Contraction Principle in Complex Valued $G$-Metric Spaces

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Abstract
In this paper, we introduce the notion of complex valued $G$-metric spaces and prove contraction principle in the newly spaces.

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1 Introduction and Preliminaries

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity. Recently, Mustafa and Sims [6], [7] have shown that most of the results concerning Dhage’s $D$-metric spaces ([2]-[5]) are invalid, therefore they introduced an improved version of the generalized metric space structure which they called $G$-metric spaces.

In 2006, Mustafa and Sims [7] introduced the concept of $G$-metric spaces as follows:

**Definition 1.1.** Let $X$ be a non-empty set and $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function $G$ is called a generalized metric or a $G$-metric on $X$ and the pair $(X, G)$ is called a $G$-metric space.

The idea of complex metric space was initiated by Azam et al. [1] to explore the idea of complex valued normed spaces and complex valued Hilbert spaces.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\prec$ on $\mathbb{C}$ as follows:

$z_1 \prec z_2$ if and only if $\text{Re}(z_1) \leq \text{Re}(z_2)$ and $\text{Im}(z_1) \leq \text{Im}(z_2)$.

It follows that $z_1 \prec z_2$ if one of the following conditions is satisfied:

(C1) $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,
(C2) $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,
(C3) $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$,
(C4) $\text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$.

In particular, we will write $z_1 \prec z_2$ if $z_1 \neq z_2$ and one of (C2), (C3) and (C4) is satisfied and we will write $z_1 \sim z_2$ if only (C4) is satisfied.
Remark 1.2. We obtained that the following statements hold:
1. If $a, b \in \mathbb{R}$ with $a \leq b$, then $az \prec bz$ for all $z \in \mathbb{C}$.
2. If $0 \not\preceq z_1 \not\preceq z_2$, then $|z_1| < |z_2|$.
3. If $z_1 \not\preceq z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Now we introduce the notion of complex valued $G$-metric space akin to the notion of complex valued metric spaces [1] as follows:

**Definition 1.3.** Let $X$ be a non-empty set and $G : X \times X \times X \to \mathbb{C}$ be a function satisfying

$$(CG1)\ G(x, y, z) = 0 \text{ if } x = y = z,$$

$$(CG2)\ 0 \prec G(x, x, y) \text{ for all } x, y \in X \text{ with } x \neq y,$$

$$(CG3)\ G(x, x, y) \preceq G(x, y, z) \text{ for all } x, y, z \in X \text{ with } y \neq z,$$

$$(CG4)\ G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots \text{ (symmetry in all three variables)},$$

$$(CG5)\ G(x, y, z) \preceq G(x, a, a) + G(a, y, z) \text{ for all } x, y, z, a \in X \text{ (rectangle inequality)}.$$  

Then the function $G$ is called a complex valued generalized metric or a complex valued $G$-metric on $X$ and the pair $(X, G)$ is called a complex valued $G$-metric space.

From $(CG5)$, the following proposition follow easily.

**Proposition 1.4.** Let $(X, G)$ be a complex valued $G$-metric space. Then for any $x, y, z \in X$

1. $G(x, y, z) \preceq G(x, x, y) + G(x, x, z),$
2. $G(x, y, y) \preceq 2G(y, x, y).$

**Definition 1.5.** Let $(X, G)$ be a complex valued $G$-metric space, let $(x_n)$ be a sequence in $X$, we say that $(x_n)$ is complex valued $G$-convergent to $x$ if for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) \prec c$ for all $n, m \geq k$. We refer to $x$ as the limit of the sequence $(x_n)$ and we write $x_n \to x$.

**Definition 1.6.** Let $(X, G)$ be a complex valued $G$-metric space. Then a sequence $(x_n)$ is called complex valued $G$-Cauchy if for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) \prec c$ for all $n, m, l \geq k$.

**Definition 1.7.** A complex valued $G$-metric space $(X, G)$ is said to be complex valued $G$-complete if every complex valued $G$-Cauchy sequence is complex valued $G$-convergent in $(X, G)$.

A point $x \in X$ is called interior point of a set $A \subset X$, whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B_G(x, r) = \{y \in X : G(x, y, y) \prec r\} \subset A.$$
A point \( x \in X \) is called limit point of a subset \( A \) of \( X \) whenever there exists \( 0 < r \in \mathbb{C} \),
\[
B_G(x, r) \cap (A/X) \neq \emptyset.
\]
\( A \) is called open whenever each element of \( A \) is an interior point of \( A \). A subset \( B \) of \( X \) is called closed whenever each limit point of \( B \) belongs to \( B \).

**Definition 1.8.** Let \((X, G)\) and \((X', G')\) be two complex valued \( G \)-metric spaces. Then a function \( f : X \to X' \) is complex valued \( G \)-continuous at a point \( x_0 \in X \) if \( f^{-1}(B_{G'}(fx_0, r)) \in \tau(G) \) for all \( r > 0 \). We say \( f \) is complex valued \( G \)-continuous if it complex valued \( G \)-continuous at all points of \( X \), that is, continuous as a function from \( X \) with \( \tau(G) \)-topology to \( X' \) with \( \tau(G') \)-topology.

Since complex valued \( G \)-metric topologies are metric topologies we have

**Proposition 1.9.** Let \((X, G)\) and \((X', G')\) be two complex valued \( G \)-metric spaces. Then a function \( f : X \to X' \) is complex valued \( G \)-continuous at a point \( x \in X \) if and only if it is complex valued \( G \)-sequentially continuous at \( x \), that is, whenever \( \{x_n\} \) is complex valued \( G \)-convergent to \( x \), we have \( fx_n \) is complex valued \( G \)-convergent to \( fx \).

### 2 Main Results

Now, we need the following propositions.

**Proposition 2.1.** Let \((X, G)\) be a complex valued \( G \)-metric space and \( \{x_n\} \) be a sequence in \( X \). Then \( \{x_n\} \) is complex valued \( G \)-convergent to \( x \) if and only if \( |G(x, x_n, x_m)| \to 0 \) as \( n, m \to \infty \).

**Proof.** Suppose that \( \{x_n\} \) is complex valued \( G \)-convergent to \( x \). For a given real number \( \epsilon > 0 \), let
\[
c = \frac{\epsilon}{\sqrt{2}} + i \frac{\epsilon}{\sqrt{2}}.
\]
Then \( 0 < c \in \mathbb{C} \) and there is a natural number \( k \) such that \( G(x, x_n, x_m) \prec c \) for all \( n, m \geq k \). Therefore, \( |G(x, x_n, x_m)| < |c| = \epsilon \) for all \( n, m \geq k \). It follows that \( |G(x, x_n, x_m)| \to 0 \) as \( n, m \to \infty \).

Conversely, suppose that \( |G(x, x_n, x_m)| \to 0 \) as \( n, m \to \infty \). Then given \( c \in \mathbb{C} \) with \( 0 < c \), there exists a real number \( \delta > 0 \) such that for \( z \in \mathbb{C} \)
\[
|z| < \delta \quad \text{implies} \quad z \prec c.
\]
For this \( \delta \), there is a natural number \( k \) such that \( |G(x, x_n, x_m)| < \delta \) for all \( n, m \geq k \). This means that \( G(x, x_n, x_m) \prec c \) for all \( n, m \geq k \). Hence \( \{x_n\} \) is complex valued \( G \)-convergent to \( x \). \( \square \)
From Propositions 1.4 and 2.1, the following proposition follows easily.

**Proposition 2.2.** Let \((X, G)\) be complex valued \(G\)-metric space, then for a sequence \(\{x_n\}\) in \(X\) and point \(x \in X\), the following are equivalent:

1. \(\{x_n\}\) is complex valued \(G\)-convergent to \(x\).
2. \(|G(x_n, x_n, x)| \to 0\) as \(n \to \infty\).
3. \(|G(x_n, x, x)| \to 0\) as \(n \to \infty\).
4. \(|G(x_m, x_n, x)| \to 0\) as \(m, n \to \infty\).

**Proposition 2.3.** Let \((X, G)\) be a complex valued \(G\)-metric space and \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is complex valued \(G\)-Cauchy sequence if and only if \(|G(x_n, x_m, x_l)| \to 0\) as \(n, m, l \to \infty\).

**Proof.** Suppose that \(\{x_n\}\) is complex valued \(G\)-Cauchy sequence. For a given real number \(\epsilon > 0\), let 
\[
c = \frac{\epsilon}{\sqrt{2}} + i \frac{\epsilon}{\sqrt{2}}.
\]
Then \(0 \prec c \in \mathbb{C}\) and there is a natural number \(k\) such that \(G(x_n, x_m, x_l) \prec c\) for all \(n, m, l \geq k\). Therefore, \(|G(x_n, x_m, x_l)| < |c| = \epsilon\) for all \(n, m, l \geq k\). It follows that \(|G(x_n, x_m, x_l)| \to 0\) as \(n, m, l \to \infty\).

Conversely, suppose that \(|G(x_n, x_m, x_l)| \to 0\) as \(n, m, l \to \infty\). Then given \(c \in \mathbb{C}\) with \(0 \prec c\), there exists a real number \(\delta > 0\) such that for \(z \in \mathbb{C}\)
\[
|z| < \delta \quad \text{implies} \quad z \prec c.
\]
For this \(\delta\), there is a natural number \(k\) such that \(|G(x_n, x_m, x_l)| < \delta\) for all \(n, m, l \geq k\). This means that \(G(x_n, x_m, x_l) \prec c\) for all \(n, m, l \geq k\). Hence \(\{x_n\}\) is complex valued \(G\)-Cauchy sequence.

**Proposition 2.4.** Let \((X, G)\) be a complex valued \(G\)-metric spaces. Then the function \(G(x, y, z)\) is jointly continuous in all three of its variables.

**Proof.** Suppose \(\{x_k\}, \{y_m\}\) and \(\{z_n\}\) are complex valued \(G\)-convergent to \(x, y\) and \(z\), respectively. Then, by (CG5) we have
\[
G(x, y, z) \preceq G(y, y_m, y_m) + G(y_m, x, z),
\]
\[
G(z, x, y_m) \preceq G(z, x_k, x_k) + G(x_k, y_m, z)
\]
and
\[
G(z, x_k, y_m) \preceq G(z, z_n, z_n) + G(z_n, y_m, x_k),
\]
so
\[
G(z, y, z) - G(x_k, y_m, z_n) \preceq G(y, y_m, y_m) + G(x, x_k, x_k) + G(z, z_n, z_n).
\]
Similarly, we have
\[ G(x_k, y_m, z_n) - G(x, y, z) \lesssim G(x_k, x, x) + G(y_m, y, y) + G(z_n, z, z). \]

Therefore by Proposition 1.4, we have
\[ |G(x_k, y_m, z_n) - G(x, y, z)| \leq 2|G(x_k, x, x) + G(y_m, y, y) + G(z_n, z, z)| \]
and hence \( |G(x_k, y_m, z_n) - G(x, y, z)| \to 0 \) as \( k, m, n \to \infty \). By Proposition 1.9, the conclusion holds.

Next, we prove contraction principle in complex valued \( G \)-metric spaces as follows:

**Theorem 2.5.** Let \( (X, G) \) be a complete complex valued \( G \)-metric space. Let \( T : X \to X \) be a contraction mappings on \( X \), i.e.,
\[ G(Tx, Ty, Tz) \lesssim kG(x, y, z) \] (2.1)
for all \( x, y, z \in X \), where \( k \in [0, 1) \). Then \( T \) has a unique fixed point.

**Proof.** Suppose that \( T \) satisfies condition (2.1). Let \( x_0 \in X \) be an arbitrary point, and define the sequence \( \{ x_n \} \) by \( x_n = T^n x_0 \). Then by (2.1), we have
\[ G(x_n, x_{n+1}, x_{n+1}) \lesssim kG(x_{n-1}, x_n, x_n). \] (2.2)
Again by (2.1), we have
\[ G(x_{n-1}, x_n, x_n) \lesssim kG(x_{n-2}, x_{n-1}, x_{n-1}). \]
Then from (2.2), we have
\[ G(x_n, x_{n+1}, x_{n+1}) \lesssim k^2G(x_{n-2}, x_{n-1}, x_{n-1}). \]
Continuing in the same way, we have
\[ G(x_n, x_{n+1}, x_{n+1}) \lesssim k^nG(x_0, x_1, x_1). \] (2.3)
Then, for all \( n, m \in \mathbb{N} \) with \( n < m \), we have by repeated use of \( (CG5) \) and (2.3) that
\[
G(x_n, x_m, x_m) \lesssim G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\
+ G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\
\lesssim (k^n + k^{n+1} + k^{n+2} + \cdots + k^{m-1})G(x_0, x_1, x_1) \\
\lesssim \frac{k^n}{1-k}G(x_0, x_1, x_1).
\]
Therefore, we have
\[ |G(x_n, x_m, x_m)| \leq \frac{k^n}{1-k} |G(x_0, x_1, x_1)|. \]

Since \( k \in [0, 1) \) if we take limits as \( n \to \infty \), then \( \frac{k^n}{1-k} |G(x_0, x_1, x_1)| \to 0 \), i.e., \( |G(x_n, x_m, x_m)| \to 0 \). For \( n, m, l \in \mathbb{N} \), From Proposition 1.4, we obtain
\[ G(x_n, x_m, x_l) \preceq G(x_n, x_m, x_m) + G(x_l, x_m, x_m). \]

Therefore,
\[ |G(x_n, x_m, x_l)| \leq |G(x_n, x_m, x_m)| + |G(x_l, x_m, x_m)|. \]
Taking limit as \( n, m, l \to \infty \), we get \( |G(x_n, x_m, x_l)| \to 0 \). So by Proposition 2.3, \( \{x_n\} \) is complex valued \( G \)-Cauchy sequence. By completeness of \( (X, G) \), there exists \( z \in X \) such that \( \{x_n\} \) is complex valued \( G \)-convergent to \( z \).

Next we prove that \( Tz = z \). Assume on the contrary that \( Tz \neq z \). Then by (2.1)
\[ G(x_{n+1}, Tz, Tz) \preceq kG(x_n, z, z) \]
and hence
\[ |G(x_{n+1}, Tz, Tz)| \leq k|G(x_n, z, z)|. \]
Taking the limit as \( n \to \infty \). By Proposition 2.4, \( G \) is continuous on its variables, we have
\[ |G(z, Tz, Tz)| \leq k|G(z, z, z)|, \]
which is a contradiction since \( k \in [0, 1) \). Thus \( Tz = z \).

Finally, to prove uniqueness, suppose that \( w (\neq z) \) is such that \( Tw = w \). Then by (2.1),
\[ G(z, w, w) = G(Tz, Tw, Tw) \preceq kG(z, w, w). \]
Therefore,
\[ |G(z, w, w)| \leq k|G(z, w, w)|. \]
Since \( k \in [0, 1) \), we have \( |G(z, w, w)| \leq 0 \). Therefore, we have \( z = w \) and thus \( z \) is a unique fixed point of \( T \). This completes the proof.

Example 2.6. Let \( X = [-1, 1] \) and \( G : X \times X \times X \to \mathbb{C} \) be complex valued \( G \)-metric space defined as follows:
\[ G(x, y, z) = |x - y| + |y - z| + |z - x| \]
for all \( x, y, z \in X \). Then \( (X, G) \) is complex valued \( G \)-metric space. Define \( T : X \to X \) as \( Tx = \frac{x}{2} \). Then \( T \) satisfy \( G(Tx, Ty, Tz) \preceq kG(x, y, z) \) holds for all \( x, y, z \in X \), where \( \frac{1}{2} \leq k < 1 \). Hence \( x = 0 \) is the unique fixed point in \( X \).

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References


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