Incremental updating approximations in dominance-based rough sets approach under the variation of the attribute set

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\textbf{A B S T R A C T}

Dominance-based Rough Sets Approach (DRSA) is a generalized model of the classical Rough Sets Theory (RST) which may handle information with preference-ordered attribute domain. The attribute set in the information system may evolve over time. Approximations of DRSA used to induce decision rules need updating for knowledge discovery and other related tasks. We firstly introduce a kind of dominance relations to substitute the equivalence relation in RST\textsuperscript{[10–12]}. DRSA attracted much attention in recent years. For example, Kotowski et al. found that the notions of rough approximations of DRSA are excessively restrictive in real-life problems. They proposed a probabilistic model for ordinal classification problems with monotonicity constraints\textsuperscript{[13]}. Dembczynski et al. reformulated the dominance principle and proposed second-order rough approximations of DRSA to handle imprecise evaluations and assignments of objects\textsuperscript{[14]}. Luo et al. proposed a limited dominance-based rough set model based on the assumption that the unknown value can only be compared with the maximal or minimal value in the domain of the corresponding attribute\textsuperscript{[15]}. Inuiguchi et al. presented a Variable-Precision Dominance-based Rough Set Approach (VP-DRSA) for attribute reduction\textsuperscript{[16]}. Yang et al. proposed the concept of a similarity dominance relation to conduct classification analysis in the incomplete information system\textsuperscript{[17]}. Then, they investigated DRSA in an incomplete interval-valued information system and employed a data complement method to transform the incomplete interval-valued information system to a traditional one\textsuperscript{[18]}. They also proposed up arrow and down arrow “optimal credible rules” based on DRSA in the incomplete information system\textsuperscript{[19]}. Typically, the collection of information is a dynamic process. The set of objects, the set of attributes and attribute values may vary with time in a dynamic information system. The update of approximations is vital to knowledge representation and reduction based on rough sets in dynamic information systems.

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1. Introduction

Rough Sets Theory (RST) introduced by Pawlak in the early 1980s\textsuperscript{[1]} is an effective mathematical tool to deal with uncertain and inconsistent information. In RST, lower and upper approximations were defined to characterize a concept, namely, a subset of the universe. By lower and upper approximations, the knowledge concealed in information systems can be expressed in the form of rules. The main advantage of RST in data analysis is that it does not need any preliminary or additional information about data\textsuperscript{[2]}. RST has been widely applied to many fields, e.g., pattern recognition\textsuperscript{[3]}, knowledge discovery\textsuperscript{[4,5]} and decision making\textsuperscript{[6,7]}. However, RST is not able to process information with preference-ordered attribute domain and decision classes\textsuperscript{[8,9]}. For example, we consider two cars, A and B, and suppose that fuel consumption of A is less than that of B while the maximum speed of A is higher than that of B. It is reasonable to believe that A is better than B. Within RST, however, the two cars will be only divided into two different classes and no preference relation between them will be considered. To handle such information, Greco et al. proposed Dominance-based Rough Sets Approach (DRSA) by using a dominance relation to substitute the equivalence relation in RST\textsuperscript{[10–12]}. DRSA attracted much attention in recent years. For example, Kotowski et al. found that the notions of rough approximations of DRSA are excessively restrictive in real-life problems. They proposed a probabilistic model for ordinal classification problems with monotonicity constraints\textsuperscript{[13]}. Dembczynski et al. reformulated the dominance principle and proposed second-order rough approximations of DRSA to handle imprecise evaluations and assignments of objects\textsuperscript{[14]}. Luo et al. proposed a limited dominance-based rough set model based on the assumption that the unknown value can only be compared with the maximal or minimal value in the domain of the corresponding attribute\textsuperscript{[15]}. Inuiguchi et al. presented a Variable-Precision Dominance-based Rough Set Approach (VP-DRSA) for attribute reduction\textsuperscript{[16]}. Yang et al. proposed the concept of a similarity dominance relation to conduct classification analysis in the incomplete information system\textsuperscript{[17]}. Then, they investigated DRSA in an incomplete interval-valued information system and employed a data complement method to transform the incomplete interval-valued information system to a traditional one\textsuperscript{[18]}. They also proposed up arrow and down arrow “optimal credible rules” based on DRSA in the incomplete information system\textsuperscript{[19]}. Typically, the collection of information is a dynamic process. The set of objects, the set of attributes and attribute values may vary with time in a dynamic information system. The update of approximations is vital to knowledge representation and reduction based on rough sets in dynamic information systems. The traditional rough sets methodologies update approximations with recomputing from scratch. However, they are inefficiency or even intractable in many real-world applications because the computational time is too long. Alternatively, an incremental updating
scheme as a kind of feasible and effective method to maintain knowledge dynamically is often employed in handling dynamic information. It can avoid unnecessary computations by utilizing previous data structures and results. There are many literature on incremental approaches based on rough sets methodologies [20–35]. Since an information system is composed of the objects, the attributes and attribute values, these incremental approaches can be divided into three classes: incremental approaches under the variation of the object set [20–28], incremental approaches under the variation of the attribute set [30–33] and incremental approaches under the variation of attribute values [34,35]. Here, we briefly overview some achievements under the variation of the attribute set as follows: Chan et al. proposed an incremental learning method for updating approximations in RST by considering adding or deleting one attribute [30]. RST is under the assumption that information systems are complete. However, missing data in information systems is common in many real-life applications. An information system with missing data is called as an Incomplete Information System (IIS). Under a characteristic relation, Li et al. proposed an incremental approach for updating approximations of a concept by considering adding or removing some attributes simultaneously in the IIS [31]. For values of attributes including both symbolic and real-valued in practical databases, Cheng proposed two incremental approaches for updating approximations in rough fuzzy set [32]. Furthermore, the incremental updating scheme was also applied in dynamic attribute reduction. Xu et al. proposed a dynamic attribute reduction algorithm in RST based on 0–1 integer programming when the multiple objects are added into an information system [36].

In the real-life applications, quantitative (e.g., from sensors) and qualitative (e.g., from manufacturing environment) data from diverse sources may be linked, thus significantly increasing the number of attributes. The integration and fusion of diverse source data have been investigated in many research fields [37–39]. However, some obtained attributes may outdate over time. Thus, the information systems need to be updated dynamically by deleting outdated attributes or adding new available attributes. In knowledge discovery and data mining based on rough set methodologies, computation of approximations is a necessary step to induce rules. In order to dynamically maintain knowledge and effectively reduce computational time, many incremental approaches have been proposed for updating approximations based on different binary relations when the attribute set evolves over time [30–33]. As a kind of important tool for handling information with preference-ordered attributes, DRSA cannot handle dynamic data directly. To our best knowledge, there are no studies for incremental updating approximations in DRSA when the attribute set varies so far. The aim of this work is to propose an incremental approach for updating approximations of DRSA when the attribute set evolves over time [30–33] and incremental approaches based on rough sets methodologies, the computation of approximations is a necessary step to induce rules. Furthermore, the incremental updating scheme was also applied in dynamic attribute reduction. Xu et al. proposed a dynamic attribute reduction algorithm in RST based on 0–1 integer programming when the multiple objects are added into an information system [36].

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2. Preliminaries

In this section, we briefly review some concepts, notations and properties of DRSA [10–13].

An information system is a 4-tuple \( S = (U, A, V, f) \), where \( U \) is a non-empty finite set of objects, called the universe. \( A = C \cup \{d\} \). \( C \) is a non-empty finite set of condition attributes, and \( d \) is a decision attribute. \( V \) is regarded as the domain of all attributes. \( f: U \times A \rightarrow V \) is an information function such that \( f(x, a) \in V_a \) \( \forall a \in A \) and \( x \in U \), where \( V_a \) is the domain of attribute \( a \).

\( \forall a \in C \), there is a preference relation on the set of objects with respect to the attribute \( a \), denoted by \( \succeq_a = \{ (x, y) \in U \times U | f(x, a) \geq f(y, a) \} \). \( \forall x, y \in U \), \( x \succeq_y y \) means “\( x \) is at least as good as \( y \) with respect to the attribute \( a \).”. For a nonempty finite attribute set \( P, P \subseteq C \), if \( x \succeq_y y \), \( \forall a \in P \), we say that \( x \) dominates \( y \) with respect to the set of attributes \( P \), denoted by \( x \dom_P y \). \( D_P \) is a dominance relation on the universe \( U \) with respect to the set of attributes \( P \).

Therefore, there are the following two sets:

- A set of objects dominating \( x \), called \( P \)-dominating set, \( D_P^P(x) = \{ y \in U \mid yD_P x \} \);
- A set of objects dominated by \( x \), called \( P \)-dominated set, \( D_P^P(x) = \{ y \in U \mid xD_P y \} \).

The universe \( U \) is divided by the decision attribute \( d \) into a family of equivalence classes with preference-ordered, called decision classes. Let \( Cl = \{ Cl_n | n \in T \} \) be a set of decision classes, \( T = \{1, \ldots, t\} \). \( \forall r, s \in T \) such that \( r > s \), the objects from \( Cl_r \) are preferred to the objects from \( Cl_s \). In DRSA, the concepts to be approximated are an upward union and a downward union of classes such that

\[ Cl_n^U = \bigcup_{n' > n} Cl_{n'}, \quad Cl_n^D = \bigcup_{n' < n} Cl_{n'}, \quad \forall n, n' \in T. \]

\( x \in Cl_n^U \) means “\( x \) belongs to at least class \( Cl_n^U \)”, and \( x \in Cl_n^D \) means “\( x \) belongs to at most class \( Cl_n^D \)”.

The lower and upper approximations of \( Cl_n^U \) are defined respectively as follows.

\[ P(Cl_n^U) = \{ x \in U \mid D_P^P(x) \subseteq Cl_n^U \}, \quad P(Cl_n^D) = \{ x \in U \mid D_P^P(x) \cap Cl_n^D \neq \emptyset \}, \]

The lower and upper approximations of \( Cl_n^U \) are defined respectively as follows.

\[ P(Cl_n^U) = \{ x \in U \mid D_P^P(x) \subseteq Cl_n^U \}, \quad P(Cl_n^D) = \{ x \in U \mid D_P^P(x) \cap Cl_n^D \neq \emptyset \}. \]

The upper and lower approximations satisfy the inclusion property.

\[ P(Cl_n^U) \subseteq Cl_n^U \subseteq P(Cl_n^D), \quad n = 2, \ldots, t \]

\[ P(Cl_1^U) \subseteq Cl_1^U \subseteq P(Cl_t^D), \quad n = 1, \ldots, t - 1. \]

The upper and lower approximations of upward and downward unions of decision classes have an important complementarity property.

\[ P(Cl_n^U) = U - P(Cl_{n-1}^D) \quad \text{and} \quad P(Cl_n^D) = U - P(Cl_{n-1}^U), \quad n = 2, \ldots, t \]

\[ P(Cl_1^U) = U - P(Cl_t^D) \quad \text{and} \quad P(Cl_1^D) = U - P(Cl_{t-1}^U), \quad n = 1, \ldots, t - 1. \]

The objects belonging to \( Cl_n^U \) and \( Cl_n^D \) with some ambiguity constitute the \( P \)-boundaries of \( Cl_n^U \) and \( Cl_n^D \), denoted by \( B_{Pn}(Cl_n^U) \) and \( B_{Pn}(Cl_n^D) \), respectively. \( B_{Pn}(Cl_n^U) \) and \( B_{Pn}(Cl_n^D) \) are represented in terms of upper and lower approximations as follows.

The remainder of this paper is organized as follows. Some basic notions of DRSA are introduced in Section 2. A dominance matrix is introduced and the definitions of \( P \)-dominating sets and \( P \)-dominated sets are rewritten in Section 3. We discuss the principles for incrementally updating approximations of DRSA in Section 4. In Section 5, we show experimental evaluations of our incremental approach on UCI data sets [40]. This paper ends with conclusions and further research topics in Section 6.
\[ B_{\mu_P}(C_{n}^P) = \overline{P}(C_{n}^P) - P(C_{n}^P) \]
\[ B_{\mu_u}(C_{n}^P) = \overline{P}(C_{n}^P) - P(C_{n}^P) \]

In [13], the definition of \( P \)-generalized decision was proposed to reflect an interval of decision classes to which an object may belong. However, the calculation of \( P \)-generalized decision is based on approximations of DRSAs. To calculate approximations of DRSAs, we revise the definition of \( P \)-generalized decision as follows.

**Definition 1.** \( \forall x \in U, \rho_Y(x) = (I_P(x), u_P(x)) \) is called as \( P \)-generalized decision of the object \( x \), where \( I_P(x) = \min \{ n \in T : D_P(x) \cap C_n \neq \emptyset \} \) and \( u_P(x) = \max \{ n \in T : D_P(x) \cap C_n = \emptyset \} \).

**Proposition 1.** \( \forall x \in U, n \in T, the following items hold

1. If \( I_P(x) \geq n \), then \( x \in P(C_n^P) \);
2. If \( I_P(x) \leq n \), then \( x \in \overline{P}(C_n^P) \);
3. If \( u_P(x) \geq n \), then \( x \in P(C_n^P) \);
4. If \( u_P(x) \leq n \), then \( x \in \overline{P}(C_n^P) \).

Proof. Let \( V_d = \{ d_1, d_2, \ldots, d_d \} \) be the domain of the decision attribute \( d \). Suppose that \( d_1 < d_2 < \cdots < d_n \) and \( d_n \in V_d \). Since \( I_P(x) = \min \{ n \in T : D_P(x) \cap C_n \neq \emptyset \} \), if \( I_P(x) \geq n \), we have \( f(y,d) \geq d_n \), \( \forall y \in D_P(x) \). It follows that \( D_P(x) \subseteq C_{I_P(x)} \Rightarrow x \in P(C_{I_P(x)}) \). Similarly, (2)-(4) hold. \( \square \)

According to **Proposition 1**, lower and upper approximations of upward and downward unions of decision classes are redefined as follows.

\[ P(C_{n}^P) = \{ x \in U : I_P(x) \geq n \} \quad (1) \]
\[ \overline{P}(C_{n}^P) = \{ x \in U : u_P(x) \geq n \} \quad (2) \]
\[ P(C_{n}^P) = \{ x \in U : u_P(x) \leq n \} \quad (3) \]
\[ \overline{P}(C_{n}^P) = \{ x \in U : I_P(x) \leq n \} \quad (4) \]

3. A dominance matrix

In DRSAs, \( \forall a \in C \), there is a preference relation \( \succeq_a \) on the universe \( U \). Let \( x, y \in U \), if \( (x,y) \in \succeq_a \), then \( f(x,a) \geq f(y,a) \); if \( (x,y) \notin \succeq_a \), then \( f(x,a) < f(y,a) \). So we can use a characteristic variant \( r_{ij}^a \) to indicate whether \( (x,y) \) belongs to \( \succeq_a \), where

\[ r_{ij}^a = \begin{cases} 1 & : f(x,a) \geq f(y,a) \\ 0 & : f(x,a) < f(y,a) \end{cases} \]

\( r_{ij}^a = 1 \) means \( (x,y) \in \succeq_a \) and \( r_{ij}^a = 0 \) means \( (x,y) \notin \succeq_a \). It follows that a matrix \( \mathbf{R}^a \) is utilized to present the preference relation \( \succeq_a \) on the universe \( U \) with respect to the attribute \( a \).

**Example 1.** Given an information system \( S = (U,C \cup \{d\},V,f) \) in Table 1, where \( U = \{x_1, x_2, \ldots, x_n\}, C = \{a_1, a_2, a_3, a_4\}, \ V_d = V_{a_1} = V_{a_2} = V_{a_3} \). From Table 1, the preference matrices of attributes \( a_1, a_2, a_3 \) and \( a_4 \) are as follows.

**Table 1**

<table>
<thead>
<tr>
<th>Object</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( x_6 )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( x_7 )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( x_8 )</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( x_9 )</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( x_{10} )</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

**Example 2.** Re-rewritten as follows:

\[ D_P(x_i) = \{ x_i \in U : \phi_i^P = |P| \} \quad (8) \]
\[ D_P(x_i) = \{ x_i \in U : \phi_i^P = |P| \} \quad (9) \]

\[ \mathbf{R}^a = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \]

The definitions of \( P \)-dominating sets and \( P \)-dominated sets are rewritten as follows:

\[ D_P(x_i) = \{ x_i \in U : \phi_i^P = |P| \} \quad (8) \]
\[ D_P(x_i) = \{ x_i \in U : \phi_i^P = |P| \} \quad (9) \]

\[ \mathbf{R}^a = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \]
4.1. Addition of some new attributes

Suppose \( Q \) is a collection of attributes added into the attribute set \( P \), \( P \cap Q = \emptyset \). For \( x, y \in U \), if \( xDy \), then \( f(x,a) \geq f(y,a) \), \( \forall a \in P \). And if \( f(x,b) \geq f(y,b) \), \( \forall b \in Q \), then \( xD_Q y \), where \( D_{P\cap Q} \) is the dominance relation with respect to the attribute set \( P \cup Q \). Then, Lemma 1 is proposed to calculate \( D_{P\cup Q}(x) \) and \( D_{P\cap Q}(x) \), \( \forall x \in U \).

**Lemma 1.** Let \( P \subset C \), \( Q \subset C \) and \( P \cap Q = \emptyset \). We have

1. \( D_{P\cup Q}(x) = D_P(x) \cap D_Q(x) \);
2. \( D_{P\cap Q}(x) = D_P(x) \cap D_Q(x) \).

**Proof.** Let \( x, y \in U \). \( D_{P\cup Q}(x) = \{ x \in U : \phi_{ij}^{P\cup Q} = |P \cup Q| \} \). Since \( P \cap Q = \emptyset \), \( |P \cup Q| = |P| + |Q| \). And since \( \phi_{ij}^{P\cup Q} = \sum_{i \in P \cup Q} p_{ij}^r = \sum_{i \in P} p_{ij}^r + \sum_{i \in Q} p_{ij}^r = \phi_{ij}^P + \phi_{ij}^Q \), if \( \phi_{ij}^{P\cup Q} = |P| + |Q| \), then \( \phi_{ij}^P = |P| \) and \( \phi_{ij}^Q = |Q| \). It follows that \( D_{P\cup Q}(x) = \{ x \in U : \phi_{ij}^{P\cup Q} = |P| + |Q| \} = \{ x \in U : \phi_{ij}^P = |P| \} \cap \{ x \in U : \phi_{ij}^Q = |Q| \} = D_P(x) \cap D_Q(x) \). Similarly, (2) holds. \( \square \)

**Example 2** (Continuation of Example 1). Let \( P = \{ a_1, a_4 \} \) and \( Q = \{ a_2, a_3 \} \).

According to \( R^p \) and \( R^q \), the dominance matrix

\[
\begin{array}{cccccccccccc}
1 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 2 \\
2 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\
1 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 2 \\
0 & 0 & 1 & 2 & 2 & 0 & 0 & 0 & 1 & 2 & 2 & 0 \\
1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 \\
0 & 0 & 1 & 2 & 2 & 0 & 0 & 1 & 0 & 2 & 2 & 0 \\
1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 \\
1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\]

According to the dominance matrix \( R^p \), \( P \)-dominating sets and \( P \)-dominated sets are calculated as follows.

\[
\begin{align*}
D_P(x_1) &= \{ x_1, x_{10} \}, & D_P(x_2) &= \{ x_1, x_3, x_4, x_5, x_6, x_9 \}; \\
D_P(x_2) &= \{ x_2, x_3, x_4, x_9, x_10 \}, & D_P(x_3) &= \{ x_2, x_4, x_6, x_9, x_9 \}; \\
D_P(x_3) &= \{ x_1, x_2, x_3, x_7, x_8, x_9, x_9 \}, & D_P(x_4) &= \{ x_2, x_4, x_6, x_7, x_8, x_9 \}; \\
D_P(x_4) &= \{ x_1, x_2, x_3, x_4, x_6, x_8, x_9 \}, & D_P(x_5) &= \{ x_4, x_6 \}; \\
D_P(x_5) &= \{ x_1, x_5, x_10 \}, & D_P(x_6) &= \{ x_4, x_5, x_6 \}; \\
D_P(x_6) &= \{ x_1, x_3, x_4, x_5, x_7, x_8, x_9, x_{10} \}, & D_P(x_7) &= \{ x_3, x_4, x_6, x_9 \}; \\
D_P(x_7) &= \{ x_1, x_2, x_3, x_4, x_8, x_{10} \}, & D_P(x_8) &= \{ x_2, x_3, x_4, x_8, x_9, x_{10} \}; \\
D_P(x_8) &= \{ x_1, x_2, x_3, x_4, x_7, x_8, x_{10} \}, & D_P(x_9) &= \{ x_3, x_4, x_6, x_{10} \}; \\
D_P(x_9) &= \{ x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_{10} \}, & D_P(x_{10}) &= \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \}; \\
\end{align*}
\]

Then, Lemma 1 is proposed to calculate \( D_{P\cup Q}(x) \) and \( D_{P\cap Q}(x) \), \( \forall x \in U \).
Proof 3. Let $P \subseteq C$, $Q \subseteq C$, $P \cap Q = \emptyset$, and $C_{n}^{\alpha}$, $n = 1, \ldots, t - 1$. We have

1. \( P \cup Q\langle C_{n}^{\alpha}\rangle = P\langle C_{n}^{\alpha}\rangle \cup Q\langle C_{n}^{\alpha}\rangle = \mathbb{Y}_{n} \),

2. \( P \cup Q\langle C_{n}^{\alpha}\rangle = P\langle C_{n}^{\alpha}\rangle \cup \mathbb{W}_{n} \),

where $\mathbb{Y}_{n} = \{ x \in P\langle C_{n}^{\alpha}\rangle : l_{p,q}(x) > n \}$ and $\mathbb{W}_{n} = \{ x \in Bn_{p}\langle C_{n}^{\alpha}\rangle : l_{p,q}(x) \leq n \}$.

Proof 4. It is similar to the proof of Proposition 2. □

Corollary 1. For each $\mathbb{V}_{n}$, $\mathbb{Z}_{n}$ in Proposition 2 and $\mathbb{X}_{n}$, $\mathbb{W}_{n}$ in Proposition 3, $n = 2, \ldots, t$, we have

\begin{align*}
\mathbb{V}_{n} &= \mathbb{W}_{n-1} \\
\mathbb{Z}_{n} &= \mathbb{X}_{n-1}
\end{align*}

Proof 5. $P \cup Q\langle C_{n}^{\alpha}\rangle = U - P \cup Q\langle C_{n-1}^{\alpha}\rangle \iff P\langle C_{n}^{\alpha}\rangle - \mathbb{Y}_{n} = U - P\langle C_{n-1}^{\alpha}\rangle \cap \mathbb{W}_{n-1} \iff P\langle C_{n}^{\alpha}\rangle = \mathbb{Y}_{n} = P\langle C_{n}^{\alpha}\rangle - \mathbb{W}_{n-1} \iff \mathbb{Y}_{n} = \mathbb{W}_{n-1}$. Similarly, $\mathbb{Z}_{n} = \mathbb{X}_{n-1}$ holds. □

Example 3 (Continuation of Example 2). $P = \{a_{1}, a_{2}\}$ and $Q = \{a_{2}, a_{3}\}$. Lower and upper approximations of upward and downward unions of decision classes with respect to $P$ are calculated as follows:

\begin{align*}
P\langle C_{2}^{\alpha}\rangle &= \{x_{2}, x_{3}, x_{4}, x_{5}, x_{6}\}, \quad P\langle C_{2}^{\alpha}\rangle = U \\
P\langle C_{2}^{\alpha}\rangle &= \{x_{10}\}, \quad P\langle C_{2}^{\alpha}\rangle = \{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{9}, x_{10}\} \\
P\langle C_{1}^{\alpha}\rangle &= \emptyset, \quad P\langle C_{1}^{\alpha}\rangle = \{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{9}\} \\
P\langle C_{2}^{\alpha}\rangle &= \{x_{4}, x_{5}, x_{6}\}, \quad P\langle C_{2}^{\alpha}\rangle = \{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}, x_{9}\} \\
P\langle C_{1}^{\alpha}\rangle &= P\langle C_{1}^{\alpha}\rangle = P\langle C_{1}^{\alpha}\rangle = U
\end{align*}

Let $P = P \cup Q$. Lower and upper approximations of upward and downward unions of decision classes with respect to $P$ are calculated as follows:

It is obvious that $P\langle C_{2}^{\alpha}\rangle = P\langle C_{2}^{\alpha}\rangle = P\langle C_{2}^{\alpha}\rangle = P\langle C_{2}^{\alpha}\rangle = U$. By Propositions 2 and 3, and Corollary 1, lower and upper approximations of $C_{2}^{\alpha}$, $C_{2}^{\alpha}$, $C_{1}^{\alpha}$, and $C_{2}^{\alpha}$ are updated as follows.

\begin{align*}
\mathbb{Y}_{2} &= \{x_{1}, x_{3}\}, \quad P\langle C_{2}^{\alpha}\rangle = P\langle C_{2}^{\alpha}\rangle = \mathbb{Y}_{2} \setminus \{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}, x_{9}, x_{10}\} \\
\mathbb{Y}_{3} &= \{x_{1}, x_{2}, x_{3}, x_{10}\}, \quad P\langle C_{2}^{\alpha}\rangle = P\langle C_{2}^{\alpha}\rangle = \mathbb{Y}_{3} = \{x_{7}, x_{9}, x_{10}\} \\
\mathbb{X}_{1} &= \{x_{1}, x_{9}\}, \quad P\langle C_{1}^{\alpha}\rangle = P\langle C_{1}^{\alpha}\rangle = \mathbb{X}_{1} = \{x_{1}, x_{4}, x_{5}, x_{6}\} \\
\mathbb{X}_{2} &= \{x_{7}, x_{9}\}, \quad P\langle C_{2}^{\alpha}\rangle = P\langle C_{2}^{\alpha}\rangle = \mathbb{X}_{2} = \{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{8}\} \\
\mathbb{Z}_{4} &= \{x_{3}, x_{5}\}, \quad P\langle C_{2}^{\alpha}\rangle = P\langle C_{2}^{\alpha}\rangle = \mathbb{Z}_{4} = \{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}, x_{9}\} \\
\mathbb{Z}_{3} &= \{x_{3}, x_{3}\}, \quad P\langle C_{2}^{\alpha}\rangle = P\langle C_{2}^{\alpha}\rangle = \mathbb{Z}_{3} = \{x_{7}, x_{9}, x_{10}\} \\
\mathbb{W}_{1} &= \{x_{1}, x_{3}\}, \quad P\langle C_{1}^{\alpha}\rangle = P\langle C_{1}^{\alpha}\rangle = \mathbb{W}_{1} = \{x_{1}, x_{3}\} \\
\mathbb{W}_{2} &= \{x_{1}, x_{2}, x_{3}, x_{10}\}, \quad P\langle C_{2}^{\alpha}\rangle = P\langle C_{2}^{\alpha}\rangle = \mathbb{W}_{2} = \{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}, x_{9}\} \\
\mathbb{W}_{3} &= \{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}, x_{9}\}, \quad P\langle C_{2}^{\alpha}\rangle = P\langle C_{2}^{\alpha}\rangle = \mathbb{W}_{3} = \{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}, x_{9}\} \\
\mathbb{W}_{4} &= \{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}, x_{9}\}, \quad P\langle C_{2}^{\alpha}\rangle = P\langle C_{2}^{\alpha}\rangle = \mathbb{W}_{4} = \{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{7}, x_{8}, x_{9}\}
\end{align*}

Based on above-mentioned Lemma 1, Propositions 2 and 3, and Corollary 1, an algorithm for updating approximations of DRSA is presented when some new attributes are added into the information system (See Algorithm 1).
Algorithm 1.

Algorithm 1: An incremental algorithm for updating approximations of DRSA when some new attributes are added into the information system

Input:
(1) P-dominating sets and P-dominated sets of all objects on the universe U.
(2) All approximations of upward and downward unions of decision classes with respect to P.
(3) A collection of some new attributes added Q.

Output:
All approximations of upward and downward unions of decision classes with respect to P \cup Q.

begin
1. Compute Q-dominating sets and Q-dominated sets of all objects on the universe U;
2. for i = 1, \ldots, |U| do
   3. \[ D_{P,Q}(x) \leftarrow D_P(x) \cap D_Q(x); \]
   4. \[ D_{\neg P,Q}(x) \leftarrow D_{\neg P}(x) \cup D_Q(x); \]
   end
3. for n = 2, \ldots, m do
   4. Compute \( \mathcal{Y}_n \) and \( \mathcal{Z}_n \);
   5. \[ P \cup \bigcap \{C_{Q,E}^k \mid E \in \mathcal{Y}_n \}; \]
   6. \[ \bigcup \{C_{Q,E}^k \mid E \in \mathcal{Z}_n \}; \]
   end
4. for n = 1, \ldots, m - 1 do
   5. \[ P \cup \bigcup \{C_{Q,E}^k \mid E \in \mathcal{Y}_{n+1} \}; \]
   6. \[ \bigcup \{C_{Q,E}^k \mid E \in \mathcal{Z}_{n+1} \}; \]
   end
5. Return \( P \cup \bigcap \{C_{Q,E}^k \}, P \cup \bigcup \{C_{Q,E}^k \}, P \cup \bigcup \{C_{Q,E}^k \} \) and \( P \cup \bigcup \{C_{Q,E}^k \}; \)
end

4.2. Deletion of some attributes

Let Q be a collection of attributes deleted from the set of attributes P, P \setminus Q \neq \emptyset. For \((x,y) \in U\), if \(x \in D_P, y \notin D_P\), then \(x \in D_{P,Q}\), where \(D_{P,Q}\) is the dominance relation with respect to the set of attributes \(P \setminus Q\). However, if \((x,y) \notin D_P\), there may exist the case that \((x,y) \in D_{P,Q}\). Thus, Lemma 2 is introduced to calculate \(D_{P,Q}(x)\) and \(D_{\neg P,Q}(x)\) by the modification of \(D_P(x)\) and \(D_{\neg P}(x)\).

Lemma 2. Let \(Q \subset P\) and \(P \setminus Q \neq \emptyset\) and \(x_i, x_j \in U\). We have

(1) \(D_{P,Q}(x) = D_P(x) \cup A_i^+\); (2) \(D_{P,Q}(x) = D_{\neg P}(x) \cup A_i^+\);

where

\[ A_i^+ = \{ x_j \in U - D_P(x) : \phi_{i,j}^P = |P| - |Q| \} \quad \text{and} \quad A_i^- = \{ x_j \in U - D_{\neg P}(x) : \phi_{i,j}^P = |P| - |Q| \} \]

Proof. Since \(D_{P,Q}(x) = \{ x_j \in U : \phi_{i,j}^P = |P| - |Q| \} \) and \(x_i \in U : \phi_{i,j}^P = |P| - |Q| \) \(= D_P(x) \cup \{ x_j \in U - D_P(x) : \phi_{i,j}^P = |P| - |Q| \}, \)
le
\[ A_i^+ = \{ x_j \in U - D_P(x) : \phi_{i,j}^P = |P| - |Q| \} \]
we have \(D_{P,Q}(x) = D_P(x) \cup A_i^+\).

Similarly, (2) holds. □

Example 4 (Continuation of Example 1). Let \(P = \{a_1, a_2, a_3\} \), \(Q = \{a_2, a_3\} \), and \(P = P \setminus Q\). P-dominating sets and P-dominated sets are obtained by updating P-dominating sets and P-dominated sets on the universe U as follows.

The dominance matrix \(R^P\) with respect to the set of attributes P is

\[
R^P = R^Q + R^Q + R^Q = \begin{bmatrix}
3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 1 \\
3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 2 & 3 & 2 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\
2 & 1 & 1 & 3 & 3 & 3 & 3 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 2 & 2 & 3 & 3 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 3 & 3 & 2 & 2 & 2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 2 & 2 & 2 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 2 & 2 & 3 & 3 & 3 & 1 & 2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 2 & 3 & 3 & 3 & 2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 2 & 2 & 2 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2 \\
\end{bmatrix}
\]

P-dominating sets and P-dominated sets of all objects on the universe U are calculated according to the dominance matrix \(R^P\) as follows.

\[ D_P(x_1) = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ D_P(x_2) = \{ x_1, x_2 \} \]
\[ D_P(x_3) = \{ x_1, x_3 \} \]
\[ D_P(x_4) = \{ x_1, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ D_P(x_5) = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ D_P(x_6) = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ D_P(x_7) = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ D_P(x_8) = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ D_P(x_9) = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ D_P(x_{10}) = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]

\[ A_1^+ = \emptyset, A_2^+ = \{ x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ A_3^+ = \emptyset, A_4^+ = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ A_5^+ = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ A_6^+ = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ A_7^+ = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ A_8^+ = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ A_9^+ = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]
\[ A_{10}^+ = \{ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \} \]

P-dominating sets and P-dominated sets of all objects on the universe U are as follows.
It is similar to the proof of Corollary 1.

Let \( Q \subset P \). For each \( Cl_n^P \), \( n = 2, \ldots, t \), we have

1. \( P - Q ( Cl_n^P ) = P ( Cl_n^P ) - F_n \);
2. \( P - Q ( Cl_n^P ) = P ( Cl_n^P ) \cup G_n \),

where \( G_n = \{ x \in P ( Cl_{n-1}^P ) : LPQ(x) < n \} \). 

\[ \frac{1}{G_n = \{ x \in P ( Cl_{n-1}^P ) : LPQ(x) < n \} \}. \]

Proof 7.

1. Since \( P - Q ( Cl_n^P ) = \{ x \in U : LPQ(x) \subseteq Cl_n^P \} = P ( Cl_n^P ) - \{ x \in P ( Cl_n^P ) : LPQ(x) < n \}, \) let \( F_n = \{ x \in P ( Cl_n^P ) : LPQ(x) < n \}, \) then we have \( P - Q ( Cl_n^P ) = P ( Cl_n^P ) U F_n \).
2. Since \( P - Q ( Cl_n^P ) = \{ x \in U : LPQ(x) \cap Cl_n^P \neq \emptyset \} = P ( Cl_n^P ) U \{ x \in U - P ( Cl_n^P ) : LPQ(x) \geq n \}, \) let \( G_n = \{ x \in P ( Cl_{n-1}^P ) : LPQ(x) \geq n \}, \) then we have \( P - Q ( Cl_n^P ) = P ( Cl_n^P ) U G_n \).

Proposition 5. Let \( Q \subset P \). For each \( Cl_n^P \), \( n = 1, \ldots, t - 1 \), we have

1. \( P - Q ( Cl_n^P ) = P ( Cl_n^P ) - J_n \);
2. \( P - Q ( Cl_n^P ) = P ( Cl_n^P ) \cup K_n \),

where \( J_n = \{ x \in P ( Cl_n^P ) : LPQ(x) < n \} \) and \( K_n = \{ x \in P ( Cl_{n-1}^P ) : LPQ(x) \geq n \} \).

Proof 8. It is similar to the proof of Proposition 4. □

Corollary 2. For each \( F_n \), \( G_n \) in Proposition 4 and \( J_n \) and \( K_n \) in Proposition 5, \( n = 2, \ldots, t \), there are

\[ F_n = \delta_{n-1} \]
\[ G_n = \beta_{n-1} \]

Proof 9. It is similar to the proof of Corollary 1. □

Example 5 (Continuation of Example 1). Let \( P = \{ a_1, a_2, a_3 \} \) and \( Q = \{ a_2, a_3 \} \). Lower and upper approximations of unions of decision classes with respect to the set of attributes \( P \) are as follows.

- Lower approximations:
  - \( P(\text{Cl}_2^Q) = \{ x_2, x_3, x_4, x_5, x_6, x_8, x_{10} \} \)
  - \( P(\text{Cl}_2^Q) = \{ x_2, x_3, x_4, x_5, x_6, x_8, x_{10} \} \)
  - \( P(\text{Cl}_2^Q) = \{ x_7, x_9, x_{10} \} \)
  - \( P(\text{Cl}_2^Q) = \{ x_1, x_3, x_6 \} \)
  - \( P(\text{Cl}_2^Q) = \{ x_1, x_3, x_6 \} \)
  - \( P(\text{Cl}_2^Q) = \{ x_1, x_3, x_6 \} \)
  - \( P(\text{Cl}_2^Q) = \{ x_1, x_3, x_6 \} \)
  - \( P(\text{Cl}_2^Q) = \{ x_1, x_3, x_6 \} \)

- Upper approximations:
  - \( P(\text{Cl}_2^Q) = \{ x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10} \} \)
  - \( P(\text{Cl}_2^Q) = \{ x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10} \} \)
  - \( P(\text{Cl}_2^Q) = \{ x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10} \} \)
  - \( P(\text{Cl}_2^Q) = \{ x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10} \} \)
  - \( P(\text{Cl}_2^Q) = \{ x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10} \} \)
  - \( P(\text{Cl}_2^Q) = \{ x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10} \} \)

5. Experimental evaluations

Some experiments are conducted to evaluate the performance of the proposed incremental approaches. We select five data sets from the machine learning data repository in Table 2, University of California at Irvine [40] as benchmark databases for performance tests. The incremental algorithms and the non-incremental algorithm are coded by C++ and run on a personal computer with Windows 7 and Inter(R) E5800@ 3.2 GHz and 2.0 GB memory.

The experiments are divided into two classes: experiments on updating approximations of DRSA when some attributes are added and experiments on updating approximations of DRSA when some attributes are deleted.

5.1. Addition of some new attributes

For this case, we make three groups of experiments to evaluate Algorithm 1, comparing the computational time between Algorithm 1 and the non-incremental algorithm and exploring whether the cardinality of the attribute set influences the performance of Algorithm 1.

In the first group of experiments, some attributes are selected randomly from the condition attributes to constitute an experimental attribute set on each of the data sets in Table 2. The cardinalities of experimental attribute sets are listed in Table 3. Based on the experimental attribute set, we compute approximations of DRSA with the non-incremental algorithm and save the results. Then we select randomly an attribute from the rest of condition attributes and add it into the experimental attribute set. Algorithm 1 and the non-incremental algorithm are employed to update approximations of DRSA, respectively. The computational time of two algorithms is shown in Table 3. For the experimental procedures, the remaining two groups of experiments are similar to the first group of experiments.

In the second group of experiments, cardinalities of experimental attribute sets are larger than that of the first group of experiments (see Table 4). Four attributes are selected randomly from the rest of condition attributes and added into the experimental attribute set. The computational time of Algorithm 1 and the non-incremental algorithm is outlined in Table 4.

In the third group of experiments, cardinalities of experimental attribute sets are larger than that of the second group of experiments (see Table 5). Eight attributes are selected randomly from the rest of condition attributes and added into the experimental attribute set. The computational time of Algorithm 1 and the non-incremental algorithm is listed in Table 5.

From Tables 3–5, the computational time of Algorithm 1 and the non-incremental algorithm rises monotonically with the increasing number of added attributes and the increasing cardinality of the attribute set. Algorithm 1 is always faster than the non-incremental algorithm in these three groups of experiments. In addition, to make sure whether and how the cardinality of the experimental attribute set influences the performance of Algorithm 1 on different data sets when numbers of attributes added are the same, the speed-up ratios between Algorithm 1 and the non-incremental algorithm are calculated based on the computational time of the second group of experiments and are shown in Fig. 1. In Fig. 1, the line indicates the trend of speed-up ratio. The x-coordinate pertains to data sets (these five data sets starting from the one with the smallest cardinality of the experimental attribute set). The y-coordinate pertains to speed-up ratio.

From Fig. 1, the trend of speed-up ratios increases when the cardinality of the experimental attribute set enlarges on different data sets. It means that the performance of Algorithm 1 rises with the increasing cardinality of the attribute set. For Robot and Movement-libras, cardinalities of experimental attribute sets are the same. However, for the number of the objects, Robot is larger than Movement-libras. Fig. 1 shows that the speed-up ratio of Movement-libras is higher than that of Robot. It implies that the size of a data set is another factor influencing the performance of Algorithm 1.

5.2. Deletion of some attributes

We also make three groups of experiments on these five data sets in Table 2 to evaluate the performance of Algorithm 2.

In the first group of experiments, we select randomly some attributes from the condition attributes to constitute the experimental attribute set on each of data sets in Table 2. The cardinalities of experimental attribute sets are shown in Table 6. Based on the

<table>
<thead>
<tr>
<th>Table 2</th>
<th>The basic information of data sets.</th>
</tr>
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<tbody>
<tr>
<td>Data Set</td>
<td>Number of objects</td>
</tr>
<tr>
<td>Sonar</td>
<td>208</td>
</tr>
<tr>
<td>SPECTF</td>
<td>267</td>
</tr>
<tr>
<td>Ionosphere</td>
<td>351</td>
</tr>
<tr>
<td>Movement-libras</td>
<td>360</td>
</tr>
<tr>
<td>Robot</td>
<td>463</td>
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</table>

<table>
<thead>
<tr>
<th>Table 3</th>
<th>A comparison of the computational time between Algorithm 1 and the non-incremental algorithm when a new attribute is added.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data set</td>
<td>Number of attributes selected</td>
</tr>
<tr>
<td>Sonar</td>
<td>15</td>
</tr>
<tr>
<td>SPECTF</td>
<td>14</td>
</tr>
<tr>
<td>Ionosphere</td>
<td>7</td>
</tr>
<tr>
<td>Movement-libras</td>
<td>20</td>
</tr>
<tr>
<td>Robot</td>
<td>20</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Table 4</th>
<th>A comparison of the computational time between Algorithm 1 and the non-incremental algorithm when four new attributes are added.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data set</td>
<td>Number of attributes selected</td>
</tr>
<tr>
<td>Sonar</td>
<td>30</td>
</tr>
<tr>
<td>SPECTF</td>
<td>21</td>
</tr>
<tr>
<td>Ionosphere</td>
<td>14</td>
</tr>
<tr>
<td>Movement-libras</td>
<td>45</td>
</tr>
<tr>
<td>Robot</td>
<td>45</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 5</th>
<th>A comparison of the computational time between Algorithm 1 and the non-incremental algorithm when eight new attributes are added.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data set</td>
<td>Number of attributes selected</td>
</tr>
<tr>
<td>Sonar</td>
<td>43</td>
</tr>
<tr>
<td>SPECTF</td>
<td>30</td>
</tr>
<tr>
<td>Ionosphere</td>
<td>21</td>
</tr>
<tr>
<td>Movement-libras</td>
<td>72</td>
</tr>
<tr>
<td>Robot</td>
<td>72</td>
</tr>
</tbody>
</table>
In the second group of experiments, cardinalities of experimental attribute sets are larger than that of the first group of experiments (see Table 6). We select randomly eight attributes from the experimental attribute set and remove them. The computational time of Algorithm 2 and the non-incremental algorithm on these five data sets is shown in Table 7.

In the third group of experiments, cardinalities of experimental attribute sets are larger than that of the second group of experiments (see Table 8). We select randomly eight attributes from the experimental attribute set and remove them. The computational time of Algorithm 2 and the non-incremental algorithm on these five data sets is shown in Table 8.

Table 6
<table>
<thead>
<tr>
<th>Data set</th>
<th>Number of attributes selected</th>
<th>Algorithm 2 (s)</th>
<th>Algorithm 1 (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sonar</td>
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<td>0.1413</td>
<td>0.0121</td>
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<td>SPECTF</td>
<td>20</td>
<td>0.2295</td>
<td>0.0186</td>
</tr>
<tr>
<td>Ionosphere</td>
<td>14</td>
<td>0.3075</td>
<td>0.0331</td>
</tr>
<tr>
<td>Movement-libras</td>
<td>45</td>
<td>1.2674</td>
<td>0.0424</td>
</tr>
<tr>
<td>Robot</td>
<td>45</td>
<td>1.5590</td>
<td>0.0571</td>
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</table>

Table 7
<table>
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<th>Data set</th>
<th>Number of attributes selected</th>
<th>Algorithm 2 (s)</th>
<th>Algorithm 1 (s)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.0348</td>
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<tr>
<td>SPECTF</td>
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<td>0.0562</td>
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<td>1.6150</td>
<td>0.1116</td>
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<tr>
<td>Robot</td>
<td>70</td>
<td>2.1216</td>
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Table 8
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<tbody>
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Incremental updating approximations of rough sets is important to decision-making and data mining in dynamic data environment. Keeping the decision classes unchanged, we analyzed the reasons that cause the variations of approximations in DRSA when some attributes were added into or deleted from an information system in this paper. We realized to update incremental approximations of DRSA without recomputing from scratch when the set of attributes varies. With some numerical examples and the experimental results, we can obtain two conclusions as follows: (1) Incrementally updating approximations of DRSA is feasible and can effectively reduce the computational time, (2) The cardinality of the attribute set and the size of data set influence the performance of the proposed incremental approaches. To handle massive data, we need to further improve our approach in order to reduce the computational time. Our future research work will focus on the development of algorithms for incrementally updating approximations on massive data sets.
Acknowledgments

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References


