A Suboptimal Filter for Continuous-Discrete Linear Systems with Parametric Uncertainties

Vladimir Shin*, Du Yong Kim*, Georgy Shevlyakov†, Kiseon Kim†

*Gwangju Institute of Science and Technology/Department of Mechatronics, Gwangju, Republic of Korea
†Gwangju Institute of Science and Technology/Department of Information and Communications, Gwangju, Republic of Korea

Abstract—We present a novel suboptimal filter addressing estimation problems that arise in continuous-discrete linear systems with parametric uncertainties. The suboptimal state estimate is formed by summing of local Kalman estimates with weights depending only on time instants. In contrast to optimal weights, the suboptimal weights do not depend on current observations, and thus the proposed filter is of a low-complexity and it can easily be implemented in real-time. High accuracy and efficiency of the suboptimal filter is demonstrated on the damper harmonic oscillator motion.

I. INTRODUCTION

We consider a linear system described by the stochastic differential equation

$$\dot{x} = F(t)x + G(t)v, \quad t \geq 0,$$  \hspace{1cm} (1)

where $x \in \mathbb{R}^q$ is the state, $v(t) \in \mathbb{R}^p$ is a zero-mean Gaussian white noise with covariance $E(v(t)v(t)^T) = Q(t)$, $x_0 \sim N(\bar{x}_0, R_0)$.

The suboptimal state estimate is given by the weighted sums

$$\hat{x}_k = H_k(t)x_k + w_k, \quad k = 0, 1, \ldots,$$  \hspace{1cm} (2)

where $y_k = y_k(t) \in \mathbb{R}^m$, $x_k = x_k(t)$, $t_k = k \cdot \Delta t$, and $\{w_k \in \mathbb{R}^p\}$ is a zero-mean white Gaussian sequence, $y_k \sim N(0, R_k(t))$. In addition, it is assumed that the matrices $F_k(t), G_k(t), Q_k(t), R_k(t)$ are known and that the mean $\bar{x}_0(\theta)$ include the unknown parameter $\theta \in \mathbb{R}^p$, which takes only a finite set of values

$$\theta \in \{\theta_1, \ldots, \theta_N\}.$$  \hspace{1cm} (3)

This finite set might be a result of discretizing a continuous parameter space $[1, 2]$. A fundamental problem associated with such systems is estimation of the state $x_k$ from the noisy observations $y_k = \{y_k : 0 \leq t_k < t\}$.

Many methods are available for the adaptation of systems. In structure adaptation, two filters are primarily used for system (1,2) [1]. In the first filter, $\theta$ is treated as a random constant vector such as $\bar{\theta}$, and the suboptimal nonlinear filters (for example, an extended Kalman filter) can be applied to estimate the composite state $\left[\begin{array}{c} x_k^T \\ \theta_k^T \end{array}\right]$, which contains $\theta$ as its component. However, it is difficult to estimate the effect of approximations made in the suboptimal realization of nonlinear filters [3, 4].

The second filter represents the adaptive Kalman filter (AKF), which separates the filtering process $x_k$ from identification of the unknown parameter $\theta$ [1, 3, 4]. In this paper we are interested in such an AKF that constitutes a partitioning of the original nonlinear filter into the bank of much simpler $N$ local filters. This AKF is also referred to as Bayesian multiple model adaptive estimation [6]. The AKF is given by a weighted sum of local Kalman filters. However, the AKF’s weights represent the conditional probabilities $p(\theta|y_i)$ which depend on current observations $y_i$ and it is rather difficult to implement the AKF in real-time for high-dimension of state vector and large number of local Kalman filters.

The objective of the present paper is to give an alternative suboptimal filter (SF). Similarly to the optimal AKF, the SF represents as a weighted sum of local Kalman filters with the weights depending only on time instants and being independent of current observations $y_k$. It gives an opportunity to design a low-complexity SF that can be easily implemented in real-time, especially in high dimension problems.

This paper is organized as follows. In Section 2, we present the AKF to continuous-discrete linear dynamic systems. In Section 3, we propose the SF for that kind of systems. The SF represents a linear combination of local Kalman filters. Each local Kalman filter is fused by the minimum mean-square criterion. In Section 4, the SF is tested in real-life system model. In Section 5, conclusions are made.

II. OPTIMAL ADAPTIVE KALMAN FILTER

The AKF for linear systems with uncertainties was firstly proposed by Magill [5] and later generalized by Lainiotis [2] to form the framework of partitioned algorithms [1]. According to Lainiotis’s partition theorem the optimal mean-square state estimate $\hat{x}_k^{opt}$ of $x_k$ and the corresponding estimation error covariance $P_k^{opt}$ are given by the weighted sums

$$\hat{x}_k^{opt} = \sum_{i=1}^{N} \hat{x}_k^{(i)} \hat{x}_k^{(i)}^T,$$  \hspace{1cm} (4)

$$P_k^{opt} = \sum_{i=1}^{N} \left[ \hat{x}_k^{(i)} - \hat{x}_k^{opt}(\hat{x}_k^{(i)} - \hat{x}_k^{opt})^T \right],$$  \hspace{1cm} (5)

where $\hat{x}_k^{(i)}$ and $P_k^{(i)}$ are the local Kalman estimate and corresponding local error covariance, which are determined by the standard continuous-discrete Kalman filter equations matched to linear system (1), (2) at fixed $\theta = \theta_i$, $i = 1, \ldots, N$[7]. Given $\theta_i$, the weights $\hat{x}_k^{(i)} = p(\theta|y_i)$ in (4), (5) represent a posteriori probabilities of $\theta_i$, which are described by the recursive Bayesian formula [1, 2]. As it was already mentioned above, the AKF can be a very costly algorithm to implement, since it
requires complex calculations of the a posteriori probabilities $p(\theta_j|\hat{\theta}_j^0)$ at each time instant $t_k > 0$.

In this work we develop an alternative SF for system (1)-(3). This filter does not require calculations of $p(\theta_j|\hat{\theta}_j^0)$ at each time instant $t_k$, and as a consequence it makes the state estimate computationally feasible for online usage. The obtained suboptimal filtering algorithm reduces the computational burden and online computational requirements.

### III. A Suboptimal Filter for Continuous-Discrete Systems

Similarly to the optimal AKF, the SF is represented by the weighted sum of the local Kalman estimates $\hat{z}_t^{(i)}$,

$$\hat{z}_t^{\text{sub}} = \sum_{i=1}^{N} c_{i}^{(i)} \hat{z}_t^{(i)} = I_n$$

where in contrast to $\hat{z}_t^{(0)}$ in (4) the new coefficients $c_{i}^{(i)}$ represent $n \times n$ weight matrices depending only on time instant and being determined from the mean-square criterion

$$bJ_t = E[|x_t(\theta) - \hat{z}_t^{\text{sub}}|^2] = \text{tr}[E[(x_t(\theta) - \hat{z}_t^{\text{sub}})(x_t(\theta) - \hat{z}_t^{\text{sub}})^T]] \rightarrow \min_{c_{i}^{(i)}}$$

**Theorem 1.** (i) The weights $c_1^{(i)}, ..., c_N^{(i)}$ are given by the following local algebraic equations

$$\sum_{i=1}^{N} c_{i}^{(i)} [P_t^{(i)} - P_t^{(0)}] = 0, \quad j = 1, ..., N - 1, \quad \sum_{i=1}^{N} c_{i}^{(i)} = I_n$$

(ii) The overall error covariance $P_t^{\text{sub}} = \text{cov}(\hat{z}_t^{\text{sub}}, \hat{z}_t^{\text{sub}})$ is given by

$$P_t^{\text{sub}} = \sum_{i=1}^{N} c_{i}^{(i)} P_t^{(i)} (c_{i}^{(i)})^T + \sum_{i=1}^{N} P_t (\theta_i) L_t^{(h,i)}$$

where $L_t^{(h,i)} = \text{cov}(\hat{z}_t^{(i)}, \hat{z}_t^{(i)})$. Equations (8), (9) depend on the local cross-covariances $P_t^{(h,i)}$, which are given in the theorem.

**Theorem 2.** The local error cross-covariances $P_t^{(h,i)}$ in (9) can be represented as

$$P_t^{(h,i)} = L_{x_{a}d}^{(h,i)} - L_{x_{a}d}^{(h,i)} T + L_{x_{a}d}^{(h)}, \quad i, j, h = 1, ..., N$$

where the second-order moments $L_{x_{a}d}^{(h)} = E[\hat{z}_t^{(h)} (\hat{z}_t^{(h)})^T]$ are determined by the following equations:

$$L_{x_{a}d}^{(i)} = F_t^{(i)} L_{x_{a}d}^{(i)} + L_{x_{a}d}^{(i)} F_t^{(i)^T} + G_t^{(i)} Q_t^{(i)/2} (Q_t^{(i)/2})^T G_t^{(i)}$$

1. **Time update between measurements:**

$$t_{k-1} \leq t \leq t_k$$

$$L_{x_{a}d}^{(i)} = F_t^{(i)} L_{x_{a}d}^{(i)} + L_{x_{a}d}^{(i)} F_t^{(i)^T} + G_t^{(i)} Q_t^{(i)/2} (Q_t^{(i)/2})^T G_t^{(i)}$$

2. **Measurement update("jump") at time $t = t_k$:**

$$\left\{ \begin{array}{l}
L_{x_{a}d}^{(i)} = F_t^{(i)} L_{x_{a}d}^{(i)} + L_{x_{a}d}^{(i)} F_t^{(i)^T},
\quad \text{at } t_{k-1} \leq t \leq t_k
\end{array} \right.$$
IV. JOINT DETECTION-ESTIMATION: THE DAMPER HARMONIC OSCILLATOR MOTION

A comparative experimental analysis of the optimal AKF and SF is demonstrated. System model of the harmonic oscillator is considered in [7, p.104]:

$$\mathbf{x}_t = \begin{bmatrix} 0 & 1 \\ -2\alpha & 0 \end{bmatrix} \mathbf{x}_{t-1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{v}_t, \quad 0 \leq t \leq 1, \quad (10)$$

where $\mathbf{x}_t = [x_{1,t}, x_{2,t}]^T$ and $x_{1,t}$ is position, and $x_{2,t}$ is its velocity, $\mathbf{v}_t$ is a zero-mean white Gaussian noise with known intensity $q$, $x_0 \sim N(\mathbf{0}, P_0)$. Position is observed with uncertainty. Then the observation model is written as

$$\mathbf{y}_k = \mathbf{H} \mathbf{x}_k + \mathbf{w}_k, \quad \mathbf{w}_k \sim N(\mathbf{0}, \mathbf{Q}),$$

$$\mathbf{H} = \begin{bmatrix} 1 & \Delta t \\ 0 & k \end{bmatrix}, \quad \Delta t = 0.01,$$  \quad (11)

where $\mathbf{w}_k \sim N(\mathbf{0}, \mathbf{Q})$, and the unknown parameter $\theta$ takes only two values, i.e.,

$$\begin{cases} \theta_1 = 1, & p(\theta_1) = 0.5 \\ \theta_2 = 0, & p(\theta_2) = 0.5 \end{cases} \quad (12)$$

This represents the observation model which takes two sensor modes with $\theta_1 = 1$ (signal-present) and $\theta_2 = 0$ (signal-absent). We compare two filters: the optimal AKF

$$\hat{x}_t^{(s)} = c_t^{(1)} \hat{x}_t(\theta_1) + c_t^{(2)} \hat{x}_t(\theta_2), \quad c_t^{(i)} = p(\theta_i|k), \quad i = 1, 2$$

and the SF

$$\hat{x}_t^{(s)} = c_t^{(1)} \hat{x}_t(\theta_1) + c_t^{(2)} \hat{x}_t(\theta_2)$$

In this case $N=2$, and the solution of linear equations (8) coincides with the Bar-Shalom-Campo formulas for the optimal combination of two correlated estimates [8],

$$c_t^{(1)} = (P_t^{(11)} - P_t^{(12)})^{-1} c_t^{(1)} (P_t^{(11)} - P_t^{(12)})^{-1},$$

$$c_t^{(2)} = (P_t^{(11)} - P_t^{(12)})^{-1} c_t^{(2)} (P_t^{(11)} - P_t^{(12)})^{-1},$$

$$P_t^{(ij)}(k) = E[(x_t - \hat{x}_t^{(i)})(x_t - \hat{x}_t^{(j)})^T], \quad i, j = 1, 2. \quad (13)$$

The performance of the SF is expressed in the terms of computation load and loss in estimation accuracy with respect to the AKF. The model parameters, noises statistics, and initial conditions are set to

$$\begin{align*}
\omega_0^2 &= 0.64, \quad \alpha &= 0.16, \quad q = 1.0, \quad r = 0.1, \\
\mathbf{x}_0 &= \begin{bmatrix} 0.0 \\ 0 \end{bmatrix}^T, \quad P_0 = \text{diag}[2.0, 1.0].
\end{align*}$$

Two cases were considered: in the first case, $\theta_1 = 1$ is the true parameter value in (11); in the second case, $\theta_2 = 0$ is the true parameter value. Figs. 1, 2 show the time histories of the mean-square errors (MSEs)

$$\begin{align*}
P_{\text{SF}}^{(i)} &= E[(x_t^{(i)} - \hat{x}_t^{(i)})^2], \\
P_{\text{SF}}^{(s)} &= E[(x_t^{(i)} - \hat{x}_t^{(s)})^2], \quad i = 1, 2.
\end{align*}$$

This is due to the fact that the suboptimal weights $c_t^{(i)}$ are pre-computed. This provides the best balance between the computational efficiency and the desired estimation accuracy.

V. CONCLUSION

In this paper, we have designed a new suboptimal filter for linear continuous-discrete dynamic systems with unknown parameters. This filter represents a linear combination of local Kalman filters with weights depending only on time instances. Each local Kalman filter is fused by the minimum mean-square criterion. The proposed low-complexity SF has a parallel structure. Since it takes a finite number of values (3), the local simulations were performed using a computer with the following specification: Intel Pentium 4 CPU 2.8GHz 512Mb RAM. The computation time for evaluation of the suboptimal estimate $\hat{x}_t^{(s)}$ is 3.8 times less than for optimal estimate $\hat{x}_t^{(o)}$. This provides the best balance between the computational efficiency and the desired estimation accuracy.
Kalman estimates $\bar{x}_{i}^{(0)}$ in (6) are separated for $i = 1, ..., N$. Therefore, the SF can be evaluated in parallel. Simulation results demonstrate high accuracy of the designed SF.

REFERENCES