Abstract—This paper characterizes the multicell precoding game where block-diagonalization (BD)-based precoding is utilized on a per-cell basis for downlink transmissions. Sharing the same frequency band, the base-station (BS) at each cell wishes to maximize the sum-rate for its connected mobile-stations (MS) with BD precoding. In this context, the paper considers a strategic non-cooperative game (SNG), where each BS greedily determines its precoding strategy in a distributed manner, based on the knowledge of the inter-cell interference (ICI) at its connected MSs. Via the game-theory framework, the existence and uniqueness of a Nash Equilibrium (NE) of this multicell game are subsequently studied. It is shown that there always exists at least one pure NE in the game, whereas the uniqueness of the NE is guaranteed under a certain condition on the ICI. The paper also characterizes the multicell precoding game where BD-Dirty Paper Coding (BD-DPC) is utilized at each BS on a per-cell basis. Simulation results then confirm our analysis on the NE’s uniqueness in the BD and BD-DPC multicell precoding games.

I. INTRODUCTION

In a multiple-input multiple-out (MIMO) system, space division multiple access (SDMA) can be applied at the base-station (BS) to simultaneously send multiple data streams to multiple mobile-stations (MS). Thus, SDMA can improve the system’s spectral efficiency, albeit appropriate precoding techniques at the BS. Research on precoding designs for a MIMO single-cell system is plentiful in the literature. Dirty paper coding (DPC) [1]–[4] has been proved as the capacity-achieving multi-user precoding strategy. Unfortunately, due to its high complexity implementation that involves random non-linear coding, DPC only remains as a theoretical benchmark. Linear precoding techniques, such as zero-forcing (ZF)-based precoding [5], become an attractive alternative because of its simplicity. In a multicell system, current designs of wireless networks adopt universal frequency reuse where all cells have the potential to use all available radio resources. However, universal frequency-reuse often comes at the cost of inter-cell interference (ICI). Thus, existing research in single-cell precoding designs may need a rework to take into account the effect of the ICI when applying to a multicell system.

Recently, the study of precoding design in a mutually interfered multicell system using game theory has attracted considerable research attention. By considering the multicell system as a strategic noncooperative game (SNG), each BS greedily adapts its precoding strategy to maximize its own utility, given the strategies from other BSs [6]–[9]. In general, these works focus on studying the existence and uniqueness of the game’s Nash equilibrium (NE). The work in [6] studied the precoding game of a two-cell multiple-input single-output (MISO) system. In the multicell MIMO system, [7] studied the competitive precoding design, where each cell selfishly maximizes its mutual information. It is noted that [6], [7] only considered the system where each BS communicates with only one MS. For the case of multiple MSs per cell, [8] studied the multicell SNG where each BS selfishly minimizes its transmit power. The work in [9] investigated the SNG where each BS utilizes ZF precoding to maximize the sum-rate to the MSs within its cell. Both studies in [8], [9] only consider the case of single-antenna MSs. Different from the previous works, this paper examines the multicell SNG under a more general setting where there are multiple MSs per cell and each MS is equipped with multiple antennas.

In this paper, we investigate the multicell precoding game where block-diagonalization (BD) precoding is applied at each BS for the downlink transmissions to its connected MSs. In particular, the BS at each cell selfishly maximizes the sum-rate to its connected MSs by the means of BD precoding [5]. The reasons that we choose to study the multicell game with BD precoding are two-fold. First, BD precoding is much simpler than DPC in implementation and capable of achieving near-DPC performance at high signal-to-noise (SNR) region. Second, one can easily obtain a closed-form water-filling (WF) solution with BD precoding, which then facilitates the characterization of the multicell game. In fact, the WF best response strategy at each BS can be interpreted as a projection onto a closed and convex set. This interpretation shall allow us to study the uniqueness of the game’s NE later on. It shall be shown that the game’s NE always exists and is guaranteed to be unique under a certain condition on the ICI. In the latter part of this paper, we also study the multicell game where each BS applies BD-Dirty Paper Coding (BD-DPC) [2] on a per-cell basis. Our analysis on the multicell BD and BD-DPC games shall be confirmed by thorough numerical simulations.

Notations: $X^H$ and $X^2$ denote the conjugate transpose (Hermitian operator) and the Moore-Penrose pseudo-inverse of the matrix $X$, respectively; $|X|_{m,n}$ stands for the $(m,n)$th entry of the matrix $X$; $\text{Tr}(X)$, $\|X\|$ and $\|X\|_F$ denote the trace, determinant and Frobenius norm of the matrix $X$, respectively; $\rho(X)$, denoting the spectral radius of the matrix $X$, is defined as $\rho(X) \triangleq \max(|\lambda_i|)$, where $\lambda_i$’s are eigenvalues of $X$; $\text{blk}(X_1,\ldots,X_K)$ denotes a square block-diagonal matrix with the main diagonal blocks as square matrices $X_1,\ldots,X_K$. 

Block Diagonalization Precoding Game in a Multiuser Multicell System

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II. THE MULTICELL BLOCK-DIAGONALIZATION PRECODING GAME

We consider a multiuser multicell downlink system with \( Q \) separate cells operating on the same frequency channel. In a particular cell, say cell-\( q \), one multiple-antenna BS is concurrently sending independent data streams to \( K_q \) remote MSs, each is equipped with multiple receive antennas. Let \( M_q \) and \( N_q \) be the numbers of antennas of the BS and the \( i \)th MS, respectively. Denote \( x_q \in \mathbb{C}^{M_q \times 1} \) as the transmitted signal vector from BS-\( q \). Assuming linear precoding at the BS, \( x_q \) can be represented as \( x_q = \sum_{i=1}^{K_q} W_{iq} s_{iq} \), where \( W_{iq} \in \mathbb{C}^{M_q \times L_q} \) is the precoding matrix and \( s_{iq} \in \mathbb{C}^{L_q \times 1} \) is the data symbol vector of length \( L_q \) intended for MS-\( i \). Without loss of generality, we assume \( E[\|s_q\|^2] = 1, \forall i, \forall q \).

At cell-\( q \), the transmission to MS-\( i \) can be modeled as

\[
y_{qi} = H_{qq} \sum_{j=1}^{K_q} W_{jq} s_{jq} + \sum_{r \neq q} H_{qr} \sum_{j=1}^{K_r} W_{rjq} s_{rjq} + z_{qi}, \quad (1)
\]

where \( H_{qqij} \in \mathbb{C}^{N_q \times M_r} \) models the channel coefficients from BS-\( r \) to MS-\( i \) in cell-\( q \), and \( z_{qi} \) models the zero-mean complex additive Gaussian noise vector with an arbitrary covariance matrix \( \mathbf{Z}_{qi} \). It is observed from (1) that the received signal at MS-\( i \) of cell-\( q \) comprises of 4 components: the direct signal \( H_{qq} W_{qiq} s_{iq} \), the intra-cell interference \( H_{qq} \sum_{j \neq i} W_{qjq} s_{jq} \), the ICI \( \sum_{r \neq q} H_{qr} \sum_{j=1}^{K_r} W_{rjq} s_{rjq} \), and the Gaussian noise \( z_{qi} \).

In this system model, it is assumed that each BS possesses full knowledge of the downlink channels to the MSs in its cell, but not the channels to the MSs in other cells. As a result, the BS cannot control its induced ICI to other cells. On the other hand, the intra-cell interference can be fully controlled or completely suppressed by the BD precoding on a per-cell basis. Note that BD precoding requires the total number of receive antennas not exceeding the number of transmit antennas at the BS, i.e., \( \sum_{i=1}^{K_q} N_q i \leq M_q, \forall q \). Herein, it is assumed that this is the case for the considered system model.

In this work, we are interested in formulating the multicell BD precoding design using the game-theory framework. In particular, we consider a SNG, where the players are the cells, the payoff functions are the sum-rates. At each cell, the BS strategically adapts its BD precoder on a per-cell basis that greedily maximizes the sum-rate to its connected MSs, subject to a constraint on its transmit power.

Let \( \Omega = \{1, \ldots, Q\} \) be the set of \( Q \) players. Denote \( Q_q = W_{qiq} W_{qiq}^H \) as the transmit covariance matrix intended for MS-\( i \) of cell-\( q \), and denote \( Q_q = \{Q_{qi}\}_{i=1}^{K_q} \) as the precoding profile for the \( K_q \) MSs in cell-\( q \). Likewise, let \( Q_{-q} = \{Q_1, \ldots, Q_{q-1}, Q_{q+1}, \ldots, Q_Q\} \) denote the precoding profile of all cells except cell-\( q \). Denote \( R_{qi}(Q_{-q}) \) as the covariance matrix of the total ICI plus background noise (IPN) at the MS-\( i \) of cell-\( q \), which is defined as

\[
R_{qi}(Q_{-q}) = \sum_{r \neq q} H_{qr} \left( \sum_{j=1}^{K_r} Q_{rj} \right) H_{qr}^H + \mathbf{Z}_{qi}. \quad (2)
\]

With BD precoding applied an a per-cell basis at BS-\( q \), the achievable data rate \( R_{qi} \) to MS-\( i \) is then given by

\[
R_{qi}(Q_{q}, Q_{-q}) = \log |I + H_{qq}^H R_{qi}^{-1}(Q_{-q}) H_{qq}|. \quad (3)
\]

Define \( R_{qi}(Q_{q}, Q_{-q}) = \sum_{j=1}^{K_q} R_{qi}(Q_{q}, Q_{-q}) \) as the payoff function of player-\( q \). Then, given a strategy profile \( Q_{-q} \) from other players, player-\( q \) selfishly maximizes its payoff function by solving the following optimization problem

\[
\begin{align*}
\text{maximize} & \quad R_q(Q_q, Q_{-q}) \\
\text{subject to} & \quad H_{qq} Q_q H_{qq}^H = 0, \forall j \neq i \\
& \quad \sum_{i=1}^{K_q} \text{Tr}(Q_{qi}) \leq P_q,
\end{align*}
\]

where \( P_q \) is the power budget at BS-\( q \). To achieve the maximum sum-rate at cell-\( q \), it is assumed that the IPN matrix \( R_{qi}(Q_{-q}) \) is perfectly measured at the corresponding MS-\( i \) and reported back to the BS. It is evidenced from the optimization problem (4) that the optimal strategy of player-\( q \) depends on the strategies of other players.

Due to the constraints \( H_{qq} Q_q H_{qq}^H = 0, \forall j \neq i \), each column of the precoder matrix \( W_q \) must be in the null space created by \( H_q = [H_{qq}^T, \ldots, H_{qq}^T \vdots H_{qq}^T \vdots \vdots H_{qq}^T]^T \). Suppose that one performs the singular value decomposition the \( (\sum_{j \neq i} \sum_{n_q} N_q n_q) \times M_q \) matrix \( H_q \), as

\[
H_q = U_q \Sigma_q V_q^H = U_q \Sigma_q \left[ \begin{smallmatrix} 0 \\ V_q \end{smallmatrix} \right] V_q^H, \quad (5)
\]

where \( \Sigma_q \) is diagonal, \( U_q \) and \( V_q \) are unitary matrices, and \( V_q \) is formed by the last \( \tilde{N}_q = M_q - \sum_{j \neq i} \tilde{N}_q \) columns of \( V_q \). Then, any precoding covariance matrix \( Q_{q} \) formulated as \( V_q D_q V_q^H \), where \( D_q \geq 0 \) is an arbitrary \( \tilde{N}_q \times \tilde{N}_q \) matrix, would make \( H_{qq} Q_{q} H_{qq}^H = 0, \forall j \neq i \). Thus, the set of admissible strategies for player-\( q \) can be defined as follows:

\[
S_q = \{Q_q \in \mathbb{S}^{M_q \times M_q} : Q_q = V_q D_q V_q^H, D_q \geq 0, \sum_{i=1}^{K_q} \text{Tr}(D_{qi}) \leq P_q \}. \quad (6)
\]

Mathematically, the game has the following structure

\[
\mathcal{G} = \left( \Omega, \{S_q\}_{q \in \Omega}, \{R_q\}_{q \in \Omega} \right). \quad (7)
\]

A Nash equilibrium (NE) of game \( \mathcal{G} \) is defined when \( R_q(Q_q, Q_{-q}^*) \geq R_q(Q_q, Q_{-q}^*), \forall Q_q \in S_q, \forall q \in \Omega. \quad (8)\)

At a NE, given the precoding strategy from other cells, a BS does not have the incentive to unilaterally change its precoding strategy, i.e., it shall achieve a lower sum-rate with the same power constraint.

III. CHARACTERIZATION OF THE GAME’S NE

In this section, we study the two most fundamental questions in analyzing a SNG: the existence and uniqueness of the game’s NE. The NE characterization allows us to predict a stable outcome of the noncooperative BD precoding design in game \( \mathcal{G} \). The existence of a pure NE in game \( \mathcal{G} \) can be deduced straightforward from the work in [10] for \( N \)-person quasi-concave games. First, as the strategy set \( S_q \) for
player-$q$ defined in (6) is compact and convex, $\forall q$. Second, the utility function $R_q(Q_q, Q_{-q})$ is a continuous function in the profile of strategies $S_q$, and concave in $Q_{q_1}, \ldots, Q_{q_k}$. Thus, Theorem 1 in [10] indicates that there always exists at least one pure NE in game $G$.

In order to study the uniqueness of a NE in game $G$, we first investigate the best response strategy at each player. As defined in $S_q$, the best response strategy of player-$q$ must be in the form $Q_q = \tilde{V}_q, D_q, \tilde{V}_q, D_q, \forall i$, where $D_q$ can be obtained from the following optimization problem

$$\maximize_{D_q, \ldots, D_{K_q}} \sum_{i=1}^{K_q} \log \left[ 1 + \tilde{V}_q R_q^{-1}(Q_q) H_{qq} \tilde{V}_q, D_q \right]$$

subject to

$$\sum_{i=1}^{K_q} \text{Tr}(D_q) \leq P_q, \quad D_q \succeq 0, \quad \forall i.$$  

By eigen-decomposing $\tilde{V}_q R_q^{-1}(Q_q) H_{qq} \tilde{V}_q = \tilde{U}_q \Lambda_q \tilde{U}_q^T$, the optimal solution to problem (9) can be easily obtained from the water-filling (WF) procedure

$$D_q \triangleq \text{WF}_q(D_{-q}) = \tilde{U}_q \left[ \mu_q I - \Lambda_q^{-1} \right]^+ \tilde{U}_q^T,$$

where the water-level $\mu_q$ is adjusted to meet the power constraint $\sum_{i=1}^{K_q} \text{Tr} \left\{ \left[ \mu_q I - \Lambda_q^{-1} \right]^+ \right\} = P_q$. Note that as $\tilde{V}_q$ only depends on in-cell channels at cell-$q$, BS-$q$ only needs to strategically adapt its precoding matrices $D_q$, $\forall i$ as in (10).

While the best response strategy of each player can be obtained in a closed-form solution in (10), the nonlinear structure in the WF operator is rather problematic in analyzing the uniqueness of the game’s NE. Fortunately, the WF operator can be interpreted as a projection onto a closed set [7]. As studied in [7] for the case of single-user MIMO WF, this interpretation is also applicable to the multi-user WF considered in problem (9).

**Lemma 1.** The optimization problem (9) is equivalent to the following optimization problem:

$$\minimize_{D_q, \ldots, D_{K_q}} \sum_{i=1}^{K_q} \left\| D_q + \left[ \tilde{V}_q R_q^{-1}(Q_q) H_{qq} \tilde{V}_q \right]^2 \right\|_F^2$$

subject to

$$\sum_{i=1}^{K_q} \text{Tr}(D_q) = P_q, \quad D_q \succeq 0.$$  

where $P_{N(H_{qq} V_q)}$ is the projection onto the null space of $H_{qq} \tilde{V}_q$, and $c_q$ is an arbitrarily large constant satisfying $c_q \geq P_q + \max_{i, \forall k} [A_{qq}]^{-1}_{kk}$.

**Proof:** Please see Appendix A. □

From Lemma 1, the WF solution in (10) is indeed the solution of the optimization (11). Thus, the block-diagonal WF solution $\text{WF}_q(D_{-q}) \triangleq \text{blk}\{\text{WF}_q(D_{-q})\}$, can be interpreted as a projection

$$\text{WF}_q(D_{-q}) = \begin{bmatrix} -\text{blk} \left[ \tilde{V}_q R_q^{-1}(Q_q) H_{qq} \tilde{V}_q \right]^2 + c_q P_{N(H_{qq} V_q)} \end{bmatrix}_{D_q},$$

where $D_q \triangleq \{D_q \in \mathbb{S}^{N_q \times N_q} : \sum_{i=1}^{K_q} \text{Tr}(D_q) = P_q\}$, is a closed and convex set.

Let $D_q \triangleq \text{blk}(D_q)$, $D = \{D_q\}_{q \in \mathcal{Q}}$, and define the multicell mapping $\text{WF}(D) = \{\text{WF}_q(D_{-q})\}_{q \in \mathcal{Q}}$. Let $e_{WF} = \|\text{WF}_q(D^{(1)}) - \text{WF}_q(D^{(2)})\|_F$ and $e_q = \|D^{(1)} - D^{(2)}\|_F$, for any given $D^{(1)} \neq D^{(2)}$ and $D^{(1)}(1), D^{(2)}(2) \in D_q, \forall q$. Then,

$$e_{WF} = \left\| \begin{bmatrix} -\text{blk} \left[ \tilde{V}_q R_q^{-1}(Q_q) H_{qq} \tilde{V}_q \right]^2 + c_q P_{N(H_{qq} V_q)} \end{bmatrix} \right\|_F$$

and

$$\leq \sum_{i=1}^{K_q} \left\| \begin{bmatrix} -\text{blk} \left[ \tilde{V}_q R_q^{-1}(Q_q) H_{qq} \tilde{V}_q \right]^2 + c_q P_{N(H_{qq} V_q)} \end{bmatrix} \right\|_F \leq \sum_{i=1}^{K_q} \left\| \begin{bmatrix} -\text{blk} \left[ \tilde{V}_q R_q^{-1}(Q_q) H_{qq} \tilde{V}_q \right]^2 + c_q P_{N(H_{qq} V_q)} \end{bmatrix} \right\|_F$$

\[ (13a) \]

where

$$\begin{align*}
\sum_{i=1}^{K_q} \left\| \begin{bmatrix} -\text{blk} \left[ \tilde{V}_q R_q^{-1}(Q_q) H_{qq} \tilde{V}_q \right]^2 + c_q P_{N(H_{qq} V_q)} \end{bmatrix} \right\|_F \\
\sum_{i=1}^{K_q} \left\| \begin{bmatrix} -\text{blk} \left[ \tilde{V}_q R_q^{-1}(Q_q) H_{qq} \tilde{V}_q \right]^2 + c_q P_{N(H_{qq} V_q)} \end{bmatrix} \right\|_F \\
\sum_{i=1}^{K_q} \left\| \begin{bmatrix} -\text{blk} \left[ \tilde{V}_q R_q^{-1}(Q_q) H_{qq} \tilde{V}_q \right]^2 + c_q P_{N(H_{qq} V_q)} \end{bmatrix} \right\|_F
\end{align*}$$

\[ (13b) \]

\[ (13c) \]

Note that the inequality (13a) holds due to the non-expansive property of the projection onto a closed and convex set [11], inequalities (13b) and (13d) hold because $X = \text{diag}\{X_1, \ldots, X_K\}$ implies that $\|X\|_F \leq \sum_{i=1}^{K_q} \|X_i\|_F$, inequality (13c) holds due to the reverse order law for the Moore-Penrose pseudo-inverse [7], and inequality (13e) holds because the Frobenius norm is consistent [12].

Define the vectors $e_{WF} = [e_{WF1}, \ldots, e_{WFQ}]^T$ and $e = [e_1, \ldots, e_Q]^T$. The set of inequalities (13f) implies that

$$0 \leq e_{WF} \leq \text{Se}.$$  

(15)

We now provide some definitions of the matrix norms and vector norms applied to study the contraction property of game $G$. Given the mapping $D^{(1)} = \text{WF}(D^{(1)})$ and a vector $w = [w_1, \ldots, w_Q]^T > 0$, the block-maximum norm [11] on the mapping $\text{WF}(D)$ is defined as
In addition, the condition G, which implies the uniqueness of the NE in game G, is higher, which then guarantees the uniqueness of the NE. Finally, define the matrix norm of a matrix \( X \) by
\[
\|X\|_{\infty, \text{vec}} = \max_{q \in \Omega} \frac{1}{w_q} \sum_{r=1}^{Q} |A_{q,r} w_r|, \quad A \in \mathbb{R}^{Q \times Q}.
\]
From the inequality (15), one has
\[
\|e W F\|_{\infty, \text{vec}} \leq \|S e\|_{\infty, \text{vec}} \leq \|S\|_{\infty, \text{mat}} \|e\|_{\infty, \text{vec}},
\]
as the induced \( \infty \)-norm \( \| \cdot \|_{\infty, \text{vec}} \) is consistent [12]. Then,
\[
\|W F(D^{(1)}) - W F(D^{(2)})\|_{F, \text{block}} = \max_{q \in \Omega} \frac{1}{w_q} \|W F_q(D^{(1)}) - W F_q(D^{(2)})\|_{F, \text{block}}
\]
\[
= \|e W F\|_{\infty, \text{vec}} \leq \|S\|_{\infty, \text{mat}} \|e\|_{\infty, \text{vec}}.
\]
Thus, if \( \|S\|_{\infty, \text{mat}} < 1 \), the W F(D) mapping is a contraction, which implies the uniqueness of the NE in game G [11]. In addition, the condition \( \|S\|_{\infty, \text{mat}} < 1 \) is also sufficient to guarantee the convergence of the NE from any starting precoding strategy \( D_q \in D_q \). Note that S is a nonnegative matrix, there always exists a positive vector w satisfying [11]
\[
(C) : \quad \|S\|_{\infty, \text{mat}} < 1 \iff \rho(S) < 1.
\]

Remark 1: Assuming the path loss fading model in this multicell system, a physical interpretation of the sufficient condition (C) is as follows. When the intra-cell BS-MS distance gets smaller relatively to the distance between the BSs, the ICI becomes less dominant. Thus, the positive off-diagonal elements of S also become smaller. This results in a smaller spectral radius of S. Therefore, as the MSs are getting closer to its connected BS, the probability of meeting condition (C) is higher, which then guarantees the uniqueness of the NE.

IV. EXTENSION TO THE MULTICELL BD-DPC PRECODING GAME

This section considers the multicell precoding game where each BS utilizes BD-DPC to the downlink transmissions to its connected MSs. It is well-known that DPC is the capacity-achieving encoding scheme for the multi-user broadcast channel [2]. In [2], a suboptimal and simpler zero-forcing DPC (ZF-DPC) scheme was proposed for single-antenna receivers that takes advantage of both DPC and ZF precoding. In ZF-DPC, the information signals sent to the multiple MSs are encoded in sequence such that the receiver at any user does not see any intra-cell interference due to the use of ZF and DPC at the BS. In this work, we apply a similar technique to the encoding process at each BS. Due to the consideration of multi-antenna receivers, the technique shall be referred to as the BD-DPC precoding.

At any BS, say BS-q, denote the encoding sequence to its \( K_q \) connected MSs as \( \pi_q = [\pi_q(1), \ldots, \pi_q(K_q)]^T \). The concept of BD-DPC can be briefly explained as followed:

- BS-q freely designs the precoder \( W_{\pi_q(1)} \) to MS-\( \pi_q(1) \).
- BS-q, having the noncausal knowledge of the codeword intended for MS-\( \pi_q(1) \), uses DPC such that MS-\( \pi_q(2) \) does not see the codeword for MS-\( \pi_q(1) \) as interference. At the same time, the precoder \( W_{\pi_q(2)} \) is designed on the null space caused by \( H_{\pi_q(1)} \) to eliminate the its induced interference to MS-\( \pi_q(1) \).
- Similarly, to encode the signal for user-i, BS-q can utilize the noncausal knowledge of the codewords for MSs \( \pi_q(1), \ldots, \pi_q(i-1) \), and design \( W_{\pi_q(i)} \) on the null space caused by \( H_{\pi_q(i(-1))} \).

Similar to game \( G \) defined in Section II, we consider a new game \( G' \), where each BS strategically adapts its BD-DPC precoders to maximize the sum-rate to its connected MSs. Mathematically, game \( G' \) can be defined as
\[
G' = \{\Omega, \{S_q'((\pi_q))\}_{q \in \Omega}; \{R_q\}_{q \in \Omega}\},
\]
The set of admissible strategies \( S_q'((\pi_q)) \) is now defined as
\[
S_q'((\pi_q)) = \left\{ Q_{\pi_q(i)} \in \mathbb{S}^{M_q \times M_q}; Q_{\pi_q(i)} = \hat{V}_{\pi_q(i)} D_{\pi_q(i)} \hat{V}_{\pi_q(i)}^H, \quad D_{\pi_q(i)} \succeq 0, \sum_{i=1}^{K_q} \text{Tr}\{D_{\pi_q(i)}\} \leq P_q \right\},
\]
where \( \hat{V}_{\pi_q(i)} \) is the null space created by \( H_{\pi_q(i)}^H \). Due to the similarity between games \( G \) and \( G' \), the characterization for game \( G' \) presented in Section III can be directly applied to game \( G' \). In particular, it can be concluded that there always exists at least one NE in game \( G' \) and the NE is unique if
\[
(C') : \quad \rho(S') < 1,
\]
where \( S' \in \mathbb{C}^{Q \times Q} \) is defined as
\[
[S']_{q,r} = \begin{cases} 
\sum_{i=1}^{K_q} \rho(\hat{V}_{\pi_q(i)} H_{\pi_q(i)}^H H_{\pi_q(i)} \hat{V}_{\pi_q(i)} D_{\pi_q(i)} \hat{V}_{\pi_q(i)}^H \hat{V}_{\pi_q(i)} H_{\pi_q(i)} \hat{V}_{\pi_q(i)}), & \text{if } r \neq q \\
0, & \text{if } r = q 
\end{cases}
\]
with \( \hat{V}_{\pi_q} \triangleq [\hat{V}_{\pi_q(1)}, \ldots, \hat{V}_{\pi_q(K_q)}] \).

Remark 2: Due to the dependence of the admissible strategy set \( S_q'((\pi_q)) \) on the encoding order \( \pi_q \) at BS-q, the characterization of game \( G' \) strictly depends on the encoding order at each BS. In addition, with different encoding orders at a BS, say BS-q, the optimal strategies to maximize the sum-rate at BS-q are also different. The condition (C') for the uniqueness of game \( G' \) also depends on the encoding order at each BS-q. In fact, for any permutation in \( \pi_1, \ldots, \pi_Q \), we have at
least a different NE of game $G'$. Given $K_q$, encoding order permutations at BS-$q$, it can be concluded that game $G'$ has at least $\prod_{q=1}^{Q} (K_q)!$ NE points.

**Remark 3:** For a particular encoding order $\pi_1, \ldots, \pi_Q$ in game $G'$, game $G'$ provides a higher degree of freedom in designing the precoder at each BS. In fact, the size of matrix $V_{\pi_q(i)}$ in game $G'$ is at least equal or larger than its counterpart $V_{q}$ in game $G$. Intuitively, the off-diagonal elements of matrix $S'$ are also larger than that of matrix $S$. As a result, it is expected that the condition for the uniqueness of the NE in game $G'$ is stricter than that in game $G$.

**V. SIMULATION RESULTS AND DISCUSSIONS**

In this section, we present some simulation results validating our studies on the uniqueness of a NE and the convergence to the NE in games $G$ and $G'$. We also present the sum-rates of the multicell system when game $G$ and $G'$ are played, in comparison to the game with DPC precoding at each BS. In the same system setting, the DPC precoding [2], [3] is performed on a per-cell basis in a non-cooperative manner (each BS selfishly maximizes its own sum-rate). For the BD-DPC precoding game, we assume a fixed encoding order from MS-1 to MS-$K_q$ at BS-$q$ and similar orders at other BSs.

Considered is a 3-cell system with 3 MSs per cell sharing the same channel frequency, as illustrated in Fig. 1. The numbers of antennas at each BS and each MS are set at $M_q = 8$ and $N_q = 2$. The same power constraint $P_q = 1$ is set at each BS, whereas the AWGN at each MS is set as $Z_{q_i} = \sigma^2 I$ with $\sigma^2 = 0.01$. The distance between any 2 BSs is normalized to 2. In each cell, the MSs are assumed to be randomly located on a circle from its connected BS with the radius of $d$. The channels from a BS to a MS are generated by using the path-loss model, where the path-loss exponent is set at 3. In each figure, each plotted point is obtained by averaging over 10,000 independent channel realizations.

Fig. 2 displays the probability of the NE’s uniqueness versus the intra-cell BS-MS distance $d$ by evaluating condition (C) for game $G$ and condition (C$'$) for game $G'$. Corresponding to a small $d$ is the low-ICI (or high-SINR) region. In contrast, at large $d$, each MS is more susceptible to higher level of ICI (and lower SINR as a result). As observed from the figure, the uniqueness of the NE (in both games $G$ and $G'$) is guaranteed if the ICI is sufficiently small, as suggested in our analytical result in Section III. In addition, the condition of the NE’s uniqueness in game $G'$ is much stricter than that in game $G$, as analyzed in Section IV.

![Fig. 1. A multicell system configuration with 3 cells, 3 users per cell.](image1)

![Fig. 2. Probability of NE’s uniqueness versus the intra-cell BS-MS distance $d$.](image2)

![Fig. 3. Network sum-rates versus the intra-cell BS-MS distance $d$.](image3)

\[\text{Fig. 1. A multicell system configuration with 3 cells, 3 users per cell.}\]

\[\text{Fig. 2. Probability of NE’s uniqueness versus the intra-cell BS-MS distance $d$.}\]

\[\text{Fig. 3. Network sum-rates versus the intra-cell BS-MS distance $d$.}\]
To illustrate the convergence of the multicell precoding games $G$ and $G'$, we select a specific channel realization and plot the achievable sum-rates versus the number of iterations of the two designs in Fig. 4. In both games, it is assumed that the BSs perform sequential precoder update at each time instance. It is observed that both games converge very quickly in a few iterations. As expected, the BD-DPC game results in a higher network sum-rate over the BD game due to the superior performance of BD-DPC precoding over BD precoding on a per-cell basis.

![Graph](image)

Fig. 4. Sum-rates versus the number of iterations (solid lines are for the BD precoding game and dashed-dot lines are for the BD-DPC precoding game).

VI. CONCLUSION

This paper studied the multicell network with universal frequency reuse where BD or BD-DPC precoding is performed on a per-cell basis. Using a game-theory framework, we investigated the conditions on the existence and uniqueness of the multicells’ NE. Simulation results confirmed with the analysis that the NE of the multicell games is unique if the ICI is sufficiently small. They also indicated that the BD-DPC multicell precoding game outperforms the BD game, and can achieve the sum-rate very close to that of the DPC precoding game, while the BD and BD-DPC precoding designs enjoy a much simpler implementation than DPC.

APPENDIX A

PROOF OF LEMMA 1

The proof for this lemma is similar to that of Lemma 1 in [7] for the case of single-user MIMO WF. Given that $V_q^H H_q^H R_q^{-1} (Q_{-q}) H_q q_q = \hat{U}_q \Lambda_q \hat{U}_q^H$, one has

$$\left( V_q^H H_q^H R_q^{-1} (Q_{-q}) H_q q_q \right)^2 = \hat{U}_q \Lambda_q^{-1} \hat{U}_q^H. \quad (27)$$

Note that $\hat{U}_q$ is a $N_q \times N_q$ unitary matrix, i.e., $\hat{U}_q \hat{U}_q^H = I$, and $\Lambda_q$ is a $N_q \times N_q$ diagonal matrix. By the assumption that $\sum_{i=1}^{K_q} N_q \leq M_q$, i.e., $N_q \leq N_q$, one may form a unitary matrix $U_q = [U_q^1 \ U_q^2]$, where $U_q^1$ is a $N_q \times (N_q - N_q)$ matrix satisfying $U_q^1 U_q^1 = 0$ and $U_q^1 U_q^1 = I$. In addition, $N(H_q q_q \ V_q) = \hat{N}(V_q^H H_q q_q^{-1} (Q_{-q}) H_q q_q \ V_q)$ implies that $P_{N_q}(H_q q_q \ V_q) = \hat{U}_q \Lambda_q^{-1} \hat{U}_q^H$. Thus, for a given $c_q$, one has

$$\left( V_q^H H_q^H R_q^{-1} (Q_{-q}) H_q q_q \ V_q \right)^2 + c_q P_{N_q}(H_q q_q \ V_q) = \hat{U}_q \Lambda_q^{-1} \hat{U}_q^H,$$

where $\Lambda_q = \text{blk}\{\Lambda_q, (1/c_q)I\}$.

The optimization problem (11) then can be rewritten as

$$\begin{array}{ll}
\text{minimize} & \sum_{k=1}^{K_q} \left\| D_{q_k} + \hat{A}_{q_k} \right\|_F^2 \\
\text{subject to} & \sum_{i=1}^{K_q} Tr \{ D_{q_k} \} = P_q, \ \hat{D}_{q_k} \succeq 0,
\end{array} \quad (28)$$

where $D_{q_k} \triangleq \hat{U}_q^H D_{q_k} \hat{U}_q$. Due to the fact the objective function is lower-bounded by diagonal matrices $\{D_{q_k}\}$, the optimal solution set $\{D_{q_k}\}$ to problem (28) have to be diagonal. Thus, the optimization (28) can be reduced to

$$\begin{array}{ll}
\text{minimize} & \sum_{k=1}^{K_q} \left\{ \left[ D_{q_k} \right]_{k,k} + \left[ \hat{A}_{q_k} \right]_{k,k} \right\}^2 \\
\text{subject to} & \sum_{i=1}^{K_q} \left[ D_{q_k} \right]_{k,k} = P_q, \ \left[ D_{q_k} \right]_{k,k} \succeq 0,
\end{array} \quad (29)$$

whose (unique) optimal solution has the WF structure such that $D_{q_k} = \left[ \mu_q I - \hat{A}_{q_k} \right]^{-1}$, where $\mu_q$ is the water-level to meet the power constraint $\sum_{k=1}^{K_q} \sum_{i=1}^{K_q} Tr \{ D_{q_k} \} = P_q$. Thus, the optimal solution to the original problem (11) is given by $D_{q_k} = \hat{U}_q^H \left[ \mu_q I - \text{blk}\{\hat{A}_{q_k}, c_q I\} \right]^{-1} \hat{U}_q^H$, if $c_q$ is chosen to be large enough such that $|\mu_q - c_q|^{-1} = 0$. As suggested in [7], choosing $c_q \geq P_q + \max_{i,k} \{\Lambda_{q_k}\}_{i,k}^{-1}$ is sufficient to meet this requirement. This concludes the proof for Lemma 1.

REFERENCES