an error less than \( \epsilon \).

**Remark 2.1.** Expression \( S(x, t, R, h) \) is symbolically computable using Mathematica 4.0; see [8].

**References**