Based on decision-theoretic rough sets (DTRS), we augment the existing model by introducing into the granular values. More specifically, we generalize a concept of the precise value of loss function to triangular fuzzy decision-theoretic rough sets (TFDTRS). Firstly, ranking the expected loss with triangular fuzzy number is analyzed. In light of Bayesian decision procedure, we calculate three thresholds and derive decision rules. The relationship between the values of the thresholds and the risk attitude index of decision maker presented in the ranking function is analyzed. With the aid of multiple attribute group decision making, we design an algorithm to determine the values of losses used in TFDTRS. It is achieved with the use of particle swarm optimization. Our study provides a solution in the aspect of determining the value of loss function of DTRS and extends its range of applications. Finally, an example is presented to elaborate on the performance of the TFDTRS model.

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1. Introduction

Rough set theory is a new mathematical tool to deal with uncertainty problem [41]. As an extension model of original rough sets, probabilistic rough sets (PRS) play a significant role in rough sets and attract the attention of many researchers [57,59]. In recent years, a series of PRS models [57] were proposed, such as 0.5-probabilistic rough sets, decision-theoretic rough sets (DTRS), variable precision rough sets (VPRS), Bayesian rough sets, parameterized rough sets, game-theoretic rough sets (GTRS), probabilistic rough set over two universes, etc. The determination for a pair of thresholds used in PRS becomes a substantial challenge [30]. The pair of thresholds in the most PRS models need a reasonable semantic interpretation [59,61,62]. By introducing game theory into PRS, Herbert and Yao [17] proposed GTRS to determine the values of thresholds used in PRS. Herbert and Yao [18] investigated the GTRS model and its capability of analyzing a major decision problem evident in the existing PRS. Azam and Yao [4] extended GTRS for formulating and analyzing multiple criteria decision making problems in rough sets. Using Shannon entropy as a measure of uncertainty, Deng and Yao [13] presented an information-theoretic approach to the interpretation and determination of thresholds used in PRS. DTRS was proposed by Yao et al. [55,56], which provided a new interpretation in the aspect of determining the threshold values. For the DTRS model, the pair of thresholds presented in PRS can be calculated by loss function with the minimum expected overall risk, where the losses are associated with the decision risk. DTRS has been applied to many domains, such as email filtering [67], investment decision [29,33], cluster [26], text classification [23], information filtering [22], web-based support systems [54], etc. Hence, DTRS has become an important research direction of rough sets.

From the viewpoint of semantics, Yao [63] reviewed several generalized (modified) models and the applications of the DTRS model. In the granulated view of the universe, Abd E1-Monsef and Kilany [1] proposed a generalized decision-theoretic model based on a general binary relation. Greco et al. [16] proposed a Bayesian decision theory for dominance-based rough set approach (DRSA), which was permitted to take into account costs of misclassification in variable consistency.
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Different attitudes of decision makers, Li and Zhou [24] proposed multi-view decision models of DTRS. Based on the misclassification cost and the test cost, Li et al. [25] designed an algorithm for searching an optimal test set of attributes with the minimum total cost. Liu et al. [31] and Lingras et al. [26] extended DTRS with two-category to multi-category. Considering the losses with probabilistic distribution, Liu et al. [32] proposed an extension of DTRS under the uniform and the normal distribution. Liu et al. [34] designed a method for estimating the conditional probability using logistic regression. Ma and Sun [35] extended Bayesian risk decision of PRS of the same universe to two universes. Yao [59] used the relative values between loss functions to express the thresholds. Yao and Zhou [60] proposed a naïve Bayesian DTRS, where the conditional probability was estimated by using the Bayes theorem with naïve probabilistic independence assumption for attributes. Under the multiple sets of decision preferences and criteria adopted by different agents, Yang and Yao [53] proposed a multi-agent DTRS model. In addition, the attribute reduction of DTRS has been discussed in [20, 58, 65, 66]. With respect to the above discussions, a critical issue of the DTRS model is assigning the loss function.

Mishra et al. [38] found that the fuzzy boundaries implied by vague information could actually help individuals perform better than when being confronted with precise information. In the realistic decision process, some influencing factors also result in decision makers not to provide precise values, e.g., limited domain knowledge of decision maker, tight deadlines, limited budgets. As an extension of precise numerical values, fuzzy set [64] is considered here to deal with vague, imprecise and uncertain problems. These observations form a cornerstone of the model to be developed in this study. Therefore, the value of loss function with the measurement of fuzzy set is more realistic. In the fuzzy set theory, membership function is a basic element. Various approaches of membership function elicitation have been discussed in [15, 19, 36, 37, 39, 43, 48, 52]. The membership function elicitation provides a solution to assign the losses of DTRS model. With respect to membership function, triangular fuzzy number is a representative one. A certain theoretically sound motivation behind the common use of triangular membership functions was analyzed in [42]. For simplicity and clarity, we assume the loss function used in DTRS model is a triangular fuzzy number. We focus on constructing triangular fuzzy decision-theoretic rough set (TFDTRS) model. In practical applications, linguistic variable is associated with the (triangular) fuzzy number [12, 15, 28, 39, 40, 43, 47]. DTRS model is a triangular fuzzy number. We focus on constructing triangular fuzzy decision-theoretic rough set (TFDTRS) model. TFDTRS model is proposed and its thresholds are analyzed in Section 3. In the frame of MAGDM, the determination of losses is considered to be an important problem. Inspired by [43, 44], we further construct a multiple attribute group decision making (MAGDM) to determine the values of losses in the context of TFDTRS. Take example for linguistic variable with triangular fuzzy number, we determine the value of the losses used in TFDTRS. The main contribution of this study can be stated as follows: (a) we provide a method to determine the losses of TFDTRS; (b) we construct a general TFDTRS model to adapt a fuzzy scenario.

The remainder of this paper is organized as follows: Section 2 provides basic concepts of DTRS and triangular fuzzy number. TFDTRS model is proposed and its thresholds are analyzed in Section 3. In the frame of MAGDM, the determination for the values of losses with triangular fuzzy number is designed in Section 4. Then, an example is given to illustrate the application of TFDTRS in Section 5. Section 6 concludes the study and elaborates on future studies.

2. Preliminaries

In this section, basic concepts of decision-theoretic rough sets and triangular fuzzy number are briefly reviewed.

2.1. Decision-theoretic rough sets (DTRS)

Based on the Bayesian decision procedure, the DTRS model is composed of 2 states and 3 actions [59, 60]. The set of states is given by \( \Omega = \{ \text{C}, \text{¬C} \} \) indicating that an object is in C and not in C, respectively. The set of actions is given by \( \mathcal{A} = \{ a_P, a_S, a_N \} \), where \( a_P \), \( a_S \), and \( a_N \) represent three actions when classifying object x, namely, deciding \( x \in \text{POS}(C) \), deciding \( x \) should be further investigated \( x \in \text{BND}(C) \), and deciding \( x \in \text{NEG}(C) \), respectively. The loss function regarding the risk or cost of actions in different states is given in Table 1.

In Table 1, \( \lambda_{PP} \), \( \lambda_{RP} \) and \( \lambda_{NP} \) denote the losses incurred for taking actions of \( a_P \), \( a_S \) and \( a_N \), respectively, when an object belongs to \( \text{C} \). Similarly, \( \lambda_{PN} \), \( \lambda_{SN} \) and \( \lambda_{NN} \) denote the losses incurred for taking the same actions when the object belongs to \( \text{¬C} \). Pr(\( \text{C} | x \)) is the conditional probability of an object \( x \) belonging to \( \text{C} \) given that the object is described by its equivalence class \( [x] \). For an object \( x \), the expected loss \( R(a_i | [x]) \) associated with taking the individual action can be expressed as:

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
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<tbody>
<tr>
<td>The loss function regarding the risk or cost of actions in different states.</td>
</tr>
<tr>
<td>( a_P )</td>
</tr>
<tr>
<td>( a_S )</td>
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<tr>
<td>( a_N )</td>
</tr>
</tbody>
</table>

\( \lambda_{PP} \) represents the loss of correctly classifying a positive instance as positive. \( \lambda_{RP} \) represents the loss of incorrectly classifying a positive instance as negative. \( \lambda_{NP} \) represents the loss of correctly classifying a negative instance as negative. \( \lambda_{SN} \) represents the loss of incorrectly classifying a negative instance as positive.
Definition 1. A fuzzy number generalizes a concept of a real number. The membership function of a fuzzy number has to satisfy the condition of convexity and normality [9]. In what follows, we review some basic definitions of triangular fuzzy number and its algebraic operations (see [5,9,49]).

2.2. Triangular fuzzy number

A fuzzy number generalizes a concept of a real number. The membership function of a fuzzy number has to satisfy the condition of convexity and normality [9]. In what follows, we review some basic definitions of triangular fuzzy number and its algebraic operations (see [5,9,49]).

\[
R(a_p[x]) = \lambda_{pp} Pr(C[x]) + \lambda_{pn} Pr(\neg C[x]),
\]
\[
R(a_b[x]) = \lambda_{bp} Pr(C[x]) + \lambda_{bn} Pr(\neg C[x]),
\]
\[
R(a_n[x]) = \lambda_{np} Pr(C[x]) + \lambda_{nn} Pr(\neg C[x]).
\]

The Bayesian decision procedure suggests the following minimum-cost decision rules:

(P0) If \( R(a_p[x]) \leq R(a_b[x]) \) and \( R(a_p[x]) \leq R(a_n[x]) \), decide \( x \in POS(C) \);

(B0) If \( R(a_b[x]) \leq R(a_p[x]) \) and \( R(a_b[x]) \leq R(a_n[x]) \), decide \( x \in BND(C) \);

(N0) If \( R(a_n[x]) \leq R(a_p[x]) \) and \( R(a_n[x]) \leq R(a_b[x]) \), decide \( x \in NEG(C) \).

Since \( Pr(C[x]) + Pr(\neg C[x]) = 1 \), we simplify the rules based only on the probability \( Pr(C[x]) \) and the loss function. By considering a reasonable kind of loss functions with the conditions

\[
\lambda_{pp} \leq \lambda_{bp} \leq \lambda_{np},
\]
\[
\lambda_{nn} \leq \lambda_{bn} \leq \lambda_{pn}.
\]

The decision rules (P0)-(N0) can be expressed concisely as follows:

(P1) If \( Pr(C[x]) \geq \alpha \) and \( Pr(C[x]) \geq \gamma \), decide \( x \in POS(C) \);

(B1) If \( Pr(C[x]) \leq \alpha \) and \( Pr(C[x]) \geq \beta \), decide \( x \in BND(C) \);

(N1) If \( Pr(C[x]) \leq \beta \) and \( Pr(C[x]) \leq \gamma \), decide \( x \in NEG(C) \).

The thresholds values \( \alpha, \beta, \gamma \) are given in the form:

\[
\alpha = \frac{(\lambda_{np} - \lambda_{bn})}{(\lambda_{pn} - \lambda_{bn}) + (\lambda_{bp} - \lambda_{pp})},
\]
\[
\beta = \frac{(\lambda_{bn} - \lambda_{nn})}{(\lambda_{bn} - \lambda_{nn}) + (\lambda_{np} - \lambda_{bp})},
\]
\[
\gamma = \frac{(\lambda_{nn} - \lambda_{bn})}{(\lambda_{nn} - \lambda_{bn}) + (\lambda_{np} - \lambda_{pp})}.
\]

2.2. Triangular fuzzy number

A fuzzy number generalizes a concept of a real number. The membership function of a fuzzy number has to satisfy the condition of convexity and normality [9]. In what follows, we review some basic definitions of triangular fuzzy number and its algebraic operations (see [5,9,49]).

Definition 1. A fuzzy number \( \tilde{M} \) on \( \mathbb{R} \) is a triangular fuzzy number, with its membership function defined as follows:

\[
\mu_{\tilde{M}}(x) = (l, m, u) = \begin{cases} 
\frac{x-l}{m-l}, & l \leq x \leq m; \\
\frac{x-u}{m-u}, & m \leq x \leq u; \\
0, & \text{otherwise.}
\end{cases}
\]

Then, algebraic operations of triangular fuzzy number are described below:

\[(l_1, m_1, u_1) \oplus (l_2, m_2, u_2) = (l_1 + l_2, m_1 + m_2, u_1 + u_2);\]
\[(l_1, m_1, u_1) \ominus (l_2, m_2, u_2) = (l_1 - u_2, m_1 - m_2, u_1 - l_2);\]
\[(l_1, m_1, u_1) \otimes (l_2, m_2, u_2) \approx (l_1 \times l_2, m_1 \times m_2, u_1 \times u_2);\]
\[(l_1, m_1, u_1) \oslash (l_2, m_2, u_2) \approx (l_1 \div u_2, m_1 \div m_2, u_1 \div l_2).\]

In particular, when the triangular fuzzy number \((l_1, m_1, u_1)\) is treated as some constant \(\lambda (\lambda \in \mathbb{R})\), the algebraic operation of multiplication is expressed in the following way:

\[
\lambda \otimes (l_2, m_2, u_2) \approx \begin{cases} 
(\lambda \times l_2, \lambda \times m_2, \lambda \times u_2), & \lambda \geq 0; \\
(\lambda \times u_2, \lambda \times m_2, \lambda \times l_2), & \lambda \leq 0.
\end{cases}
\]
3. Triangular fuzzy decision-theoretic rough sets (TFDTRS)

In this section, we consider the loss function of DTRS model is a triangular fuzzy number and construct a TFDTRS model. Triangular fuzzy number provides us a new strategy to measure the loss function. In light of the decision procedure of DTRS, the TFDTRS model comprises three main steps: (a) constructing loss function matrix with triangular fuzzy number; (b) ranking the expected loss; (c) forming decision rules. The flow of processing in the TFDTRS model is shown in Fig. 1.

3.1. Loss function matrix with triangular fuzzy number

For the loss function regarding the risk or cost of actions in different states, its value with triangular fuzzy number is given in Table 2.

In Table 2, \( \lambda_{PP} = (l_{PP}, m_{PP}, u_{PP}) \), \( \lambda_{BP} = (l_{BP}, m_{BP}, u_{BP}) \) and \( \lambda_{NP} = (l_{NP}, m_{NP}, u_{NP}) \) denote the losses incurred for taking actions of \( a_P \), \( a_B \) and \( a_N \), respectively, when an object belongs to \( C \). Similarly, \( \lambda_{PN} = (l_{PN}, m_{PN}, u_{PN}) \), \( \lambda_{BN} = (l_{BN}, m_{BN}, u_{BN}) \) and \( \lambda_{NN} = (l_{NN}, m_{NN}, u_{NN}) \) denote the losses incurred for taking the same actions when the object belongs to \( \neg C \). On the basis of conditions (2), the losses satisfy the following constraints:

\[
\begin{align*}
    l_{PP} &\leq l_{BP} < l_{NP} ; \\
    m_{PP} &\leq m_{BP} < m_{NP} ; \\
    u_{PP} &\leq u_{BP} < u_{NP} ; \\
    l_{NN} &\leq l_{BN} < l_{PN} ; \\
    m_{NN} &\leq m_{BN} < m_{PN} ; \\
    u_{NN} &\leq u_{BN} < u_{PN} .
\end{align*}
\]

According to (3)-(5), a certain type of the relationships among \( \lambda_{PP}, \lambda_{BP} \) and \( \lambda_{NP} \) is shown in Fig. 2.

Similarly, a certain type of the relationships among \( \lambda_{PN}, \lambda_{BN} \) and \( \lambda_{NN} \) is simply shown in Fig. 3, according to (6)-(8).
For an object $x$, the expected loss $R(a_i|x)$ associated with taking the individual action can be expressed as:

$$R(a_i|x) = (l_{P}, m_{P}, u_{P})Pr(C|x) \oplus (l_{N}, m_{N}, u_{N})Pr(\neg C|x),$$

where $\rho$ is the risk attitude index of decision maker, representing the degree of optimism of a decision maker.

Since $Pr(C|x) + Pr(\neg C|x) = 1$, we calculate the expected losses based on the conditional probability $Pr(C|x)$ and algebraic operations of triangular fuzzy numbers:

$$R(a_i|x) = (l_{P}Pr(C|x) + l_{N}(1 - Pr(C|x)), m_{P}Pr(C|x) + m_{N}(1 - Pr(C|x)), u_{P}Pr(C|x) + u_{N}(1 - Pr(C|x))).$$

Since $\rho$ indicates a higher degree of optimism. In particular, when $\rho = 0$ and $\rho = 1$, the value of $r(M)$ represents the view points of a pessimistic and optimistic decision maker, respectively. When $\rho = 0.5$, the value of $r(M)$ represents the view points of a moderate decision maker.

3.2. Ranking the expected losses

Ranking the expected losses is an important process of decision making in TFDTRS. Ranking function is an efficient element during ranking fuzzy numbers, which maps the fuzzy numbers into the real line [21]. This section ranks the expected loss with integral value method. By comparing with other ranking methods, some features of integral value method are summarized.

3.2.1. Ranking the expected losses with integral value method

Integral value method [27] converts the fuzzy number into a single real number and rank fuzzy numbers using these real numbers. It takes into consideration decision maker’s risk attitude. Kumar et al. [21] modified the method presented in [27] and proposed a new approach for ranking of L-R type generalization fuzzy numbers. According to the results reported in [21], ranking function of triangular fuzzy number $M = (l, m, u)$ is given as follows:

$$r(M) = \frac{1}{2}[\rho(l + m) + (1 - \rho)(m + u)].$$

where the parameter $\rho$ is the risk attitude index of decision maker, representing the degree of optimism of a decision maker. With respect to the loss function, a larger value of $\rho$ indicates a higher degree of optimism. In particular, when $\rho = 0$ and $\rho = 1$, the value of $r(M)$ represents the view points of a pessimistic and optimistic decision maker, respectively. When $\rho = 0.5$, the value of $r(M)$ represents the view points of a moderate decision maker.
Example 1. Let the expected loss \( R(\tilde{a}_P[x]) = (1, 2, 3) \). Under the integral value method, the ranking function of triangular fuzzy number \( R(\tilde{a}_P[x]) \) is expressed as

\[
r(\tilde{R}(\tilde{a}_P[x])) = \frac{1}{2}[3\rho + 5(1 - \rho)].
\]

For a pessimistic decision maker, with \( \rho = 0 \), \( r(\tilde{R}(\tilde{a}_P[x]))_1 = 2.5 \). For a moderate decision maker, with \( \rho = 0.5 \), \( r(\tilde{R}(\tilde{a}_P[x]))_1 = 2 \). For an optimistic decision maker, with \( \rho = 1 \), \( r(\tilde{R}(\tilde{a}_P[x]))_1 = 1.5 \). The pessimistic decision maker prefers that the value of the expected loss is close to the right part of triangular fuzzy number, while the optimistic decision maker considers that it is inclined to the left part of triangular fuzzy number. The moderate decision maker believes the expected loss function value is in the middle.

Based on the integral value method and (10), the expected losses are calculated as follows:

\[
r(\tilde{R}(\tilde{a}_P[x]))_1 = \frac{1}{2}\{\rho[l_{PP}Pr(C[x]) + l_{NP}(1 - Pr(C[x])) + \beta_{PP}Pr(C[x]) + \beta_{PN}(1 - Pr(C[x]))] + (1 - \rho)[m_{PP}Pr(C[x]) + m_{PN}(1 - Pr(C[x])) + u_{PP}Pr(C[x]) + u_{PN}(1 - Pr(C[x]))]\}
\[
= r(\tilde{\lambda}_{pp})_1 Pr(C[x]) + r(\tilde{\lambda}_{np})_1(1 - Pr(C[x])),
\]

\[
r(\tilde{R}(\tilde{a}_P[x]))_1 = \frac{1}{2}\{\rho[l_{BP}Pr(C[x]) + l_{BN}(1 - Pr(C[x])) + \beta_{BP}Pr(C[x]) + \beta_{BN}(1 - Pr(C[x]))] + (1 - \rho)[m_{BP}Pr(C[x]) + m_{BN}(1 - Pr(C[x])) + u_{BP}Pr(C[x]) + u_{BN}(1 - Pr(C[x]))]\}
\[
= r(\tilde{\lambda}_{bp})_1 Pr(C[x]) + r(\tilde{\lambda}_{bn})_1(1 - Pr(C[x])),
\]

\[
r(\tilde{R}(\tilde{a}_P[x]))_1 = \frac{1}{2}\{\rho[l_{NP}Pr(C[x]) + l_{NN}(1 - Pr(C[x])) + \beta_{NP}Pr(C[x]) + \beta_{NN}(1 - Pr(C[x]))] + (1 - \rho)[m_{NP}Pr(C[x]) + m_{NN}(1 - Pr(C[x])) + u_{NP}Pr(C[x]) + u_{NN}(1 - Pr(C[x]))]\}
\[
= r(\tilde{\lambda}_{np})_1 Pr(C[x]) + r(\tilde{\lambda}_{nn})_1(1 - Pr(C[x]))).
\]

where

\[
r(\tilde{\lambda}_{pp})_1 = \frac{1}{2}\{\rho[l_{PP} + m_{PP}] + (1 - \rho)(m_{PP} + u_{PP})\};
\]

\[
r(\tilde{\lambda}_{pn})_1 = \frac{1}{2}\{\rho[l_{PN} + m_{PN}] + (1 - \rho)(m_{PN} + u_{PN})\};
\]

\[
r(\tilde{\lambda}_{bp})_1 = \frac{1}{2}\{\rho[l_{BP} + m_{BP}] + (1 - \rho)(m_{BP} + u_{BP})\};
\]

\[
r(\tilde{\lambda}_{bn})_1 = \frac{1}{2}\{\rho[l_{BN} + m_{BN}] + (1 - \rho)(m_{BN} + u_{BN})\};
\]

\[
r(\tilde{\lambda}_{np})_1 = \frac{1}{2}\{\rho[l_{NP} + m_{NP}] + (1 - \rho)(m_{NP} + u_{NP})\};
\]

\[
r(\tilde{\lambda}_{nn})_1 = \frac{1}{2}\{\rho[l_{NN} + m_{NN}] + (1 - \rho)(m_{NN} + u_{NN})\}.
\]

3.2.2. Integral value method vis-a-vis other ranking methods

Some ranking methods of the fuzzy numbers have been developed in the literatures [2, 3, 7, 8, 10, 21, 27, 45, 50]. Considering the divisions and latest developments of ranking function, five representative methods are discussed. For triangular fuzzy number, its ranking functions with these ranking methods are analyzed in Appendix A, except integral value method. The five representative ranking functions with their corresponding methods are listed in Table 3.

Comparing with other ranking methods, the characteristics of integral value method are summarized as follows:

(a) Integral value method ranks the fuzzy numbers by transforming fuzzy number into a single number. It is the same with other ranking methods.

(b) In the aspect of ranking triangular fuzzy number, distance minimization method and deviation degree method are special cases of the integral value method.

• When the value of \( \rho \) is assigned 0.5, the expression of \( R(\tilde{M}) \) is the same with \( R(\tilde{M})_1 \).

• If two triangular fuzzy numbers are ranked using deviation degree method, the ranking result is essentially same with the distance minimization method (see Appendix B).
(c) With respect to different heights and different spreads method and distance method using circumcenter of centroids method, decision rules derived from the integral value method are very simpler and easier to apply in the TFDTRS model. The conditional probability $Pr(C|x)$ of the decision rules (P1)-(N1) is a critical component. In the context of TFDTRS model, the expression of $Pr(C|x)$ is linear in the integral value method, while the expressions of $Pr(C|x)$ in $r(M)_4$ and $r(M)_5$ are nonlinear.

**Example 2.** Let $\lambda_{AP} = (1, 2, 3), \lambda_{BP} = (2, 3, 4), \lambda_{AP} = (3, 4, 5), \lambda_{BN} = (1, 2, 4), \lambda_{BN} = (3, 4, 5), \lambda_{NN} = (4, 5, 6)$. On the basis of (10), the expected losses are calculated as follows:

\[
R(\hat{a}P|x) = (1, 2, 4 - Pr(C|x)); \\
R(\hat{a}q|x) = (3 - Pr(C|x), 4 - Pr(C|x), 5 - Pr(C|x)); \\
R(\hat{a}n|x) = (4 - Pr(C|x), 5 - Pr(C|x), 6 - Pr(C|x)).
\]

Here, we take $R(\hat{a}P|x)$ and $R(\hat{a}n|x)$ as an example to illustrate the third characteristic of integral value method. For the integral value method,

\[
r(\hat{R}(\hat{a}P|x))_1 = \frac{1}{2} [3\rho + (1 - \rho)(6 - Pr(C[x]))],
\]

\[
r(\hat{R}(\hat{a}P|x))_1 = \frac{1}{2} [\rho(9 - 2Pr(C[x])) + (1 - \rho)(11 - 2Pr(C[x]))].
\]

(14)

For the different heights and different spreads method,

\[
r(\hat{R}(\hat{a}P|x))_4 = \frac{9 - Pr(C[x])}{4(6 - Pr(C[x])) + \sqrt{(5 - Pr(C[x]))^2 + 2(1 - Pr(C[x]))^2 + (7 - 3Pr(C[x]))^2}},
\]

\[
r(\hat{R}(\hat{a}q|x))_4 = \frac{20 - 4Pr(C[x])}{4(6 - Pr(C[x])) + \sqrt{32}}.
\]

(15)

For the distance method using circumcenter of centroids method,

\[
r(\hat{R}(\hat{a}P|x))_5 = \sqrt{\frac{13 - Pr(C[x])}{6} + \frac{(4Pr(C[x]) - 3)^2}{12}},
\]

\[
r(\hat{R}(\hat{a}q|x))_5 = \sqrt{\frac{30 - 6Pr(C[x])}{6} + \frac{1}{12}}.
\]

(16)

Under this case, the expressions of $Pr(C|x)$ in $r(\hat{R}(\hat{a}P|x))_4$ and $r(\hat{R}(\hat{a}P|x))_4$ (or $r(\hat{R}(\hat{a}P|x))_5$ and $r(\hat{R}(\hat{a}P|x))_5$) are nonlinear. If $r(\hat{R}(\hat{a}P|x))_4$ and $r(\hat{R}(\hat{a}P|x))_4$ (or $r(\hat{R}(\hat{a}P|x))_5$ and $r(\hat{R}(\hat{a}P|x))_5$) are transformed into the formations of decision rules (P1)-(N1), they need to add more constraint conditions than integral value method. However, the expressions of $Pr(C|x)$ under the integral value method is linear and it can be directly derived rules without the need for more constraint conditions.

(d) Integral value method also takes into account the risk attribute of decision maker.
With the above-mentioned analysis, the integral value method is suitable to rank the expected losses used in the TFDTRS model.

3.3. Decision rules

Continuing with the ranking functions of the expected losses under integral value method, we induce their corresponding decision rules. On the basis of [12], the decision rules (P₀) - (N₀) can be re-expressed as follows:

\( (P'₀) \) if \( r(\tilde{R}(a_P([x])))_1 \leq r(\tilde{R}(a_B([x])))_1 \) and \( r(\tilde{R}(a_P([x])))_1 \leq r(\tilde{R}(a_N([x])))_1 \), decide \( x \in \text{POS}(C) \);

\( (B'₀) \) if \( r(\tilde{R}(a_B([x])))_1 \leq r(\tilde{R}(a_P([x])))_1 \) and \( r(\tilde{R}(a_B([x])))_1 \leq r(\tilde{R}(a_N([x])))_1 \), decide \( x \in \text{BND}(C) \);

\( (N'₀) \) if \( r(\tilde{R}(a_N([x])))_1 \leq r(\tilde{R}(a_P([x])))_1 \) and \( r(\tilde{R}(a_N([x])))_1 \leq r(\tilde{R}(a_B([x])))_1 \), decide \( x \in \text{NEG}(C) \).

Furthermore, according to the decision rules (P₁) - (N₁), the three thresholds of TFDTRS model are calculated as follows (the derivation process sees Appendix C):

\[
\alpha = \frac{r(\lambda_{PN})_1 - r(\lambda_{BN})_1}{r(\lambda_{PN})_1 - r(\lambda_{BN})_1 + (r(\lambda_{BP})_1 - r(\lambda_{PP})_1)}, \\
\beta = \frac{r(\lambda_{BN})_1 - r(\lambda_{NN})_1}{r(\lambda_{BN})_1 - r(\lambda_{NN})_1 + (r(\lambda_{NP})_1 - r(\lambda_{PP})_1)}, \\
\gamma = \frac{r(\lambda_{PN})_1 - r(\lambda_{NN})_1}{r(\lambda_{PN})_1 - r(\lambda_{NN})_1 + (r(\lambda_{NP})_1 - r(\lambda_{PP})_1)}. 
\]  (17)

If the loss function in Table 2 is precise, we have: \( l_P = m_P = u_P, l_N = m_N = u_N, l_P = m_P = u_P, l_N = m_N = u_N \). According to (13) we calculate the value of the threshold \( \alpha \) based on (17)

\[
r(\lambda_{PN})_1 = \frac{1}{2} \rho (l_P + m_P) + (1 - \rho)(m_P + u_P) = \frac{1}{2} \rho (l_P + m_P) + (1 - \rho)(l_P + u_P) = l_P.
\]

\[
r(\lambda_{BN})_1 = l_B, r(\lambda_{BP})_1 = l_B, r(\lambda_{PP})_1 = l_P. \]

The expression of \( \alpha \) is

\[
\alpha = \frac{(l_P - l_B)}{(l_N - l_P) + (l_B - l_P)}.
\]

Similarity, the expressions of \( \beta \) and \( \gamma \) are straightforward

\[
\beta = \frac{(l_B - l_N)}{(l_B - l_P) + (l_N - l_P)}, \quad \gamma = \frac{(l_P - l_N)}{(l_P - l_N) + (l_N - l_B)}.
\]

At this point, we observe that the results are the same with DTRS, i.e., TFDTRS is an extension of DTRS.

**Proposition 1.** Suppose that the loss function presented in TFDTRS is a single and precise numeric, then the three thresholds \( \alpha, \beta, \gamma \) in (17) are unrelated to the risk attitude index of decision maker \( \rho \).

Considering the risk attitude of decision maker, Li and Zhou [24] proposed an optimistic decision, an equable decision and a pessimistic decision model of DTRS. The risk attitude index of decision maker in the integral value method is an important component of ranking fuzzy numbers. It influences the values of three thresholds. Specially, when \( \rho = 0 \), the three thresholds under the pessimistic decision maker can be expressed as:

\[
\alpha_p = \frac{(m_P + u_P) - (m_B + u_B)}{((m_P + u_P) - (m_B + u_B)) + ((m_B + u_B) - (m_P + u_P))},
\]

\[
\beta_p = \frac{(m_B + u_B) - (m_N + u_N)}{((m_B + u_B) - (m_N + u_N)) + ((m_N + u_N) - (m_B + u_B))},
\]

\[
\gamma_p = \frac{(m_P + u_P) - (m_N + u_N)}{((m_P + u_P) - (m_N + u_N)) + ((m_N + u_N) - (m_P + u_P))}.
\]
When $\rho = 0.5$, the three thresholds under the moderate decision maker can be expressed as:

\[
\begin{align*}
\alpha_m &= \frac{(l_{PN} + m_{PN} - l_{BN} + m_{BN})}{(l_{PN} + 2m_{PN} + u_{PN}) - (l_{BN} + 2m_{BN} + u_{BN}) - ((l_{BB} + 2m_{BP} + u_{BP}) - (l_{PP} + 2m_{PP} + u_{PP}))}, \\
\beta_m &= \frac{(l_{BN} + 2m_{BN} + u_{BN}) - (l_{BN} + 2m_{BN} + u_{BN}) + ((l_{BN} + 2m_{BN} + u_{BN}) - (l_{BN} + 2m_{BN} + u_{BN}))}{(l_{BN} + 2m_{BN} + u_{BN}) - (l_{BN} + 2m_{BN} + u_{BN}) - ((l_{BN} + 2m_{BN} + u_{BN}) - (l_{BN} + 2m_{BN} + u_{BN}))}, \\
\gamma_m &= \frac{(l_{PN} + 2m_{PN} + u_{PN}) - (l_{NN} + 2m_{NN} + u_{NN}) + ((l_{NP} + 2m_{NP} + u_{NP}) - (l_{PP} + 2m_{PP} + u_{PP}))}{(l_{PN} + 2m_{PN} + u_{PN}) - (l_{NN} + 2m_{NN} + u_{NN}) - ((l_{PN} + 2m_{NP} + u_{NP}) - (l_{PP} + 2m_{PP} + u_{PP}))}.
\end{align*}
\]

When $\rho = 1$, the three thresholds under the optimistic decision maker can be expressed as:

\[
\begin{align*}
\alpha_o &= \frac{(l_{PN} + m_{PN})}{(l_{PN} + m_{PN}) - (l_{BN} + m_{BN}) - ((l_{BP} + m_{BP}) - (l_{PP} + m_{PP}))}, \\
\beta_o &= \frac{(l_{BN} + m_{BN}) - (l_{NN} + m_{NN})}{(l_{BN} + m_{BN}) - (l_{NN} + m_{NN}) + ((l_{BP} + m_{BP}) - (l_{PP} + m_{PP}))}, \\
\gamma_o &= \frac{(l_{PN} + m_{PN}) - (l_{NN} + m_{NN})}{(l_{PN} + m_{PN}) - (l_{NN} + m_{NN}) + ((l_{NP} + m_{NP}) - (l_{BP} + m_{BP}))}.
\end{align*}
\]

In accordance with the discussions proposed by Yao [57, 59, 61], as a well-defined boundary region, the conditions of rule (B1) suggest that $\alpha > \beta$, that is,

\[
\frac{(r(\lambda_{BP})_1 - r(\lambda_{PP})_1)}{(r(\lambda_{BN})_1 - r(\lambda_{NN})_1)} < \frac{(r(\lambda_{NP})_1 - r(\lambda_{BP})_1)}{(r(\lambda_{BN})_1 - r(\lambda_{NN})_1)}.
\]  \tag{18}

It implies that $0 \leq \beta < \gamma < \alpha \leq 1$. The following simplified three-way decision rules are obtained:

(P2) If $Pr(C|x) \geq \alpha$, decide $x \in \text{POS}(C)$;
(B2) If $\beta < Pr(C|x) < \alpha$, decide $x \in \text{BND}(C)$;
(N2) If $Pr(C|x) \leq \beta$, decide $x \in \text{NEG}(C)$.

For the pessimistic decision maker, with $\rho = 0$, the corresponding three-way decision rules are:

If $Pr(C|x) \geq \alpha_p$, decide $x \in \text{POS}(C)$;
If $\beta_p < Pr(C|x) < \alpha_p$, decide $x \in \text{BND}(C)$;
If $Pr(C|x) \leq \beta_p$, decide $x \in \text{NEG}(C)$.

For the moderate decision maker, with $\rho = 0.5$, the corresponding three-way decision rules are:

If $Pr(C|x) \geq \alpha_m$, decide $x \in \text{POS}(C)$;
If $\beta_m < Pr(C|x) < \alpha_m$, decide $x \in \text{BND}(C)$;
If $Pr(C|x) \leq \beta_m$, decide $x \in \text{NEG}(C)$.

For the optimistic decision maker, with $\rho = 1$, the corresponding three-way decision rules are:

If $Pr(C|x) \geq \alpha_o$, decide $x \in \text{POS}(C)$;
If $\beta_o < Pr(C|x) < \alpha_o$, decide $x \in \text{BND}(C)$;
If $Pr(C|x) \leq \beta_o$, decide $x \in \text{NEG}(C)$.

Besides (18), we obtain another condition $\alpha \leq \beta$, that is,

\[
\frac{(r(\lambda_{BP})_1 - r(\lambda_{PP})_1)}{(r(\lambda_{BN})_1 - r(\lambda_{NN})_1)} \geq \frac{(r(\lambda_{NP})_1 - r(\lambda_{BP})_1)}{(r(\lambda_{BN})_1 - r(\lambda_{NN})_1)}.
\]  \tag{19}

It implies that $0 \leq \alpha \leq \gamma \leq \beta \leq 1$. In this case, the decision rules (P2)-(N2) can be rewritten as follows:

(P2) If $Pr(C|x) \geq \gamma$, decide $x \in \text{POS}(C)$;
(N2) If $Pr(C|x) < \gamma$, decide $x \in \text{NEG}(C)$.
where the three-way decision rules \((P_2)-(N_2)\) change to the two-way. \(\gamma\) can also be represented by \(\gamma_p\), \(\gamma_m\) or \(\gamma_o\) under the different risk attitudes.

4. The determination for the values of losses used in the TFDTRS model

In Section 3, TFDTRS is constructed in order to adapt the fuzzy scenario. During the application of TFDTRS, the determination for the values of losses is a pivotal issue. In this section, we discuss the determination for the values of losses used in the TFDTRS model and design a method to support the decision of TFDTRS presented in Section 3. The fuzzy form of linguistic variables exists in practical applications [12,15,28,39,40,47,52]. Nguyen et al. [40] discussed a genetic design of linguistic terms for fuzzy rule based classifiers. They established a method to design genetically the linguistic terms along with their fuzzy sets. Trivino and Sugeno [47] dealt with the development of computational systems, which were able to provide users with meaningful linguistic descriptions of phenomena. Each linguistic variable can be corresponding to a fuzzy number [28].

For example, the membership functions of linguistic variables \{Absolute low, Very low, Low, Fairly low, Medium, Fairly high, High, Very high, Absolute high\} are visualized in Fig. 4.

As the distribution shown in Fig. 4, the scales of linguistic variables with triangular fuzzy numbers are measured using an uniform distribution. From the viewpoint of granulation, Pedrycz and Song [43] regulated the scales of linguistic variables to improve the inconsistency among multiple experts in the multiple attribute group decision making (MAGDM). The optimum scales of linguistic variables was not an uniform distribution. In the MAGDM problem, the degree of consensus among experts is used to measure the effect of the group decision result, i.e., the inconsistency. The modified evaluations of experts play an important role in reaching the consensus [43,44,51]. In the studies reported in [43,44], the inconsistency is the fitness function of particle swarm optimization (PSO). It need not change the evaluation information of experts, but optimizes the scales of linguistic variables based on the predetermined uniform distribution. Based on the results presented in [11,43,44,51], we choose the linguistic variables with triangular fuzzy numbers to discuss the determination for the values of losses in the context of TFDTRS model. It can be considered as a MAGDM problem, where the losses \(\lambda_{\bullet \bullet}\) constitute a set of attributes and they are evaluated by multiple experts. Under a suitable consistency of MAGDM, the losses of alternatives are determined by aggregating the evaluations of experts.

Before determining the values of losses we construct the MAGDM problem in the background of TFDTRS model. Let \(A = \{a_1, a_2, \ldots, a_6\}\) be a set of evaluation attributes, which is corresponding to \(\lambda_{PP}, \lambda_{BP}, \lambda_{NP}, \lambda_{PN}, \lambda_{BN}\) and \(\lambda_{NN}\), respectively. Let \(O = \{o_1, o_2, \ldots, o_m\}\) be a discrete set of alternatives. Let \(S\) be a linguistic evaluation set with odd cardinality, which is denoted as \(S = \{s_0, s_1, \ldots, s_t\}\). \(E = \{e_1, e_2, \ldots, e_t\}\) is a set of experts and \(W = \{w_1, w_2, \ldots, w_t\}\) is a weight vector of experts, where \(\sum_{k=1}^{t} w_k = 1\) and \(w_k \geq 0\).

The decision matrix of expert is denoted as \(D_k = (M^k_{ij})_{m\times6}\), where \(k\) is the mark of experts. \(M^k_{ij}\) is a value in the form of linguistic variable given by the expert \(e_k\) for the alternative \(o_i\) with respect to the attribute \(a_j\) where \(1 \leq k \leq t\), \(1 \leq i \leq m\) and \(1 \leq j \leq 6\). According to the relationship between the linguistic variables and triangular fuzzy numbers, the values in the decision matrix \(D_k\) are transformed into triangular fuzzy numbers, denoted by \(D_k = (N^k_{ij})_{m\times6}\), where \(N^k_{ij} = (l^k_{ij}, m^k_{ij}, u^k_{ij})\). The corresponding group decision matrix is denoted as \(G = (N_{ij})_{m\times6}\), where

\[
N_{ij} = \sum_{k=1}^{t} w_k N^k_{ij}.
\]

According to the results presented in [43], the effectiveness of inconsistency among experts was verified by the Monte Carlo simulation method, which transformed the high level granulation formalism into the low level granulation, viz. used
In the Monte Carlo simulations, the triangular fuzzy numbers of decision matrixes are replaced with the drawn precise numeric and the corresponding membership functions of the sample drawn are generated. Let \( Z \) denote the number of the sample drawn. Here, we adopt the distance method \([51]\) to measure the inconsistency of experts. Based on the membership function of linguistic variable in the sample \( z \), the inconsistency measure of individual to group decision is defined:

\[
d(e_k, G)_z = \frac{\mu_z \sum_{i=1}^{m} \sum_{j=1}^{n} |N^{e_k}_{zij} - N_{zij}|}{\sum_{i=1}^{m} \mu_z}.
\]

(21)

where \( N^{e_k}_{zij} \) and \( N_{zij} \) drawn from the sample are precise and \( \mu_z \) is the minimization of the membership function for all decision matrixes in the sample \( z \), \( d(e_k, G)_z \) is the distance between \( e_k \) and \( G \) in the sample \( z \). The smaller of \( d(e_k, G)_z \) shows the better of the consistency between \( e_k \) and \( G \). The fitness function of the PSO algorithm to be minimized is the sum of inconsistency measure, namely

\[
Q = \frac{1}{Z} \sum_{i=1}^{Z} \sum_{k=1}^{I} d(e_k, G)_z.
\]

(22)

In light of \([14,43,44,46]\), we design an algorithm to determine the values of loss functions used in TFDTRS model under the linguistic variables with triangular fuzzy numbers:

\begin{enumerate}
  \item Step 1: Construction of the MAGDM in the context of TFDTRS: determine the number of experts and their weights. Select reasonable scales of linguistic variables with triangular fuzzy numbers, \( S = \{s_0, s_1, \ldots, s_6\} \). In light of the set of alternatives with respect to the set of attributes, obtain the evaluation results and construct corresponding decision matrix for every expert.
  \item Step 2: Setting the parameters of PSO, including the fitness function, the population of the particle, the dimensional space of each particle and its basic updated rules of the velocity and the position.
  \item Step 3: Based on the scales of linguistic variables, optimize the scales of linguistic variables using the PSO algorithm until it arrives at the total number of generations, where we need to determine the number of sample drawn \( Z \) and employ Monte Carlo simulation method.
  \item Step 4: With respect to the optimum scales of linguistic variables and (20), construct the group decision matrix and determine the values of losses for each alternative with triangular fuzzy numbers.
\end{enumerate}

**Example 3.** Suppose that there are six public-private partnership (PPP) highway investment projects, i.e., \( o_1, o_2, \ldots, o_6 \). The linguistic variables and their original scales are visualized in Fig. 4. The projects are evaluated by four experts \( (e_1, e_2, e_3, e_4) \). They have the same weight and the decision matrix of four experts are listed in Tables 4–7.

Let \( Z = 500 \). According to \([14,43,46]\), the parameters of PSO are set up: the fitness function is \( (22) \), the population of the particle is 100, the dimensional space of each particle is 8, the total number of generations is 500. The progression of PSO algorithm is quantified in terms of the fitness function obtained in successive generations, see Fig. 5.

The PSO algorithm produces the optimal cutoff points as follows: \( 3.07, 3.14, 3.20, 3.30, 3.35, 3.38 \) and \( 3.64 \). The fitness function under the optimum state is 0.2607. However, the results with the uniform distribution (that is \( 1, 2, 3, 4, 5, 6, 7 \)) is 0.3354. The optimum scales of linguistic variables are described in Fig. 6.

<table>
<thead>
<tr>
<th>Table 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>The decision matrix ( D_1 ).</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>( o_1 )</td>
</tr>
<tr>
<td>( o_2 )</td>
</tr>
<tr>
<td>( o_3 )</td>
</tr>
<tr>
<td>( o_4 )</td>
</tr>
<tr>
<td>( o_5 )</td>
</tr>
<tr>
<td>( o_6 )</td>
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</table>

<table>
<thead>
<tr>
<th>Table 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>The decision matrix ( D_2 ).</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>( o_1 )</td>
</tr>
<tr>
<td>( o_2 )</td>
</tr>
<tr>
<td>( o_3 )</td>
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<tr>
<td>( o_4 )</td>
</tr>
<tr>
<td>( o_5 )</td>
</tr>
<tr>
<td>( o_6 )</td>
</tr>
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Table 6
The decision matrix $D_3$.

<table>
<thead>
<tr>
<th>$o_i$</th>
<th>$\tilde{\lambda}_{PP}$</th>
<th>$\tilde{\lambda}_{BP}$</th>
<th>$\tilde{\lambda}_{NP}$</th>
<th>$\tilde{\lambda}_{BN}$</th>
<th>$\tilde{\lambda}_{PN}$</th>
<th>$\tilde{\lambda}_{NN}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$o_1$</td>
<td>Absolute low</td>
<td>Very low</td>
<td>Fairly low</td>
<td>Absolute high</td>
<td>Fairly low</td>
<td>Very low</td>
</tr>
<tr>
<td>$o_2$</td>
<td>Very low</td>
<td>Low</td>
<td>Medium</td>
<td>High</td>
<td>Fairly high</td>
<td>Very low</td>
</tr>
<tr>
<td>$o_3$</td>
<td>Very low</td>
<td>Fairly low</td>
<td>High</td>
<td>Medium</td>
<td>Fairly low</td>
<td>Very low</td>
</tr>
<tr>
<td>$o_4$</td>
<td>Absolute low</td>
<td>Absolute low</td>
<td>Fairly low</td>
<td>High</td>
<td>Fairly low</td>
<td>Very low</td>
</tr>
<tr>
<td>$o_5$</td>
<td>Absolute low</td>
<td>Low</td>
<td>Very high</td>
<td>Absolute high</td>
<td>Very high</td>
<td>Absolute low</td>
</tr>
<tr>
<td>$o_6$</td>
<td>Very low</td>
<td>Fairly high</td>
<td>High</td>
<td>Low</td>
<td>Very low</td>
<td>Very low</td>
</tr>
</tbody>
</table>

Table 7
The decision matrix $D_4$.

<table>
<thead>
<tr>
<th>$o_i$</th>
<th>$\tilde{\lambda}_{PP}$</th>
<th>$\tilde{\lambda}_{BP}$</th>
<th>$\tilde{\lambda}_{NP}$</th>
<th>$\tilde{\lambda}_{BN}$</th>
<th>$\tilde{\lambda}_{PN}$</th>
<th>$\tilde{\lambda}_{NN}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$o_1$</td>
<td>Low</td>
<td>Fairly low</td>
<td>High</td>
<td>Very high</td>
<td>High</td>
<td>Low</td>
</tr>
<tr>
<td>$o_2$</td>
<td>Very low</td>
<td>Very high</td>
<td>Absolute high</td>
<td>Absolute high</td>
<td>Very high</td>
<td>Fairly low</td>
</tr>
<tr>
<td>$o_3$</td>
<td>Absolute low</td>
<td>Low</td>
<td>Fairly high</td>
<td>Very high</td>
<td>Low</td>
<td>Low</td>
</tr>
<tr>
<td>$o_4$</td>
<td>Absolute low</td>
<td>Fairly high</td>
<td>Very high</td>
<td>Absolute high</td>
<td>High</td>
<td>Fairly low</td>
</tr>
<tr>
<td>$o_5$</td>
<td>Fairly low</td>
<td>Medium</td>
<td>Very high</td>
<td>High</td>
<td>Absolute low</td>
<td>Absolute low</td>
</tr>
<tr>
<td>$o_6$</td>
<td>Low</td>
<td>Medium</td>
<td>Absolute high</td>
<td>Absolute high</td>
<td>Fairly high</td>
<td>Low</td>
</tr>
</tbody>
</table>

Fig. 5. The values of the fitness function in successive generations of the PSO.

Fig. 6. The optimum scales of linguistic variables.

Remark 1. From Fig. 6, the optimal cutoff points between 0 and 8 are 3.07, 3.14, 3.20, 3.30, 3.35, 3.38 and 3.64 along the horizontal ordinate. To improve the inconsistency, the optimal cutoff points become an unbalance situation so that reduce the inconsistency of group decision making.
Finally, the group decision matrix is calculated based on the optimal results of PSO, (20) and algebraic operations of triangular fuzzy numbers. The group decision matrix is shown in Table 8.

From Table 8, the values of losses for each project are determined.

### 5. An illustrative example

In this section, we illustrate the TFDTRS model by an example of decision-making in public-private partnership (PPP) project investment. A PPP project is funded and operated through a partnership of government and one or more private sectors according to their contract. On the one hand, the government can reduce financial expenditure and efficiently allocate limited resource. On the other hand, the private sectors can benefit from the PPP project by using their technology, fund and professional knowledge. PPP is a common model in the project finance and has been adopted by many countries.

During the implementation of a PPP project, there are numerous risk and uncertain factors. Some unsuccessful projects have been reported in [6]. For PPP projects in China, Xu et al. [52] have summed up 17 critical risk factors and 6 critical risk groups by empirical research studies. Take the 6 critical risk groups for example, they include macromacroeconomic risk, construction and operation risk, government maturity risk, market environment risk, economic viability risk and government intervention. Thus, risk assessment of the PPP project is necessary. Considering risk assessment and the TFDTRS model, this section focuses on analyzing decision-making of PPP project investment for the private sectors. The procedure for decision-making of PPP project investment based on TFDTRS model is described in Fig. 7.

During the risk assessment of PPP project investment, we have two states $\Omega = \{ C, \neg C \}$ indicating that the project is a good one and a bad one, respectively. The set of actions for the project $x$ is given by $A = \{ a_p, a_g, a_N \}$, where $a_p$, $a_g$, and

### Table 8

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>$\lambda_{pp}$</th>
<th>$\lambda_{xp}$</th>
<th>$\lambda_{xg}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>(1.5334, 2.3371, 3.1545)</td>
<td>(2.0407, 2.3215, 3.3474)</td>
<td>(3.207, 3.8384, 4.5767)</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>(0.3064, 2.3405)</td>
<td>(2.4387, 3.0007, 4.4323)</td>
<td>(5.3212, 6.8220, 8.3856)</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>(0.2305, 3.1222)</td>
<td>(3.1991, 3.3227, 4.4657)</td>
<td>(3.4132, 4.5302, 4.6665)</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>(0.7851, 0.8012, 1.3265)</td>
<td>(2.4496, 2.4856, 3.3490)</td>
<td>(3.3426, 5.4375, 6.6644)</td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>(1.5520, 2.3532, 3.1792)</td>
<td>(3.1808, 3.0365, 4.4409)</td>
<td>(3.4033, 4.6466, 8.3885)</td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>(0.7668, 1.5520, 3.1201)</td>
<td>(2.3941, 2.3164, 3.2707)</td>
<td>(3.4132, 4.5302, 4.6665)</td>
</tr>
</tbody>
</table>

### Table 9

The thresholds of PPP highway investment projects.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$\beta$</th>
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<tbody>
<tr>
<td>0</td>
<td>0.7743</td>
<td>0.6950</td>
<td>0.5826</td>
<td>0.6951</td>
<td>0.4848</td>
<td>0.3738</td>
</tr>
<tr>
<td>0.1</td>
<td>0.7560</td>
<td>0.6923</td>
<td>0.6061</td>
<td>0.6728</td>
<td>0.4763</td>
<td>0.3692</td>
</tr>
<tr>
<td>0.2</td>
<td>0.7371</td>
<td>0.6896</td>
<td>0.6282</td>
<td>0.6505</td>
<td>0.4674</td>
<td>0.3643</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7176</td>
<td>0.6869</td>
<td>0.6491</td>
<td>0.6283</td>
<td>0.4582</td>
<td>0.3591</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6975</td>
<td>0.6843</td>
<td>0.6687</td>
<td>0.6062</td>
<td>0.4486</td>
<td>0.3536</td>
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<td>0.6786</td>
<td>0.6816</td>
<td>0.6887</td>
<td>0.5842</td>
<td>0.4386</td>
<td>0.3477</td>
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<tr>
<td>0.6</td>
<td>0.6550</td>
<td>0.6790</td>
<td>0.7048</td>
<td>0.5623</td>
<td>0.4282</td>
<td>0.3413</td>
</tr>
<tr>
<td>0.7</td>
<td>0.6326</td>
<td>0.6764</td>
<td>0.7215</td>
<td>0.5404</td>
<td>0.4173</td>
<td>0.3345</td>
</tr>
<tr>
<td>0.8</td>
<td>0.6095</td>
<td>0.6738</td>
<td>0.7373</td>
<td>0.5186</td>
<td>0.4060</td>
<td>0.3272</td>
</tr>
<tr>
<td>0.9</td>
<td>0.5854</td>
<td>0.6713</td>
<td>0.7523</td>
<td>0.4969</td>
<td>0.3941</td>
<td>0.3193</td>
</tr>
<tr>
<td>1</td>
<td>0.5604</td>
<td>0.6687</td>
<td>0.7666</td>
<td>0.4752</td>
<td>0.3818</td>
<td>0.3107</td>
</tr>
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<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$\beta$</th>
</tr>
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<tbody>
<tr>
<td>0.7495</td>
<td>0.5883</td>
<td>0.3686</td>
</tr>
<tr>
<td>0.1286</td>
<td>0.5653</td>
<td>0.3860</td>
</tr>
<tr>
<td>0.2073</td>
<td>0.5623</td>
<td>0.4033</td>
</tr>
<tr>
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<td>0.5593</td>
<td>0.4209</td>
</tr>
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<tr>
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<td>0.6191</td>
<td>0.5499</td>
<td>0.4747</td>
</tr>
<tr>
<td>0.5961</td>
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<td>0.4931</td>
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<tr>
<td>0.5729</td>
<td>0.5435</td>
<td>0.5116</td>
</tr>
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<td>0.5402</td>
<td>0.5304</td>
</tr>
<tr>
<td>0.5252</td>
<td>0.5368</td>
<td>0.5494</td>
</tr>
</tbody>
</table>
\( a_N \) represent investment, need further investigated and do not investment, respectively. In the TFDTRS model, there are six parameters, \( \tilde{\lambda}_{PP} \), \( \tilde{\lambda}_{BP} \) and \( \tilde{\lambda}_{NP} \) denote the losses incurred for taking actions of investment, need further investigated and do not investment respectively, when the project belongs to a good project. Similarly, \( \tilde{\lambda}_{PN}, \tilde{\lambda}_{BN} \) and \( \tilde{\lambda}_{NN} \) denote the losses incurred for taking actions of investment, need further investigated and do not investment when the project belongs to a bad project. The overall risk of PPP project investment is hard quantified. For the PPP projects, the losses regarding the risk or cost of actions in different states can be obtained by questionnaire survey. When experts fill in the questionnaire, they need to integrate all risk factors and evaluate the loss functions with linguistic variables. In this example, we continue with the results of Example 3 to illustrate the TFDTRS model, including the original scales of linguistic variables with uniform distribution, decision matrices, the optimum scales of linguistic variables and group decision matrix. Given a project \( x \), the conditional probability \( Pr(C|\{x\}) \) can be determined according to original information or prior information. For clarity, the conditional probability for 6 PPP highway investment projects are given as follows: \( Pr(C|\{o_1\}) = 0.8000, Pr(C|\{o_2\}) = 0.3600, Pr(C|\{o_3\}) = 0.4100, Pr(C|\{o_4\}) = 0.5308, Pr(C|\{o_5\}) = 0.3800 \) and \( Pr(C|\{o_6\}) = 0.4180 \).

In Table 8, we observe that the values of losses of each PPP project meet conditions (3)–(8). According to (17), the thresholds \( \alpha, \gamma \) and \( \beta \) of each project are calculated. Table 9 lists the calculating results of the thresholds of PPP highway investment projects, where the risk attitude index of decision maker \( \rho \) changes from 0 to 1 with a step size of 0.1.

With respect to the results presented in Table 9, the variation characteristics of the thresholds with the increasing of risk attitude index of decision maker to each PPP investment project are described in Fig. 8. From Fig. 8, the value of \( \rho \) impact the values of \( \alpha, \gamma \) and \( \beta \). The value of \( \gamma \) is always between \( \alpha \) and \( \beta \). For the projects \( o_1, o_3, o_4, o_5 \) and \( o_6 \), the values of \( \alpha \) are decreasing with the increasing of \( \rho \), while the values of \( \beta \) are also increasing. According to (18) and (19), their decision rules can change from the three-way decision rules \( (P_2)-(N_2) \) to the two-way decision rules \( (P'_2)-(N'_2) \). For the project \( o_2 \), the values of \( \alpha \) are decreasing with the increasing of \( \rho \), and the values of \( \beta \) are also decreasing. According to (18), its decision rules are still the three-way decision rules \( (P_2)-(N_2) \).
Fig. 8. The variation characteristics of the thresholds with the increasing of risk attribute index of decision maker to each PPP investment project.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>POS($C$)</th>
<th>BND($C$)</th>
<th>NEG($C$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${o_1}$</td>
<td>${o_4, o_6}$</td>
<td>${o_2, o_3, o_5}$</td>
</tr>
<tr>
<td>0.1</td>
<td>${o_1}$</td>
<td>${o_4, o_6}$</td>
<td>${o_2, o_3, o_5}$</td>
</tr>
<tr>
<td>0.2</td>
<td>${o_1}$</td>
<td>${o_2, o_4, o_6}$</td>
<td>${o_2, o_3, o_5}$</td>
</tr>
<tr>
<td>0.3</td>
<td>${o_1}$</td>
<td>${o_2, o_4, o_6}$</td>
<td>${o_2, o_3, o_5}$</td>
</tr>
<tr>
<td>0.4</td>
<td>${o_1}$</td>
<td>${o_2, o_4, o_6}$</td>
<td>${o_2, o_3, o_5}$</td>
</tr>
<tr>
<td>0.5</td>
<td>${o_1}$</td>
<td>${o_2, o_4, o_6}$</td>
<td>${o_2, o_3, o_5}$</td>
</tr>
<tr>
<td>0.6</td>
<td>${o_1}$</td>
<td>${o_2, o_4, o_6}$</td>
<td>${o_2, o_3, o_5}$</td>
</tr>
<tr>
<td>0.7</td>
<td>${o_1}$</td>
<td>${o_2, o_4, o_6}$</td>
<td>${o_2, o_3, o_5}$</td>
</tr>
<tr>
<td>0.8</td>
<td>${o_1}$</td>
<td>${o_2, o_4, o_6}$</td>
<td>${o_2, o_3, o_5}$</td>
</tr>
<tr>
<td>0.9</td>
<td>${o_1, o_3}$</td>
<td>${o_2, o_4}$</td>
<td>${o_1, o_3, o_6}$</td>
</tr>
<tr>
<td>1</td>
<td>${o_1, o_3, o_6}$</td>
<td>${o_2}$</td>
<td>${o_4, o_5}$</td>
</tr>
</tbody>
</table>

Given a project $x$, we can determine its concrete decision by comparing the conditional probability $Pr(C|x)$ with the thresholds $(\alpha, \gamma, \beta)$. Based on the decision rules $(P_2)-(N_2)$ and $(P_2')-(N_2')$, the concrete decisions of PPP highway investment projects in the different risk attitude index of decision maker are shown in Table 10.

From Table 10, the risk attitude of decision maker can influence the investment decision of each project. For example, when $\rho = 0$, the pessimistic decision maker considers the projects $o_1$ can be invested, but $o_2, o_3$ and $o_5$ should not be invested. Meanwhile, the projects $o_4$ and $o_6$ need be further investigated. When $\rho = 0.5$, the moderate decision maker considers the project $o_1$ can be invested, but $o_3$ and $o_5$ should not be invested. Simultaneously, the projects $o_2, o_4$ and $o_6$ need further investigated. When $\rho = 1$, the optimistic decision maker decides the projects $o_1, o_3$ and $o_6$ can be invested, but $o_4$ and $o_5$ should not be invested. Meanwhile, the project $o_2$ need be further investigated. With the higher degree of optimism, the projects in the region of POS($C$) are increasing, while the projects in the region of NEG($C$) are decreasing. The reason is that the optimistic decision maker may be confidence in the project investment decision, but the pessimistic decision maker hesitates to make a decision.

6. Conclusions

In this paper, the TFDTRS model is proposed by considering the losses being expressed by triangular fuzzy numbers. The algorithm for determining the values of losses used in TFDTRS is designed to support the application of TFDTRS. An example of PPP project investment is utilized to illustrate the performance of the TFDTRS. Our study provides a method to determine the loss function of DTRS and make it adapt to a fuzzy scenario. TFDTRS can be very useful in dealing with many management
decision problems in the context of risk and uncertainty, such as venture investment, the selection of manufacturing partner, market segmentation, and government decision. Future research work may focus on developing new DTRS models involving a characterization of the decision scenarios in terms of more general information granularity.

Acknowledgements

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Appendix A

In the following, the representative ranking fuzzy number methods are briefly summarized and applied them to rank triangular fuzzy numbers.

(1) Distance minimization method

According to DTRS model and [3], the ranking function of triangular fuzzy numbers $\tilde{M} = (l, m, u)$ is given as follows:

$$r(\tilde{M})_2 = \frac{1}{4}(l + 2m + u).$$

(A.1)

(2) Deviation degree method

According to DTRS model and [2], the ranking function of triangular fuzzy numbers $\tilde{M} = (l, m, u)$ is given as follows:

$$r(\tilde{M})_3 = \frac{l + 2m + u - 4x_{\min}}{2 + 4x_{\max} - (l + 2m + u)}.$$  

(A.2)

where $x_{\min}$ and $x_{\max}$ are the infimum and supremum of support set of these fuzzy numbers, see [2, 50].

(3) Different heights and different spreads method

According to DTRS model and [7], the ranking function of triangular fuzzy numbers $\tilde{M} = (l, m, u)$ is given as follows:

$$x_{\tilde{M}} = \frac{l + 2m + u}{4};$$

$$STD_{\tilde{M}} = \sqrt{\frac{(l - x_{\tilde{M}})^2 + 2(m - x_{\tilde{M}})^2 + (u - x_{\tilde{M}})^2}{3}};$$

$$r(\tilde{M})_4 = \frac{x_{\tilde{M}}}{1 + STD_{\tilde{M}}}.$$  

(A.3)

where $\tilde{M}$ is a standardized triangular fuzzy number. In [7], the standardized approach of generalized fuzzy number is explained in details.

(4) Distance method using circumcenter of centroids

According to DTRS model and [45], the ranking function of triangular fuzzy numbers $\tilde{M} = (l, m, u)$ is given as follows:

$$r(\tilde{M})_5 = \left(\frac{l + 4m + u}{6}\right)^2 + \left(\frac{4(l - m)(u - m) + 5}{12}\right).$$  

(A.4)

For any two triangular fuzzy number $\tilde{A}$ and $\tilde{B}$, the rules of ranking triangular fuzzy numbers are obtained based on their ranking functions $r(\tilde{A})_i$ and $r(\tilde{B})_i$:

(i) $\tilde{A} > \tilde{B}$ iff $r(\tilde{A})_i > r(\tilde{B})_i$;

(ii) $\tilde{A} < \tilde{B}$ iff $r(\tilde{A})_i < r(\tilde{B})_i$;

(iii) $\tilde{A} \sim \tilde{B}$ iff $r(\tilde{A})_i = r(\tilde{B})_i$.

(A.5)

where $i = 1, 2, \ldots, 5$. The order " $\geq$ " and " $\leq$ " can be formulated as:

$\tilde{A} \geq \tilde{B}$ iff $\tilde{A} > \tilde{B}$ or $\tilde{A} \sim \tilde{B}$;

$\tilde{A} \leq \tilde{B}$ iff $\tilde{A} < \tilde{B}$ or $\tilde{A} \sim \tilde{B}$.
Appendix B

Proposition 2. If two triangular fuzzy numbers are ranked using deviation degree method, the ranking result is essentially same with the distance minimization method.

Proof. Based on (A.2), we can get:

\[ l + 2x + u - 4x_{\min} \geq 0, \]
\[ 2 + 4x_{\max} - (l + 2x + u) > 0. \]

Suppose that two triangular fuzzy numbers are denoted by \( \vec{A} = (l_A, m_A, u_A) \) and \( \vec{B} = (l_B, m_B, u_B) \), the ranking functions of \( \vec{A} \) and \( \vec{B} \) using deviation degree method are given as follows:

\[ r(\vec{A})_3 = \frac{l_A + 2m_A + u_A - 4x_{\min}}{2 + 4x_{\max} - (l_A + 2m_A + u_A)}, \]
\[ r(\vec{B})_3 = \frac{l_B + 2m_B + u_B - 4x_{\min}}{2 + 4x_{\max} - (l_B + 2m_B + u_B)}. \]

If \( \vec{A} \geq \vec{B} \), we can get:

\[
\begin{align*}
\iff r(\vec{A})_3 &\geq r(\vec{B})_3 \\
\iff \frac{l_A + 2m_A + u_A - 4x_{\min}}{2 + 4x_{\max} - (l_A + 2m_A + u_A)} &\geq \frac{l_B + 2m_B + u_B - 4x_{\min}}{2 + 4x_{\max} - (l_B + 2m_B + u_B)} \\
\iff (l_A + 2m_A + u_A - 4x_{\min})(2 + 4x_{\max} - (l_B + 2m_B + u_B)) &\geq (l_B + 2m_B + u_B - 4x_{\min})(2 + 4x_{\max} - (l_A + 2m_A + u_A)) \\
\iff (l_A + 2m_A + u_A)(2 + 4x_{\max} - 4x_{\min}) &\geq (l_B + 2m_B + u_B)(2 + 4x_{\max} - 4x_{\min}) \\
\iff (l_A + 2m_A + u_A) &\geq (l_B + 2m_B + u_B) \\
\iff \frac{1}{4} (l_A + 2m_A + u_A) &\geq \frac{1}{4} (l_B + 2m_B + u_B)
\end{align*}
\]

Similarly, if \( \vec{A} \leq \vec{B} \), we can get:

\[
\begin{align*}
\iff r(\vec{A})_3 &\leq r(\vec{B})_3 \\
\iff (l_A + 2m_A + u_A) &\leq (l_B + 2m_B + u_B) \\
\iff \frac{1}{4} (l_A + 2m_A + u_A) &\leq \frac{1}{4} (l_B + 2m_B + u_B)
\end{align*}
\]

Therefore, the ranking result of \( \vec{A} \) and \( \vec{B} \) depends on the comparison of \( (l_A + 2m_A + u_A) \) and \( (l_B + 2m_B + u_B) \), which is same with the distance minimization method. The statement in Proposition 2 holds. \( \square \)

Appendix C

Under conditions of (3)–(8), we can simplify the decision rules \((P_0')-(N_0')\). For the rule \((P_0')\), the first condition can be expressed as:

\[
\begin{align*}
r(\vec{C}[x]) &\leq r(\vec{B}[x]) \\
\iff r(\vec{C}[x]) &\leq r(\vec{P}[x]) r(\vec{B}[x]) + r(\vec{C}[x]) (1 - r(\vec{B}[x])) \\
\iff r(\vec{C}[x]) &\geq \frac{r(\vec{P}[x]) - r(\vec{B}[x])}{r(\vec{P}[x]) - r(\vec{B}[x])}
\end{align*}
\]
Similarly, the second condition of rule \((P'_0)\) can be expressed as:

\[
\begin{align*}
& r(\tilde{R}(a_B|[[x]]))_1 \leq r(\tilde{R}(a_N|[[x]]))_1 \\
\iff & r(\tilde{\lambda}_{NP})_1 Pr(C|[[x]]) + r(\tilde{\lambda}_{PP})_1 (1 - Pr(C|[[x]])) \leq r(\tilde{\lambda}_{NP})_1 Pr(C|[[x]]) + \tilde{\varphi}_{NN}_1 (1 - Pr(C|[[x]])) \\
\iff & Pr(C|[[x]]) \geq \frac{r(\tilde{\lambda}_{PN})_1 - r(\tilde{\lambda}_{BN})_1}{(r(\tilde{\lambda}_{PN})_1 - r(\tilde{\lambda}_{BN})_1) + (r(\tilde{\lambda}_{NP})_1 - r(\tilde{\lambda}_{BP})_1)}.
\end{align*}
\]

The first condition of rule \((B'_0)\) is the converse of the first condition of rule \((P'_0)\). It follows,

\[
\begin{align*}
& r(\tilde{R}(a_B|[[x]]))_1 \leq r(\tilde{R}(a_N|[[x]]))_1 \\
\iff & Pr(C|[[x]]) \leq \frac{r(\tilde{\lambda}_{PN})_1 - r(\tilde{\lambda}_{BN})_1}{(r(\tilde{\lambda}_{PN})_1 - r(\tilde{\lambda}_{BN})_1) + (r(\tilde{\lambda}_{NP})_1 - r(\tilde{\lambda}_{BP})_1)}.
\end{align*}
\]

For the second condition of rule \((B'_0)\), we have:

\[
\begin{align*}
& r(\tilde{R}(a_B|[[x]]))_1 \leq r(\tilde{R}(a_N|[[x]]))_1 \\
\iff & r(\tilde{\lambda}_{BN})_1 Pr(C|[[x]]) \leq r(\tilde{\lambda}_{NP})_1 Pr(C|[[x]]) + \tilde{\varphi}_{NN}_1 (1 - Pr(C|[[x]])) \\
\iff & Pr(C|[[x]]) \geq \frac{r(\tilde{\lambda}_{BN})_1 - \tilde{\varphi}_{NN}_1}{(r(\tilde{\lambda}_{BN})_1 - \tilde{\varphi}_{NN}_1) + (r(\tilde{\lambda}_{NP})_1 - r(\tilde{\lambda}_{BP})_1)}.
\end{align*}
\]

The first condition of rule \((N'_0)\) is the converse of the second condition of rule \((P'_0)\) and the second condition of rule \((N'_0)\) is the converse of the second condition of rule \((B'_0)\). It follows,

\[
\begin{align*}
& r(\tilde{R}(a_N|[[x]]))_1 \leq r(\tilde{R}(a_B|[[x]]))_1 \\
\iff & Pr(C|[[x]]) \leq \frac{r(\tilde{\lambda}_{BN})_1 - \tilde{\varphi}_{NN}_1}{(r(\tilde{\lambda}_{BN})_1 - \tilde{\varphi}_{NN}_1) + (r(\tilde{\lambda}_{NP})_1 - r(\tilde{\lambda}_{BP})_1)}.
\end{align*}
\]

According to the decision rules \((P_1)-(N_1)\) in Section 2, we denote the three expressions in these conditions by the following three parameters:

\[
\begin{align*}
\alpha &= \frac{r(\tilde{\lambda}_{PN})_1 - r(\tilde{\lambda}_{BN})_1}{(r(\tilde{\lambda}_{PN})_1 - r(\tilde{\lambda}_{BN})_1) + (r(\tilde{\lambda}_{BP})_1 - r(\tilde{\lambda}_{PP})_1)} , \\
\beta &= \frac{r(\tilde{\lambda}_{BN})_1 - \tilde{\varphi}_{NN}_1}{(r(\tilde{\lambda}_{BN})_1 - \tilde{\varphi}_{NN}_1) + (r(\tilde{\lambda}_{NP})_1 - r(\tilde{\lambda}_{BP})_1)} , \\
\gamma &= \frac{r(\tilde{\lambda}_{BN})_1 - \tilde{\varphi}_{NN}_1}{(r(\tilde{\lambda}_{BN})_1 - \tilde{\varphi}_{NN}_1) + (r(\tilde{\lambda}_{NP})_1 - r(\tilde{\lambda}_{PP})_1)} .
\end{align*}
\]

**References**


