BIN PACKING:
ASYMPTOTICALLY EXACT APPROACH

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1.

The one-dimensional problem of bin packing (see [1]) can be formulated as follows. Given a list of objects (items) \( S = \{ a_i \mid i = 1, \ldots, n \} \) and the bin capacity \( B \), it is required to pack the objects from the list \( S \) into minimal number of bins in such a manner that the sum of weights of the objects in each bin not exceed \( B \).

In what follows the notation \( a_i \) stands for both an object with the number \( i \) and its weight. It is intrinsic to suppose that \( 0 < a_i \leq B \) for all \( i = 1, \ldots, n \). This problem is NP-difficult (see [2]). Already in first works dealing with the study of bin packing, there was proposed a number of low-cost approximation algorithms and obtained guaranteed bounds for its relative error in the worst case (ibid.). There was noted also that the behavior of these algorithms “on the average” essentially differs from the obtained bounds. Seemingly, this fact stimulated the appearance of works dealing with probabilistic investigation of the problem. As a rule, these works contain either an estimation of the ratio between the mean value of the number of bins required by certain approximation algorithm and the expected optimal number of bins (see [3]–[4]), or an investigation of the probabilistic properties of the optimal solution (see [5]).

In the present article we investigate the quality of the approximation algorithm by means of the idea of constructing algorithms with \((\varepsilon, \delta)\)-estimations which was proposed in [6]. The essence of such an approach is presented in Section 2. In Section 3 we describe the distribution functions for weights of objects, which are considered in what follows. In this case we talk about so-called \( B \)-asymmetric and \( B \)-regular lists. In Section 4 we describe an approximation algorithm \( \mathcal{A} \) for solving bin packing problem and give the proof of the optimality of this algorithm in the case of determined \( B \)-asymmetric and \( B \)-regular lists. In Sections 5–7 we investigate the behavior of a modified approximation algorithm \( \hat{\mathcal{A}} \) on lists with random weights of objects. In Section 5 we prove, for example, a theorem on the upper bound for the probability of the algorithm \( \hat{\mathcal{A}} \) failing. In Sections 6 and 7 we establish the conditions of asymptotical exactness of \( \hat{\mathcal{A}} \) as concerns the \( B \)-asymmetric and \( B \)-regular distribution functions, respectively.

2.

Let \( \mathcal{A} \) be a certain algorithm for solving optimization problem on minimum. We denote by \( F_{\mathcal{A}}(S) \) and \( F^*(S) \) the meanings of the goal functions on the solution of a problem \( S \) obtained by means of the algorithm \( \mathcal{A} \) and on the optimal solution, respectively. We say that the algorithm \( \mathcal{A} \) satisfies estimations \((\varepsilon_n, \delta_n)\) on a class of problems \( \mathcal{K}_n \) of the dimension \( n \) if for all \( S \in \mathcal{K}_n \) there is valid the inequality

\[
Pr\{F_{\mathcal{A}}(S) > (1 + \varepsilon_n)F^*(S)\} \leq \delta_n.
\]

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The algorithm $\mathcal{A}$ is said to be *asymptotically exact* on a class of problems $\mathcal{K}$ if there exist sequences $(\varepsilon_n), (\delta_n)$ such that for any $n$ the algorithm $\mathcal{A}$ satisfies the estimations $(\varepsilon_n, \delta_n)$ on a subset $\mathcal{K}_n \subset \mathcal{K}$ of problems of the dimension $n$ and $\varepsilon_n \to 0, \delta_n \to 0$ as $n \to \infty$.

One can consider the parameters $(\varepsilon_n)$ and $(\delta_n)$ as bounds for the relative error and the probability of failing of the algorithm, respectively.

The technique of $(\varepsilon_n, \delta_n)$-estimations made it possible to establish conditions for asymptotical exactness of the low-cost approximation algorithms for solving a number of difficult problems of discrete optimization (see [7]-[12]), including the bin packing problem for random lists with the symmetric and non-increasing functions of weight distribution (see [9], [11]). In the present article we investigate a modified algorithm for bin packing and justify its asymptotical exactness for a class of problems which is essentially more extensive.

3.

In what follows we suppose that $a_i \in I_B$, where $I_B = \{1, 2, \ldots, B\}$ ($B$ is integer). Let us introduce into consideration the characteristic function (c.f.) $\chi_S$ of a list $S$: $\chi_S(r)$ — the quantity of objects of the weight $r$ in the list $S$, $r \in I_B$. We shall use the notation: $\lfloor x \rfloor$ is the largest integer which do not exceed $x$; $\lceil x \rceil$ is the least integer which are greater than or equal to $x$; $\{x\}$ is the fractional part of $x$ ($\{x\} = x - \lfloor x \rfloor$).

A non-negative function $f$ of integer-valued argument $r \in I_B$ is said to be $b$-asymmetric ($b$-symmetric) if $f(r) \leq f(b - r)$ ($f(r) = f(b - r)$) for $r = 1, \ldots, \lfloor b/2 \rfloor$ and $f(r) = 0$ for $r > b$.

Obviously, the class of $B$-asymmetric functions includes also the $B$-symmetric functions, the non-decreasing functions, the functions of a constant value. In addition, the asymmetry of a list is equivalent to its representability in the form of the maximal $B$-symmetric sublist and a remaining sublist of objects with their weights exceeding $B/2$.

A function $f$ is said to be $B$-regular for $B = 2^Q$, $Q$ is integer, if it can be represented as the sum of $2^k$-symmetric functions $f^{(k)}$, $k \in I_Q$,

$$f(r) = \sum_{k=1}^{Q} f^{(k)}(r), \quad r \in I_B.$$  

The decomposition of a $B$-regular function $f$ into $2^k$-symmetric components $f^{(k)}$, $k \in I_Q$, can be found via the following recurrent relations:

$$f^{(k)}(r) = \begin{cases} f_k(2^k - r) & \text{for } 1 \leq r < 2^{k-1}, \\ f_k(r) & \text{for } 2^{k-1} \leq r \leq 2^k, \\ 0 & \text{for } r > 2^k, \end{cases} \quad f_{k-1}(r) = \begin{cases} f_k(r) - f^{(k)}(r) & \text{for } 1 \leq r \leq 2^{k-1}, \\ 0 & \text{for } r > 2^{k-1}, \end{cases}$$  

(1)

where $f_Q(r) = f(r)$ for $r \in I_B$.

Relations (1) imply: first, that the functions $f_k$ are the $2^k$-regular ones; second, the necessary and sufficient condition for the $B$-regularity of a function $f$ is the fulfillment of the inequalities: $f_k(r) \geq f^{(k)}(2^k - r), \quad r = 1, \ldots, 2^{k-1}, \quad k \in I_Q$; third, the decomposition of a $B$-regular function can be written in the form

$$f(r) = \sum_{k=0}^{B} f^{(k)}(r), \quad r \in I_B,$$

where

$$k_B = Q; \quad k_r = \min \{ k \mid r < 2^k \} = \lceil \log_2 (r + 1) \rceil, \quad 1 \leq r < B.$$

An important subclass of the $B$-regular functions is the set of the non-increasing functions $f(r), \quad r \in I_B$, where $B = 2^Q$, $Q$ being an integer.
Lemma 1 ([9]). A non-increasing function \( f(r), r \in I_B \), (where \( B = 2^Q \), \( Q \) is integer) is \( B \)-regular.

4.

By a \( b \)-bundle we shall call either one object with the weight \( b \) or a pair of objects with the weights \( r \) and \( b-r \) for \( r = 1, \ldots, [b/2] \).

Let us describe an approximation algorithm \( A \) for bin packing of objects, which was suggested by the authors in [9]. If \( B = 2^Q \), \( Q \) is integer, then the algorithm \( A \) consecutively for \( k = Q, Q-1, \ldots, 1 \) forms a list of \( 2^k \)-bundles and packs this list into bins by the principle of the \( NFD \)-algorithm ("next feasible with preliminary ordering by lack of increase"). The objects from a sublist \( S' \subset S \), which cannot be placed into full bins, are packed by means of the \( NF \)-algorithm ("next feasible"). The description of the algorithms \( NFD \) and \( NF \) can be found in [1], [2].

In the case of \( B \)-asymmetric lists and for \( B \neq 2^Q \), \( Q \) being integer, we use a simplified version of the algorithm \( A \); we fill bins by \( B \)-bundles and locate the rest objects \( a_i \in S' \) by means of the \( NF \)-algorithm.

If \( B \leq cn \), where \( c > 0 \) is a constant, then the working time of the algorithm \( A \) is linear with respect to \( n \). It is clear that for \( \sum \{ a_i \mid i \in S' \} \leq B \) the algorithm \( A \) gives us the exact solution of the problem.

Lemma 2 ([9]). For \( B \)-regular lists \( S \), the algorithm \( A \) finds the exact solution of the bin packing problem with the value of the goal function \( F_A = F_S^* \) being equal to

\[
\left\lfloor \frac{1}{B} \sum_{r=1}^{B} r \chi_S(r) \right\rfloor.
\]

Lemma 3. For \( B \)-asymmetric lists \( S \), the simplified algorithm \( A \) finds the exact solution of the bin packing problem with the value of the goal function being equal to

\[
\sum_{r=\left\lfloor (B+1)/2 \right\rfloor}^{B} \chi_S(r) + \left\lfloor \left( \frac{B+1}{2} \right) \chi_S([B/2]) \right\rfloor.
\]

Proof. Indeed, for \( B \)-asymmetric lists, the simplified algorithm \( A \) packs the objects into a minimal number of bins. For odd \( B \), this number is equal to the quantity \( \sum \{ \chi_S(r) \mid B/2 \leq r \leq B \} \) of all those objects whose weight exceeds the half of the capacity of a bin. For even \( B \), we must add to this number \( \lfloor 0.5 \chi_S(B/2) \rfloor \) bins containing the objects of the weight \( B/2 \). Then we can write the additional term in the form \( \lfloor 0.5(B+1) \rfloor \chi_S([B/2]) \), which gives us 0 and \( \lfloor 0.5 \chi_S(B/2) \rfloor \), for odd and even \( B \), respectively. \( \square \)

5.

Now we turn to a bin packing with random lists \( S = \{ a_i \mid i \in I_n \} \), where the weights of objects are independent random variables with the discrete distribution function

\[
p_r = Pr \{ a_i = r \} \geq 0, \quad r = 1, \ldots, B; \quad \sum_{r=1}^{B} p_r = 1.
\]

In what follows we shall consider the class \( K_n^r \) of bin packing problems with the following rule of forming the random lists: during \( n \) consecutive independent tests the next object \( a_i \) gets into one of weight classes \( r = 1, \ldots, B \) in accordance with the distribution function (2). In that situation the values \( \chi_S(1), \ldots, \chi_S(B) \) of the characteristic function of list \( S \) are, generally speaking, dependent random variables because they are connected via relation \( \sum \{ \chi_S(r) \mid r \in I_B \} = n \).

There arises an intrinsic question on the possibility of low-cost algorithms for solving problems of the class \( K_n^r \) in the case where the distribution functions satisfy some properties similar to those which enable us to apply effective exact algorithm \( A \) for the determined lists. This question
does not have a trivial answer, because, in the general situation, the corresponding property of the distribution function \( p_r \) is not preserved for the characteristic functions \( \chi_S(r) \) of a concrete realization of the random list.

Let us note that, in the situation where the weights of objects are random variables, the definition of the class of problems is related to the form of the distribution function (2).

In order to solve the problems from the class \( K'_n \) we apply a modified approximation algorithm \( \tilde{A} \), which uses a solution obtained via the algorithm \( A \) for a certain estimative list \( \tilde{S} \) under the condition that it majorizes the initial list \( S \), i.e., for any weight class \( r = 1, \ldots, B \) there are fulfilled the inequalities

\[
\chi_S(r) \leq \chi_{\tilde{S}}(r).
\]

In the set of random lists \( S \) we select a subset \( S_\Delta \) of the lists satisfying with arbitrary \( r \in I_B \) the following relations:

\[
|\chi_S(r) - np_r| \leq \Delta_r,
\]

where \( \Delta_r = 2(np_r(1 - p_r) \ln n)^{1/2} \).

We apply the algorithm \( \tilde{A} \) only in the case where a concrete realization of a random list belongs to the set \( S_\Delta \). Otherwise we say that the algorithm fails.

Thus, for a list from \( S_\Delta \), we apply first the algorithm \( A \) to estimative list \( \tilde{S} \). Then we remove from full bins all objects which do not belong to the initial list \( S \), and thus obtain solution of the problem with a number of bins which equals \( F_\tilde{A} \).

For estimation of the probability of the algorithm failing we need inequalities for probabilities of large deviations of sum \( S_n = \sum_{i=1}^n \xi_i \) of the independent random variables \( \xi_1, \ldots, \xi_n \), which are distributed by the Bernoulli law

\[
Pr\{\xi_i = 1\} = p; \quad Pr\{\xi_i = 0\} = 1 - p, \quad i = 1, \ldots, n.
\]

**Lemma 4** ([13], Chap. 5, p. 131). For \( \Delta \geq 0 \),

\[
Pr\{S_n - np \geq \Delta\} \leq \exp(-nH(p + \Delta/n)), \quad Pr\{S_n - np \leq -\Delta\} \leq \exp(-nH(p - \Delta/n)),
\]

where \( H(x) = x \ln(x/p) + (1 - x) \ln((1 - x)/(1 - p)) \).

By virtue of the relations

\[
-nH(p + \Delta/n) = -\frac{\Delta^2}{2np(1 - p)} + o(1), \quad -nH(p - \Delta/n) = -\frac{\Delta^2}{2np(1 - p)} + o(1),
\]

which are valid for \( \Delta = o(n^{2/3}) \) (see [13], Chap. 5, p. 131), we obtain

**Corollary.** For \( \Delta = o(n^{2/3}) \), there is valid the inequality

\[
Pr\{|S_n - np| \geq \Delta\} \leq c \exp\left(-\frac{\Delta^2}{2np(1 - p)}\right),
\]

where \( c = 2(1 + o(1)) \).

**Theorem 1.** The probability of the algorithm \( \tilde{A} \) failing on the class of problems \( K'_n \) is bounded above by the value \( O(B/n^2) \).

**Proof.** If \( A_r \) stands for the event defined by (4), then for the probability of the algorithm \( \tilde{A} \) failing we have

\[
\delta_{\tilde{A}} = Pr\left\{ \bigcup_{r=1}^B \tilde{A}_r \right\} \leq \sum_{r=1}^B Pr(\tilde{A}_r),
\]

where \( \tilde{A}_r \) means the event opposite to \( A_r \).
Since $\Delta_r \leq 2(np_r(1 - p_r) \ln n)^{1/2} = o(n^{2/3})$, in accordance with Corollary of Lemma 4 we obtain that
\[
\delta_{\tilde{A}} \leq c^{B} \sum_{r=1}^{B} \exp \left( -\frac{\Delta_r^2}{2np_r(1 - p_r)} \right) = c^{B} \sum_{r=1}^{B} \exp \left( -\frac{4np_r(1 - p_r) \ln n}{2np_r(1 - p_r)} \right) = \frac{cB}{n^2}. \quad \Box
\]

Theorem 1 immediately implies the following

**Corollary.** If $B = o(n/ \ln n)$, then $\delta_{\tilde{A}} = o \left( \frac{1}{n \ln n} \right)$.

Now let us turn to the estimation of exactness of the algorithm $\tilde{A}$ on the class of problems $\mathcal{K}'_n$.

6.

In the case of a $B$-asymmetric function $p_r$, $r = 1, \ldots, B$, we apply the algorithm $\tilde{A}$ under assumption that the characteristic function of estimative list $\tilde{S}$ is equal to
\[
\chi_{\tilde{S}}(r) = \left\lfloor np_r + \Delta_r \right\rfloor, \quad r = 1, \ldots, B.
\]

Property (3) of the estimative list $\tilde{S}$ follows from definition (5) and conditions (4) for the algorithm $\tilde{A}$ functioning.

**Theorem 2.** The algorithm $\tilde{A}$ on the class $\mathcal{K}'_n$ of bin packing problems with $B$-asymmetric function $p_r$ has the following bound of relative error
\[
\varepsilon_{\tilde{A}} = O \left( \sqrt{\frac{B}{n/ \ln n}} \right)
\]
and is asymptotically exact for $B = o(n/ \ln n)$.

**Proof.** Note that, in the case $\Delta_r \leq 2(np_r(1 - p_r) \ln n)^{1/2}$, the $B$-asymmetry of the function $p_r$ yields the same property for the vector $\Delta_r$, because in that situation $(p_r - p_{B-r})(1 - p_r - p_{B-r}) \leq 0$, or $p_r(1 - p_r) \leq p_{B-r}(1 - p_{B-r})$, and, consequently, $\Delta_r \leq \Delta_{B-r}$, $r = 1, \ldots, \lfloor B/2 \rfloor$. Therefore, the characteristic function (5) of the estimative list $\tilde{S}$ is $B$-asymmetric as the integer part of a linear combination of the two $B$-asymmetric functions.

Let us estimate the relative error $\varepsilon_{\tilde{A}} = (F_{\tilde{A}} - F_{\tilde{S}})/F_{\tilde{S}}$ of the algorithm $\tilde{A}$. For convenience, we introduce the notation
\[
\Sigma_r x_r = \sum_{r=1}^{B} x_r.
\]

As a lower bound for the minimal value of the goal function for odd $B$ we take the number $\Sigma_r \chi_{S}(r)$ (i.e., the quantity of objects which have weights greater than the half of bin capacity). For even $B$ we add the value $\chi_{S}(B/2)/2$ to this number. Due to (4) this gives us the following lower bound
\[
F_{\tilde{S}}^* \geq \Sigma_r \chi_{S}(r) + \chi_{S} \left( \left\lfloor \frac{B}{2} \right\rfloor \right) \frac{B + 1}{2} \geq \\
\geq \Sigma_r (np_r - \Delta_r) + (np_{B/2} - \Delta_{B/2}) \frac{B + 1}{2} = \\
= n \left( \Sigma_r p_r + p_{B/2} \left\lfloor \frac{B + 1}{2} \right\rfloor \right) - \Sigma_r \Delta_r.
\]
Now let us obtain the upper bound for the value of the goal function after applying the algorithm \( \tilde{\mathcal{A}} \):

\[
F_{\tilde{\mathcal{A}}} \leq F_Z^* = \Sigma_r \chi_2^*(r) + \left\lfloor \chi_2^* \left( \left\lfloor \frac{B}{2} \right\rfloor \right) \right\rfloor \left( \frac{B + 1}{2} \right) \right\rfloor \leq \\
\leq \Sigma_r (np_r + \Delta_r) + (np_{B/2} + \Delta_{B/2}) \left( \frac{B + 1}{2} \right) + \frac{1}{2}.
\]

We estimate \( \varepsilon_{\tilde{\mathcal{A}}} \) with regard for the fact that \( B \)-asymmetric function satisfies the inequality \( \Sigma_r p_r + p_{B/2}\{(B + 1)/2\} \geq 1/2 \). As a result, we have

\[
\varepsilon_{\tilde{\mathcal{A}}} \leq \frac{\Sigma_r (np_r + \Delta_r) + (np_{B/2} + \Delta_{B/2}) \left( \frac{B + 1}{2} \right) + \frac{1}{2}}{n(\Sigma_r p_r + p_{B/2})\left( \frac{B + 1}{2} \right)} - 1 \leq \frac{2\Sigma_r \Delta_r + 1/2}{n/2 - \Sigma_r \Delta_r}.
\]

Owing to the inequalities \( \Delta_r \leq 2np_r \ln n \) and \( \Sigma_r \sqrt{p_r} \leq \sqrt{B} \), we obtain that

\[
\varepsilon_{\tilde{\mathcal{A}}} \leq \frac{4\sqrt{n} \ln n \Sigma_r \sqrt{p_r} + 1/2}{n/2 - 2\sqrt{n} \ln n \Sigma_r \sqrt{p_r}} \leq 8\sqrt{\frac{B}{n/\ln n}} \left( 1 + \frac{1}{\sqrt{\ln n}} \right) = O\left( \sqrt{\frac{B}{n/\ln n}} \right).
\]

By virtue of Corollary of Theorem 1 we have \( \delta_{\tilde{\mathcal{A}}} = o(1/(n \ln n)) \). Thus, the algorithm \( \tilde{\mathcal{A}} \) under assumptions of this proposition satisfies the bounds \( \varepsilon_{\tilde{\mathcal{A}}} \to 0, \delta_{\tilde{\mathcal{A}}} \to 0 \) as \( n \to \infty \). Consequently, it is asymptotically exact. \( \square \)

**Remark.** If the maximal size of an object is bounded by the value \( R \) which is less than \( B \), then we can refine Theorem 2 by replacing \( B \) with \( R \).

7.

Now let us pass to the analysis of exactness of the algorithm \( \tilde{\mathcal{A}} \) on the class \( \mathcal{K}_n \) of problems with \( B \)-regular distribution function \( p_r \).

We write out the decomposition of the function \( p_r \) into \( 2^k \)-symmetric components \( p^{(k)}_r \), \( k = 1, \ldots, Q \),

\[
p_r = \sum_{k=k_r}^Q p^{(k)}_r, \quad r = 1, \ldots, B. \tag{6}
\]

We use the following notation for the functions

\[
\varphi(x) = nx + 2(\ln x/n)^{1/2}, \quad 0 \leq x \leq 1;
\]

\[
g^{(k)}_r \begin{cases} 
\lfloor \varphi(p^{(Q)}_r) \rfloor & \text{for } k = Q, \\
\lfloor \varphi(p^{(k)}_r) \rfloor & \text{for } 1 \leq k < Q \end{cases} \quad (r = 1, \ldots, B)
\]

and define the list \( \tilde{S} \) via its characteristic function

\[
\chi_{\tilde{S}}(r) = \sum_{k=k_r}^Q g^{(k)}_r, \quad r = 1, \ldots, B. \tag{7}
\]

**Lemma 5.** The list \( \tilde{S} \), defined by (7), is \( B \)-regular one. If the algorithm \( \tilde{\mathcal{A}} \) operates, then the latter list is also an estimative list.

**Proof.** The \( 2^k \)-symmetry of the functions \( p^{(k)}_r \) implies that the functions \( \sqrt{p^{(k)}_r} \) are \( 2^k \)-symmetric. Obviously, the functions \( \varphi(p^{(k)}_r) \) are \( 2^k \)-symmetric as linear combinations of the \( 2^k \)-symmetric functions \( p^{(k)}_r \) and \( \sqrt{p^{(k)}_r} \). Hence the \( 2^k \)-symmetry of the functions \( g^{(k)}_r \) also follows. On the whole, these facts imply the \( B \)-regularity of the mentioned list \( \tilde{S} \) by virtue of the representability of its characteristic function \( \chi_{\tilde{S}}(r) \) in the form of the sum of \( 2^k \)-symmetric functions.
Let us show that the list \( \tilde{S} \) is the estimative one if the algorithm \( \tilde{A} \) operates. Indeed, in that case there are fulfilled the inequalities

\[
\chi_S(r) \leq \left[ n p_r + \Delta_r \right], \quad r = 1, \ldots, B.
\]

On the other hand, formula (6) for decomposition of the distribution function and the inequalities

\[
\left( \sum_i x_i \right)^{1/2} \leq \sum_i x_i^{1/2}, \quad x_i \geq 0,
\]

justify the validity of inequalities (3):

\[
\left[ n p_r + \Delta_r \right] = \left[ n p_r + 2(n p_r (1 - p_r) \ln n)^{1/2} \right] \leq
\leq \left[ n p_r + 2(n p_r \ln n)^{1/2} \right] = \left[ n \sum_{k=k_r}^Q p_r^{(k)} + 2 \left( n \ln n \sum_{k=k_r}^Q p_r^{(k)} \right)^{1/2} \right] \leq
\leq \left[ n \sum_{k=k_r}^Q p_r^{(k)} + 2 \sqrt{n \ln n} \sum_{k=k_r}^Q \sqrt{p_r^{(k)}} \right] = \left[ n \sum_{k=k_r}^Q \varphi(p_r^{(k)}) \right] \leq
\leq \sum_{k=k_r}^Q g_r^{(k)} = \chi_{\tilde{S}}(r), \quad r = 1, \ldots, B. \quad \square
\]

We introduce the following notation

\[
\varepsilon_n = \left( \frac{B/\alpha_p}{n/\ln n} \right)^{1/2},
\]

where \( \alpha_p = \frac{1}{B} \sum r p_r \) is the ratio of the mean value of the weight of objects and the capacity of bin which restricts maximal possible weight of an object. Obviously, \( 0 < \alpha_p \leq 1 \).

**Lemma 6.** If a problem from the class \( K'_B \) has \( B \)-regular distribution function and its list belongs to the set \( S_\Delta \), then it satisfies the following lower bound for the optimal value of the goal function

\[
F^*_S \geq n\alpha_p (1 - \sqrt{3}\varepsilon_n).
\]

**Proof.**

\[
F^*_S \geq \frac{1}{B} \sum_{r=1}^B r\chi_S(r) \geq \frac{1}{B} \sum_{r=1}^B r(n p_r - \Delta_r) \geq
\geq n\alpha_p - \frac{2}{B} \sqrt{n \ln n} \sum_{r=1}^B r \sqrt{p_r} \geq n\alpha_p - \frac{2}{B} \sqrt{n \ln n} \left( \sum_{r=1}^B r \sum_{r=1}^B r p_r \right)^{1/2} \geq
\geq n\alpha_p - \frac{2}{B} \sqrt{n \ln n} \left( \sum_{r=1}^B \frac{B(B + 1)}{2} B \alpha_p \right)^{1/2} \geq n\alpha_p - \sqrt{3n \ln n} \sqrt{B \alpha_p} = n\alpha_p (1 - \sqrt{3}\varepsilon_n). \quad \square
\]

**Lemma 7.** For any problem from the class \( K'_B \), with \( B \)-regular distribution function, the approximation algorithm \( \tilde{A} \) with the estimative list \( \tilde{S} \) (7) guarantees (in the case of its functioning) the upper bound for an approximate value of the goal function

\[
F^*_A \leq n\alpha_p (1 + 2\varepsilon_n + \varepsilon^2_n / \ln n).
\]

**Proof.**

\[
F^*_A \leq F^*_S \leq \frac{1}{B} \sum_{r=1}^B r\chi_{\tilde{S}}(r) + 1 \leq \frac{1}{B} \sum_{r=1}^B r \sum_{k=k_r}^Q g_r^{(k)} + 1 \leq
\leq \frac{1}{B} \sum_{r=1}^B r \left( Q - k_r + \sum_{k=k_r}^Q \varphi(p_r^{(k)}) \right) + 1 = \frac{1}{B} \sum_{r=1}^B r (Q - k_r) +
\]

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\[ + \frac{1}{B} \sum_{r=1}^{B} \sum_{k=k_r}^{Q} (np_r^{(k)}) + 2\sqrt{n \ln(np_r^{(k)})} + 1 = \frac{1}{B}(\Sigma_1 + \Sigma_2 + \Sigma_3) + 1, \]

where

\[
\Sigma_1 = \sum_{r=1}^{B} r(Q - k_r) = \sum_{k=1}^{Q-1} (Q - k) \sum_{r=2}^{2k-1} r = \sum_{k=1}^{Q-1} (Q - k) \frac{3 \cdot 2^{k-1} - 1}{2} 2^{k-1} \leq \frac{3}{8} \sum_{k=1}^{Q} (Q - k)4^k \leq \frac{3}{8} B^2 \sum_{k=1}^{\infty} \frac{k}{4^k} \leq \frac{3}{8} B^2 \sum_{k=1}^{\infty} \frac{0.5 \cdot 2^k}{4^k} = \frac{3}{16} B^2 \sum_{k=1}^{\infty} \frac{1}{2^k} < \frac{B^2}{4};
\]

\[
\Sigma_2 = n \sum_{r=1}^{B} r \sum_{k=k_r}^{Q} p_r^{(k)} = n \sum_{r=1}^{B} r \cdot p_r = nB\alpha_p;
\]

\[
\Sigma_3 = 2\sqrt{n \ln n} \sum_{r=1}^{B} \sum_{k=k_r}^{Q} \sqrt{p_r^{(k)}} \leq 2\sqrt{n \ln n} \sum_{r=1}^{B} r \sqrt{(Q - k_r + 1)p_r} \leq 2\sqrt{n \ln n} (nB\alpha_p (\Sigma_1 + \sum_{r=1}^{B} r))^{1/2} \leq 2B \sqrt{n \ln n} \sqrt{\alpha_p \left(\frac{B}{4} + \frac{B+1}{2}\right)} \leq 2B \sqrt{n \ln n} B\alpha_p,
\]

hence, for \( B > 1 \)

\[
F_A \leq n\alpha_p + 2\sqrt{n \ln n} B\alpha_p + B/4 + 1 < \]

\[
< n\alpha_p \left(1 + 2\sqrt{\frac{B\alpha_p}{n \ln n}} + \frac{B/\alpha_p}{n}\right) = n\alpha_p \left(1 + 2\tilde{\varepsilon}_n + \tilde{\varepsilon}_n^2 / \ln n\right). \]

\[ \square \]

**Theorem 3.** The algorithm \( \tilde{A} \) on the class \( \mathcal{K}_n \) of bin packing problems with \( B \)-regular function of distribution of weights is asymptotically exact for \( B/\alpha_p = o(\ln n) \).

**Proof.** Since \( B \leq B/\alpha_p \), we have that \( B = o(n/\ln n) \) and, in accordance with Corollary of Theorem 1, the probability of the algorithm \( \tilde{A} \) failing is bounded by the value \( o(1/(n/\ln n)) \rightarrow 0 \) as \( n \to \infty \). Thus, our realization of the random list \( S \) satisfies inequalities (3) and (4) with a probability tending to 1 as \( n \) increases. Let us estimate the error of the obtained solution with regard for Lemma 6.

\[
\varepsilon_A = \frac{F_A - F_S}{F_S} \leq \frac{n\alpha_p (1 + 2\tilde{\varepsilon}_n + \tilde{\varepsilon}_n^2 / \ln n)}{n\alpha_p (1 - \sqrt{3}\tilde{\varepsilon}_n)} - 1 = \frac{(2 + \sqrt{3})\tilde{\varepsilon}_n (1 + (2 - \sqrt{3})\tilde{\varepsilon}_n / \ln n)}{1 - \sqrt{3}\tilde{\varepsilon}_n} = O(\tilde{\varepsilon}_n) \to 0 \text{ for } n \to \infty. \]

\[ \square \]

**References**


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