Abstract—Exponent-rules are known to describe experimentally how the observables related to incandescent lamps alter with the change in the rated voltage. Pedagogical derivation of the rules for typical observables like the life, lumens, power, etc., in the case of vacuum bulbs is presented. This is achieved by assuming a steady-state operation of the bulb, parameterizing intrinsic properties of tungsten as suitable powers of temperature, and eliminating the temperature between those observables for which the exponent-rules are sought.

Index Terms—Exponent-rules, incandescent vacuum lamps.

I. INTRODUCTION

Incandescent electric bulbs [1] can illuminate not only a room but also the minds of electrical engineering and physics students in a class because of the interplay [2] among the branches of heat, electricity, and optics. This can be corroborated by the sizeable number of papers [1] which have appeared in pedagogical journals and by the topics covered therein. In particular, mention can be made of the historical perspective [3], [15], [16], temperature and color of the filament [4], efficiency and efficacy of the lamp [5], switching time [6], mortality statistics and life of the bulb [1], [7], thermal expansion of the filament [8], etc.

However, an important topic not covered in teaching journals so far is the exponent-rules [9]–[12] for bulbs. These rules are a beautiful set of empirical relations [13] (Table I) among various observables viz life, lumens, lumen per watt, voltage, current, power, and resistance. They serve to specify how the concerned observable changes when the bulb is operated at other than the rated voltage. For example, one of the rules (cf. Table I) telling about the way in which the consumed power alters is

\[
\frac{\text{watts}}{\text{volts}} = \left(\frac{\text{volts}}{\text{volts}}\right)^n; \quad n = 1.58
\]

where the upper case letters refer to the rated voltage and lower case letters to general values. The importance of exponent-rules for students stems from the fact that their theoretical analysis can clarify several concepts of physics and electrical engineering.

The aim of this paper is to give a simple derivation of the exponent-rules and the values of the corresponding exponents for vacuum lamps. This is achieved through the following two principles. 1) As is well known, the filament of incandescent lamp is usually made of tungsten and its temperature will change as soon as the operating voltage is altered. Now, as will become apparent in the sequel, most intrinsic physical properties of the metal tungsten can be parameterized as suitable powers of temperature; and 2) when the cold filament has become sufficiently hot to achieve its state of full brilliance, the input electrical power will go predominantly into the Stefan radiative channel.

In Section II, the basic theoretical expressions for the bulb observables are presented. The derivation of the actual exponent-rules and their numerical illustrations will be taken up in Section III. Finally, Section IV discusses the results.

II. THEORY

A. Notations and Parameterizations

Consider at room temperature \( T_0 \) a filament in the form of metallic wire having length \( L_0 \), radius \( r_0 \), surface area \( A_0 = 2\pi r_0 L_0 \), density \( \rho \), and mass \( M = \pi r_0^2 L_0 \). Suppose to a bulb having this filament an electrical voltage \( V \) is applied; then within a time of the order of 0.1 s the current shoots to its normal value and the temperature increases to a steady value \( T \), typically of the order of 3000 K. Many intrinsic properties of the metal are quite sensitive functions of the temperature and it
will be convenient to parameterize them in a manner described below.

The emissivity \( \varepsilon \) and electrical resistivity \( \rho \) can be parameterized as

\[
\varepsilon = \alpha_1 T^{\beta_1}, \quad (1) \\
\rho = \alpha_2 T^{\beta_2}. \quad (2)
\]

Since the coefficient of linear expansion of tungsten \([14]\) is negligibly small, the change in filament’s length and radius can be ignored, so that the resistance \( R \) varies according to

\[
R = \alpha_3 T^{\beta_3}; \quad \beta_3 = \beta_2. \quad (3)
\]

At high temperatures, thermionic emission of atoms happens and the corresponding Richardson evaporation rate \( J \) per unit area is reasonably approximated by

\[
J = \alpha_4 T^{\beta_4}. \quad (4)
\]

Finally, light emission at incandescence occurs in accordance with Planck’s radiation formula for a grey body. However, when this radiation is detected by the human eye the visual response is maximum at wavelength \( \lambda_m = 555 \) nm. It is known that the corresponding Planck’s energy density will depend on the factor

\[
[\exp(hc/\lambda_m kT) - 1]^{-1} \approx \exp[-hc/\lambda_m kT] = \alpha_5 T^{\beta_5}. \quad (5)
\]

The multiplicative factors \( \alpha_1 \cdots \alpha_5 \) will not be needed in the subsequent work because only the ratios among the observables are of interest. However, the exponents \( \beta_1 \cdots \beta_5 \) will be important as will become apparent in the algebraic formulation/numerical work to follow.

### B. Bulb Observables

It is customary to assume that the input electrical power goes predominantly into Stefan’s radiative channel leading to the steady-state condition

\[
V^2/R = \sigma A_0 T^{\beta_1}. \quad (6)
\]

where \( \sigma \) is the Stefan’s constant. Hence, the power \( P \), voltage \( V \), and current \( I \) vary according to

\[
P = V^2/R = \alpha_6 T^{\beta_6}; \quad \beta_6 = \beta_1 + 4 \quad (7) \\
V = \sigma_1 T^{\beta_7}; \quad \beta_7 = (\beta_2 + \beta_3 + 4)/2 \quad (8) \\
I = V/R = \alpha_8 T^{\beta_8}; \quad \beta_8 = (\beta_1 - \beta_2 + 4)/2 \quad (9)
\]

where use has been made of the parameterizations \((1, 2, 3)\). Next the question of the life \( \tau \) of the bulb will be taken up. Remembering that an evaporation rate \( J \) is happening per unit area over surface area \( A_0 \), the time taken for mass \( M \) to evaporate would be \([7]\)

\[
\tau = M/J A_0 = \rho_0 d_0/2I = \alpha_9 T^{\beta_9} \quad \beta_9 = \beta_4 \quad (10)
\]

with the help of \((4)\). Next, the calculation of the total visible light output \( Q \) (lumens) becomes relevant. Employing the standard Planck’s distribution one can write \([5]\)

\[
Q = \int_{\lambda_0}^{\lambda_f} 683 S(\lambda) \in (\lambda, T; \lambda_0, \lambda_f) d\lambda \\
= \frac{683 \lambda_0^5 \exp(\frac{hc}{\lambda_m kT})}{\lambda_0^5 \exp(\frac{hc}{\lambda_m kT}) - 1} \int_{\lambda_0}^{\lambda_f} S(\lambda) d\lambda \\
= \alpha_{10} T^{\beta_{10}}; \quad \beta_{10} = \beta_2 + \beta_5 \quad (11)
\]

in view of \((1)\) and \((5)\) and the fact that integral over \( S(\lambda) \) is a constant. Finally, the efficacy \( e \) (lumen per watt) of the bulb reads

\[
e = Q/P = \alpha_{11} T^{\beta_{11}}; \quad \beta_{11} = \beta_6 - 4. \quad (12)
\]

The next section will now take up the derivation of, and numerical work on, the exponents.

## III. EXPONENT-RULES

### A. Derivation

The rule which links the life to lumens will be considered first. Eliminating \( T \) between \((10)\) and \((12)\) gives

\[
\tau \alpha_1/Q^a; \quad a = \beta_6/\beta_{10}. \quad (14)
\]

Considering this proportionality at the rated value designated by capital letters, and at general value distinguished by small letters and taking the ratio yields

\[
\frac{\text{life}}{\text{LIFE}} = \left( \frac{\text{LUMENS}}{\text{lumens}} \right)^a
\]

as desired. Similarly all the remaining exponent-rules can be derived via suitable eliminations of temperature between observable-pairs as specified in Table II. It may be remarked that out of the 14 exponents tabulated only four \( (\beta_1, \beta_2, \beta_4, \beta_5) \) are independent.

### B. Numerical Work

The data on emissivity \( \varepsilon \), resistivity \( \rho \), and evaporation rate \( J \) given by Richardson formula for tungsten were taken from CRC Handbook \([14]\) over the temperature range 2100–3400 °C, because the bulbs listed in General Electric Catalog \([13]\) are known to operate in this region. The corresponding \( \alpha \) and \( \beta \) parameters \([\text{cf. (1), (2), and (4)}]\) were obtained by unweighted least square fit. The numerical values of the function \( \exp(-hc/\lambda_m kT) \) were also subjected to similar least square fit procedure to get \( \alpha_5 \) and
The good agreement between the theoretical and experimental exponents [cf. (8)] implies that the corresponding relative change in the temperature \(\delta T/T \approx \beta_2^{-1} \delta V/V\) should not exceed about 4%, e.g., for a bulb operating at \(T = 3000K\) one should have \(\delta T \leq 100K\). This feature may be understood from the fact that, apart from the resistivity, other intrinsic properties of tungsten do not have a strict power law dependence on the temperature. In general, more complicated polynomials and exponentials may occur.

**REFERENCES**


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