Effect of rain on removal of a gaseous pollutant and two different particulate matters from the atmosphere of a city

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Received 27 December 2006; received in revised form 19 October 2007; accepted 23 October 2007

Abstract

A nonlinear five-dimensional mathematical model is proposed and analyzed to study the removal of a gaseous pollutant and two different particulate matters by rain from the atmosphere of a city. The atmosphere, during rain, is assumed to consist of five interacting phases namely, the raindrops phase, the gaseous pollutant phase, its absorbed phase and the phases of two different particulate matters, one being formed by the gaseous pollutant. We assume that the gaseous pollutant is removed from the atmosphere by the processes of absorption while the two particulate matters are removed only by the process of impaction with different removal rates. By analyzing the model, it is shown that under appropriate conditions, these pollutants can be removed from the atmosphere and their equilibrium levels, remaining in the atmosphere, would depend mainly upon the rates of emission of pollutants, growth rate of raindrops, the rate of raindrops falling on the ground, etc. It is found that if the rates of conversion of gaseous pollutant into the particulate matter and rainfall are very large, then the gaseous pollutants would be removed completely from the atmosphere.

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Keywords: Nonlinear model; Gaseous pollutants; Particulate matters; Precipitation scavenging; Stability

1. Introduction

Air pollution is one of the most challenging problems which our cities face today as their atmospheres are getting highly polluted due to discharges of gaseous and particulate matters from various household, industrial and vehicular sources. Gases like SO\textsubscript{2} and NO\textsubscript{2} released from industries, power plants etc., when reach high into the atmosphere and combine with moisture, form acid rain which is very harmful to ecosystems. Larger particulate matters, ranging from 2.5 microns to 100 microns in diameter, usually from smoke and dust released by industrial processes, construction works, road traffic, etc. pollute the atmosphere almost continuously. The smaller particulate matters less than 2.5 microns in diameter come from the combustion of fossil fuels and cause acute changes in lung function, respiratory and
heart diseases, etc. Thus, the removal of pollutants from the atmosphere is an important problem to be solved. Nature helps us in this endeavour by providing a removal mechanism for pollutants in the form of rain where atmospheric gases are absorbed and particulate matters are trapped in raindrops falling on the ground.

In experimental studies, it has been shown that the pollutants are removed from the atmosphere by precipitation [1, 4,5,9,17,19–22]. In particular, Sharma et al. [22] measured the concentration of suspended particulate matters in Kanpur city, U.P., India and found considerable decrease in their concentrations during and after monsoon. Davies [4] studied the removal of SO₂ by precipitation in an industrial area of Sheffield, U.K. and found significant reduction in its concentration after rain. Similar results have been reported by Pandey et al. [17] for Varanasi, India, see also, Ravindra et al. [21] etc.

Some investigations have also been conducted to study the dynamics of removal of pollutants by rain or fog [2,3,5,6,8,10–15,18,23,28]. Engelmann [5], one of the pioneers in this field, has calculated the effect of precipitation scavenging on removal of pollutants from the atmosphere. Hales [8] presented some fundamentals for the analysis of precipitation scavenging emphasizing the importance of reversible phenomena. The process of trace gases scavenging is reversible in nature and the phenomenon of absorption and desorption may cause a redistribution of pollutants in the atmosphere. Slinn [28] has analytically considered the redistribution of gas plume caused by reversible washout and presented solutions in some cases. Pandis and Seinfeld [18] have studied the interactions between equilibration process and wet or dry deposition. Three cases have been considered to study interaction and deposition processes. The first case has been related to a gas phase species which can be reversibly transferred to aerosol phase. In the second case, the gas phase species in presence of droplets of liquid water (fog) has been transferred reversibly to the aqueous phase. In the third case, two gases reacted to give a volatile aerosol phase. They have found interesting relationship in all the three cases and, in particular, found in the second case (i.e. in the fog episode), the deposition of gaseous species increased by as much as three times in comparison to the dry case.

It is pointed out here that the removal of pollutants by rain is a nonlinear process as it involves nonlinear interactions of various phases in the atmosphere [14,16,24]. In the atmosphere, during rain, the absorption/impaction of pollutants is proportional to the number density of raindrops as well as the cumulative concentrations of pollutants representing the nonlinear phenomenon. Shukla et al. [24] presented a nonlinear mathematical model for the removal of gaseous air pollutants by rain to see the effect of precipitation on the equilibrium levels of pollutants in the atmosphere.

In view of the above, in this paper we propose a five-dimensional nonlinear mathematical model to study the removal of a gaseous pollutant and two different particulate matters (one being formed by the gaseous pollutant) of different sizes from the atmosphere by precipitation. One of the aims of this investigation is to model and analyze the situations studied by Pandis and Seinfeld [18] related to gas phase transfer to aerosol phase and their depletion by rain. The model is analyzed using the stability theory of nonlinear differential equations. A numerical study is also performed to see the role of key parameters on the removal process.

2. Model

We consider the polluted atmosphere of a city where rain is taking place. We assume that five interacting phases (the raindrops, the gaseous pollutants, two different particulate matters, one being formed by the gaseous pollutant and the phase of gaseous pollutant absorbed in raindrops) exist in the atmosphere of a city.

It is pointed out here that during high intensity of rain, the contact time of gaseous phase with raindrops phase is very short before falling on the ground and therefore it may be reasonable to assume that the interaction between these phases is governed by the simple law of mass action. A similar situation also arises in the case of interaction of particulate matters with raindrops phase. It is assumed here that the gaseous pollutant is removed by rainfall as well as by other causes such as effect of gravity, interactions with plant leaves, buildings, etc. However, the removal of particulate matters takes place due to the processes of impaction by raindrops, gravity etc. Further, it is possible that a fraction of gaseous pollutant may re-enter into the atmosphere by a reversible process and some recycling phenomenon may occur which should be considered in the modeling process.

Let \( C_r(t) \) be the number density of raindrops in the atmosphere, \( C(t) \), \( C_{p1}(t) \) and \( C_{p2}(t) \) be the concentrations of a gaseous pollutant, a particulate matter formed by gaseous pollutant and a different particulate matter emitted directly into the atmosphere, respectively. It is assumed that the density \( C_r \) of raindrops may deplete naturally as well as by interaction with the gaseous pollutant which is assumed to be proportional to the raindrops density \( C_r \) as well as the concentration of gaseous pollutants (i.e. \( C_r C \)), \( q \) is the growth rate of raindrops assumed to be a constant (intensity
of rain), \( r_0 \) is the natural deposition rate coefficient of the density of raindrops, \( r \) is the removal rate coefficient of the density of raindrops due to interaction with \( C \). The term \( rC_r \) represents the evaporation of raindrops due to heated gaseous pollutant from its source during emission, where \( r \) may be a function of temperature but it is assumed to be a constant in our study.

Thus, using the above notations and considerations, the dynamics of raindrops density \( C_r \) is assumed to be governed by the following equation,

\[
\frac{dC_r}{dr} = q - r_0C_r - r C_r \tag{1}
\]

with \( C_r(0) \geq 0 \).

To write the other equations of the model, it is assumed that \( Q_0 \) and \( Q \) are respectively the constant emission rates of a gaseous pollutant and the other particulate matter emitted directly from an external source, with their natural depletion (dry deposition) rates \( \delta_0 \) and \( \delta_2 \) \( C_p2 \) respectively. We consider that the gaseous pollutant is converted into the particulate matter (\( C_{p1} \)) by the rate \( \delta \). Further, the absorption/impaction rates of these pollutants are assumed to be proportional to the number density of raindrops as well as the concentrations of respective pollutants (i.e. \( \alpha_1 C_r \), \( \alpha_1 C_{p1} C_r \) and \( \alpha_2 C_{p2} C_r \)). The gaseous pollutant in the absorbed phase may be removed by the rate \( k C_a \) and a fraction of it (i.e. \( \theta \kappa C_a \)) may re-enter into the atmosphere by reversible reaction. It is also assumed that removal of the gaseous pollutant in the absorbed phase is proportional to its concentration in the absorbed phase and the number density of raindrops (i.e. \( \nu C_r C_a \)) and a fraction of it (i.e. \( \pi \nu C_r C_a \)) may also re-enter into the atmosphere again by a reversible process due to which the concentration of the gaseous pollutant in the atmosphere increases. The constants \( 0 \leq \theta, \pi \leq 1 \) are the reversible rate coefficients.

In view of the above, the dynamics of the variables in these phases can be written by the following system of differential equations,

\[
\frac{dC}{dr} = Q_0 - (\delta_0 + \delta) C - \alpha C C_r + \theta k C_a + \pi \nu C_r C_a \tag{2}
\]

\[
\frac{dC_{p1}}{dr} = \delta C - \delta_1 C_{p1} - \alpha_1 C_{p1} C_r \tag{3}
\]

\[
\frac{dC_{p2}}{dr} = Q - \delta_2 C_{p2} - \alpha_2 C_{p2} C_r \tag{4}
\]

\[
\frac{dC_a}{dr} = \alpha C C_r - k C_a - \nu C_r C_a \tag{5}
\]

\( C(0) \geq 0 \), \( C_{p1}(0) \geq 0 \), \( C_{p2}(0) \geq 0 \), \( C_a(0) \geq 0 \).

The Eqs. (1)–(5) constitute the proposed model.

In Eqs. (2)–(5), the constants \( \delta_0 \), \( \delta_1 \), \( \delta_2 \) and \( k \) are natural removal rate coefficients of quantities \( C \), \( C_{p1} \), \( C_{p2} \) and \( C_a \) respectively, \( \alpha \) is the removal rate coefficient of \( C \) due to absorption, \( \alpha_1 \) and \( \alpha_2 \) are the removal rate coefficients of \( C_{p1} \) and \( C_{p2} \) respectively due to impaction and \( \nu \) is the removal rate coefficient of the absorbed phase of the gaseous pollutant.

It may be noted here that if the coefficients \( \alpha, \alpha_1, \alpha_2 \) are very large then the growth rates of variables \( C \), \( C_{p1} \) and \( C_{p2} \) in (2)–(4) become negative implying that all pollutants from the atmosphere would be removed. Also for large \( \nu > \alpha \), the growth rate of \( C_a \) becomes negative, then the formation of absorbed phase is very transient and it may not exist.

In the following, we analyze the model (1)–(5) using the stability theory of differential equations. We need bounds of dependent variables involved in the model. For this, we state the region of attraction as follows, without proof [7,25–27].

The set

\[
\Omega = \{(C_r, C, C_{p1}, C_{p2}, C_a) : 0 \leq C_r \leq \frac{q}{r_0}, 0 \leq C + C_a \leq \frac{Q_0}{\delta m}, 0 \leq C_{p1} \leq \delta \frac{Q_0}{\delta_1 \delta m}, 0 \leq C_{p2} \leq \frac{Q}{\delta_2} \}
\]

attracts all solutions originating from the interior of the positive octant, where \( \delta_m = \min\{(\delta_0 + \delta), (1 - \theta)k\} \).
3. Equilibrium and stability analysis

The model has only one nonnegative equilibrium namely $E^*(C^r, C^*, C^p_1, C^p_2, C^a)$ where $C^r, C^*, C^p_1, C^p_2$ and $C^a$ are positive solutions of the following system of algebraic equations,

$$C_r = \frac{q}{r_0 + rC} \quad (6)$$

$$C = \frac{Q_0(k + \nu C_r)}{(\delta_0 + \delta) k + \{(\delta_0 + \delta) v + (1 - \theta)\alpha k \} C_r + (1 - \pi)\alpha v C_r^2} = f(C_r) \quad (7)$$

$$C^p_1 = \frac{\delta f(C_r)}{(\delta_1 + \alpha_1 C_r)} \quad (8)$$

$$C^p_2 = \frac{Q}{(\delta_2 + \alpha_2 C_r)} \quad (9)$$

$$C_a = \frac{\alpha C_r f(C_r)}{k + \nu C_r} \quad (10)$$

To show the existence of $E^*$, from Eqs. (6) and (7), we write,

$$F(C_r) = q - r_0 C_r - r C_r f(C_r). \quad (11)$$

It would be sufficient if we show that $F(C_r) = 0$ has one and only one root.

To prove this, from Eq. (11), we have

$$F(0) = q > 0 \quad \text{and} \quad F\left(\frac{q}{r_0}\right) < 0.$$ 

Also, $F'(C_r) = -[r_0 + r\{C_r f'(C_r) + f(C_r)\}] < 0$, provided the following condition is satisfied.

$$r_0 + r\{C_r f'(C_r) + f(C_r)\} > 0. \quad (12)$$

This condition is feasible for large $r_0$.

Thus, $F(C_r) = 0$ has exactly one root (say $C^*_r$) between 0 and $\frac{q}{r_0}$ under condition (12). Using $C^*_r$, the values of $C^*, C^p_1, C^p_2$ and $C^a$ can be found from Eqs. (7)–(10) respectively.

Now we check the characteristics of equilibrium values of various variables with respect to parameter $q$, the growth rate of raindrops.

3.1. Variation of $C^*_r$ with $q$

Differentiating Eq. (6) with respect to $q$ and using Assumption (12), we get $\frac{\partial C^*_r}{\partial q} > 0$.

Thus $C^*_r$ increases with increase in $q$. Similarly we can show that $\frac{\partial C^*_r}{\partial \delta} < 0$.

3.2. Variation of $C^*$ with $q$

From Eq. (7), we get $\frac{\partial C^*_r}{\partial q} < 0$ and since $\frac{\partial C^*_r}{\partial q} > 0$ it follows that $\frac{\partial C^*_r}{\partial q} < 0$.

Therefore $C^*$ decreases with increase in $q$.

3.3. Variation of $C^*$ with $\delta$

From Eq. (7) it can be checked that $\frac{\partial C^*_r}{\partial \delta} < 0$. This shows that $C^*$ decreases as $\delta$ increases.

3.4. Variation of $C^*_{p1}$ with $q$

From Eq. (8), we have $\frac{\partial C^*_{p1}}{\partial q} < 0$ and since $\frac{\partial C^*_r}{\partial q} > 0$, therefore $\frac{\partial C^*_{p1}}{\partial q} < 0$.

Thus $C^*_{p1}$ decreases with increase in $q$. 

3.5. Variation of \( C^*_p \) with \( q \)

From Eq. (9), it can be easily seen that \( \frac{\partial C^*_p}{\partial q} < 0 \) showing that \( C^*_p \) decreases with increase in \( q \).

Hence, as the growth rate of raindrops \( q \) increases i.e. as the rain intensity increases, the concentrations of pollutants (gaseous as well as particulate matters) decrease.

To study the stability behavior of the equilibrium point, we state the following theorems.

**Theorem 1.** Let the following inequalities hold,
\[
[r C^*_r + (\alpha C^*_r - \pi \nu C^*_a)]^2 < \frac{q}{4 C^*_r} (\delta_0 + \delta + \alpha C^*_r)
\]
\[
\frac{9(\theta k + \pi \nu C^*_r)^2}{(\delta_0 + \delta + \alpha C^*_r)} < (k + \nu C^*_r)^2 \min \left\{ \frac{q}{C^*_r(\alpha C^*_r - \nu C^*_a)^2}, \frac{(\delta_0 + \delta + \alpha C^*_r)}{(\alpha C^*_r)^2} \right\}
\]
then \( E^* \) is locally asymptotically stable. (See Appendix A for proof).

**Theorem 2.** Let the following inequalities hold,
\[
\left[ \frac{q}{r_0} + \alpha C^*_r + \pi \nu \frac{Q_0}{\delta_m} \right]^2 < \frac{q}{4 C^*_r} (\delta_0 + \delta)
\]
\[
\frac{9(\theta k + \pi \nu C^*_r)^2}{(\delta_0 + \delta)} < (k + \nu C^*_r)^2 \min \left\{ \frac{q}{C^*_r(\alpha + \nu \delta_m)^2}, \frac{(\delta_0 + \delta)}{(\alpha C^*_r)^2} \right\}
\]
then \( E^* \) is nonlinearly asymptotically stable. (See Appendix B for proof).

**Remarks.**

(i) It is noted from inequalities (15) and (16) that if \( \delta_0, \delta \) are very large (i.e. \( \delta_0 \to \infty, \delta \to \infty \)) then these inequalities are satisfied easily showing that these parameters have stabilizing effects on the system.

(ii) It is further noted that if the reversible rate coefficients of the gaseous pollutant is small i.e. \( \theta \) and \( \pi \) tend to zero, then the inequality (16) is automatically satisfied indicating the stabilized behavior of the system by reversible processes.

(iii) We also note that if \( r, \alpha \) and \( \nu \) are small then the possibilities of satisfying conditions (13) and (15) are greater showing that removal rate parameters have stabilizing effects on the system.

The above stability theorems and the discussions imply that the concentrations of the pollutants, gaseous as well as particulate matters, in the atmosphere decrease as the growth rate of raindrops increases. Further, if appropriate conditions are satisfied then the gaseous pollutants from the atmosphere would be removed by rain for large conversion rate and for large rainfall rate.

4. Numerical example

It is noted here that our main aim is to show, through a nonlinear model and its qualitative analysis, the effect of rain on the removal of a gaseous pollutant and two particulate matters, one being formed from the gaseous pollutant. It is, therefore, desirable that we must show the existence of equilibrium values of variables of the model as well as the feasibility of stability conditions numerically for a set of parameters. Since the field data for the proposed model is not available for comparison, the units for various variables and parameters are not included in their values used for computation.

Thus, we use the following values of parameters for computation,
\[
q = 10, \ r_0 = 0.2, \ Q_0 = 3.0, \ Q = 1.0, \ r = 0.00003, \ \delta_0 = 0.05, \ \delta = 0.20, \ \delta_1 = 0.25, \ \delta_2 = 0.30 \\
\alpha = 0.70, \ k = 0.40, \ \nu = 0.55, \ \theta = 0.00003, \ \pi = 0.0000002, \ \alpha_1 = 0.70, \ \alpha_2 = 0.60.
\]

The equilibrium values of different variables in \( E^* \) are calculated from (6)–(10) as follows:
\[
C^*_r = 49.9993617, \quad C^*_a = 0.0851075, \quad C^*_{p1} = 0.0004828, \\
C^*_p = 0.0330037, \quad C^*_a = 0.1067656.
\]
The eigenvalues of the matrix in (A3) corresponding to \( E^* \) are obtained as follows,

\[ -35.249553, -30.299617, -0.200000, -35.249639, -27.899565. \]

Since all the eigenvalues corresponding to \( E^* \) are negative, therefore \( E^* \) locally asymptotically stable.

It has been checked that for the above set of parameters, the local as well as nonlinear stability conditions (13)–(16) respectively are satisfied.

5. Numerical simulation and discussion of results

In this section, we present the solutions of the differential equations (1)–(5) by using the Runge–Kutta fourth order method and the values of parameters given in Section 4. Using MAPLE 7.0, the phase plane diagrams of \( C_r-C \) and \( C-C_{p1} \) has been drawn in Figs. 1 and 2 showing the nonlinear behavior of \( E^* \) with respect to these parameters values.
In Figs. 3–6 the variation of concentrations of $C$, $C_{p1}$, $C_{p2}$ and $C_a$ with time $t$ are shown for different values of growth rate of raindrops (i.e. $q = 0, 10, 20$). From these figures, it can be concluded that the concentrations of these pollutants decrease for large $t$ and they decrease further as $q$ increases. These results show that the amount of a particular pollutant removed from the atmosphere is dependent on the duration of rainfall. They also imply that as the intensity of rain $q$ increases, the concentrations of various pollutants decrease. These conclusions are qualitatively in line with the experimental finding of Davies [4], Pandey et al. [17] and Sharma et al. [22]. From Figs. 3–5 it is also noted if there is no rain ($q = 0$) then concentrations of these pollutants would continue to increase in the atmosphere. From Fig. 3, it is further noted that $C$ decreases as $\delta$ increases and for large rainfall $C$ may tend to zero. In Figs. 7–10 the concentrations of $C$, $C_{p1}$, $C_{p2}$ and $C_a$ with time $t$ are depicted for various values of $\alpha$, $\alpha_1$, $\alpha_2$ and $\nu$ respectively keeping other parameters fixed. The results depicted in Fig. 7 shows that the concentration of the gaseous pollutant decreases as the removal rate coefficient $\alpha$ increases showing the role of absorption. Similarly Figs. 8 and 9 show that both the concentrations of particulate matters decrease as the impaction parameters $\alpha_1$ and $\alpha_2$ increase showing the role of impaction. The same result is obvious in Fig. 10 with respect to the parameter $\nu$.

6. Conclusions

The mathematical modeling and study of removal of pollutants (gaseous/particulate matters) from the atmosphere of a city by rain scavenging is an important problem which must be studied as it would enhance our understanding in protecting ancient monuments, buildings and populations from pollutants and toxic gases by using aerial sprays of water or other liquids.
Therefore in this paper, we have proposed and analyzed a nonlinear model for the removal of a gaseous pollutant and two different particulate matters (one being formed by the gaseous pollutant) from the atmosphere of a city by rain. It has been assumed that the removal of gaseous pollutant takes place by the process of absorption by raindrops falling on the ground while the removal of particulate matters by the processes of impaction and entrapment by falling raindrops. It has been shown qualitatively and numerically that when the pollutants are emitted at a constant rate, these pollutants can be washed out from the atmosphere under appropriate conditions and the equilibrium level of these pollutants would depend upon the rates of emissions, the growth rate of raindrops and removal parameters.
It has been noted when there is no rain the concentrations of pollutants in the atmosphere would continuously increase. It is found that the equilibrium values of the gaseous pollutant and that of particulate matters in the atmosphere are much smaller after rain than their corresponding values before rain and if the conversion rate of gaseous pollutant to the particulate matter is very large, it may be completely removed from the atmosphere due to heavy rain. These results have been found to be in line with the experimental observations of Davies [4], Pandey et al. [17] and Sharma et al. [22].

Appendix A

Proof of Theorem 1. To establish the local stability of \( E^* \), let us consider the following positive definite function,

\[
V = \frac{1}{2} \left( k_0 C_r^2 + k_1 C_1^2 + k_2 C_{r11}^2 + k_3 C_{p21}^2 + k_4 C_{a1}^2 \right)
\]

where \( C_r, C_1, C_{r11}, C_{p21} \) and \( C_{a1} \) are small perturbations about \( E^* \) as

\[
C_r = C_r^* + C_{r1}, \quad C = C^* + C_1, \quad C_{r11} = C_{r11}^* + C_{r111}, \quad C_{p21} = C_{p21}^* + C_{p211}, \quad C_{a1} = C_{a1}^* + C_{a11}.
\]
Differentiating (A.1) with respect to 't' we get

\[ \dot{V} = k_0 r C_r \dot{C}_r + k_1 C_1 \dot{C}_1 + k_2 C_{r11} \dot{C}_{r11} + k_3 C_{r21} \dot{C}_{r21} + k_4 C_{a1} \dot{C}_{a1}. \]  

(A.2)

The linearized system of Eqs. (1)–(5) corresponding to \( E^* \) is written as follows,

\[
\begin{bmatrix}
\dot{C}_{r1} \\
\dot{C}_1 \\
\dot{C}_{p11} \\
\dot{C}_{p21} \\
\dot{C}_{a1}
\end{bmatrix}
= 
\begin{bmatrix}
-\frac{q}{C_r} & -r C_r^* & 0 & 0 & 0 \\
-(\alpha C_r^* - \pi \nu C_a^*) & -(\delta_0 + \delta + \alpha C_r^*) & 0 & 0 & 0 \\
-\alpha C_{r1}^* & -\delta & -(\delta_1 + \alpha C_r^*) & 0 & 0 \\
(\alpha C_r^* - \nu C_a^*) & C_r^* & 0 & 0 & -(k + \nu C_r^*) \\
\end{bmatrix}
\begin{bmatrix}
C_{r1} \\
C_1 \\
C_{p11} \\
C_{p21} \\
C_{a1}
\end{bmatrix}.
\]  

(A.3)

Using the above relations, from (A.2), we have

\[
\dot{V} = -\frac{q}{C_r} k_0 r C_r^2 - k_1 (\delta_0 + \delta + \alpha C_r^*) C_r^2 - k_2 (\delta_1 + \alpha_1 C_r^*) C_{r11}^2 - k_3 (\delta_2 + \alpha_2 C_r^*) C_{p21}^2
\]

\[
- k_4 (C_r^*) C_{a1}^2 - [k_0 r C_r^* + k_1 (\alpha C_r^* - \pi \nu C_a^*)] C_{r1} C_r - k_2 \alpha_1 C_{r1}^* C_{p11} - k_3 \alpha_2 C_{p21} C_{r1} C_{p21}
\]

\[
+ k_4 (\alpha C_r^* - \nu C_a^*) C_{r1} C_{a1} + k_2 \delta C_{1} C_{r11} + k_1 (\theta k + \pi \nu C_r^*) C_{r1} C_{a1} + k_4 \alpha C_r^* C_{r1} C_{a1}.
\]

\( \dot{V} \) will be negative definite provided the conditions,

\[
[k_0 r C_r^* + k_1 (\alpha C_r^* - \pi \nu C_a^*)]^2 < \frac{1}{4} \frac{q}{C_r} k_0 k_1 (\delta_0 + \delta + \alpha C_r^*) \]  

(A.4)

\[
k_2 |\alpha_1 C_{p11}^*|^2 < \frac{q}{2C_r^*} k_0 (\delta_0 + \alpha_1 C_r^*) \]  

(A.5)

\[
k_3 |\alpha_2 C_{p21}^*|^2 < \frac{q}{C_r^*} k_0 (\delta_2 + \alpha_2 C_r^*) \]  

(A.6)

\[
k_4 |(\alpha C_r^* - \nu C_a^*)|^2 < \frac{q}{3C_r^*} k_0 (k + \nu C_r^*) \]  

(A.7)
\[
k_2\delta^2 < \frac{1}{2}k_1(\delta_0 + \delta + \alpha C_r^*)(\delta_1 + \alpha_1 C_r^*) \quad (A.8)
\]
\[
[k_1(k \theta + \pi \nu C_r^*)]^2 < \frac{1}{3}k_1k_4(\delta_0 + \delta + \alpha C_r^*)(k + \nu C_r^*) \quad (A.9)
\]
\[
[k_4\alpha C_r^*]^2 < \frac{1}{3}k_1k_4(\delta_0 + \delta + \alpha C_r^*)(k + \nu C_r^*), \quad (A.10)
\]

are satisfied. Now choosing \( k_0 = k_1 = 1, k_2 < \min \left\{ \frac{q}{2C_r^*(\alpha_1 C_r^*)^2}, \frac{(\delta_0 + \delta + \alpha C_r^*)^2}{2\delta^2}, \frac{(\delta_0 + \delta + \alpha C_r^*)(\delta_1 + \alpha_1 C_r^*)}{2\delta^2} \right\} \) and \( k_3 < \frac{q}{4C_r^*(\alpha C_r^*)^2} \) the above inequalities reduce to,
\[
[r C_r^* + (\alpha C_r^* - \pi \nu C_a^*)]^2 < \frac{q}{4C_r^*}(\delta_0 + \delta + \alpha C_r^*) \quad (A.11)
\]
\[
9(\theta k + \pi \nu C_r^* )(\delta_0 + \delta + \alpha C_r^*)^2 < (k + \nu C_r^*)^2 \min \left\{ \frac{q}{C_r^*(\alpha C_r^* - \nu C_a^*)^2}, \frac{(\delta_0 + \delta + \alpha C_r^*)}{(\alpha C_r^*)^2} \right\}. \quad (A.12)
\]

Hence under the above conditions, \( \dot{V} \) will be negative definite showing that \( V \) is a Lyapunov’s function and hence the theorem.

**Appendix B**

**Proof of Theorem 2.** Consider the following positive definite function about \( E^* \),
\[
U = \frac{1}{2} [m_0(C_r - C_r^*)^2 + m_1(C - C^*)^2 + m_2(C_p1 - C_p1^*)^2 + m_3(C_p2 - C_p2^*)^2 + m_4(C_a - C_a^*)^2] \quad (B.1)
\]
and differentiating with respect to \( t \) we get,
\[
\dot{U} = m_0(C_r - C_r^*) \dot{C_r} + m_1(C - C^*) \dot{C} + m_2(C_p1 - C_p1^*) \dot{C}_p1 + m_3(C_p2 - C_p2^*) \dot{C}_p2 + m_4(C_a - C_a^*) \dot{C}_a
\]
\[
\dot{U} = (C_r - C_r^*)m_0 C_r + m_1(C - C^*) \dot{C} + m_2(C_p1 - C_p1^*)(\dot{C}_p1) + m_3(C_p2 - C_p2^*)(\dot{C}_p2) + m_4(C_a - C_a^*)(\dot{C}_a)
\]
After some algebraic manipulations, it can be written as
\[
\dot{U} = -m_1\alpha C_r(C - C^*)^2 - \frac{q}{C_r^*}m_0 (C_r - C_r^*)^2 - m_1(\delta_0 + \delta)(C - C^*)^2
\]
\[
- m_2(\delta_1 + \alpha_1 C_r^*)(C_p1 - C_p1^*)^2 - m_3(\delta_2 + \alpha_2 C_r^*)(C_p2 - C_p2^*)^2 - m_4(k + \nu C_r^*)(C_a - C_a^*)^2
\]
\[
- [m_0r C_r + m_1(\alpha C_r^* - \pi \nu C_a) ](C_r - C_r^*)(C - C^*) - m_2\alpha_1 C_p1(C_r - C_r^*)(C_p1 - C_p1^*)
\]
\[
- m_3\alpha_2 C_p2(C_r - C_r^*)(C_p2 - C_p2^*) + m_2\delta (C - C^*)(C_p1 - C_p1^*)
\]
\[
+ m_4(\alpha C - \nu C_a)(C_r - C_r^*)(C_a - C_a^*)
\]
\[
+ m_1(\theta k + \pi \nu C_r^*)(C - C^*)(C_a^* - C_a^*) + m_4\alpha C_r^*(C - C^*)(C_a^* - C_a^*)
\]

Now, \( \dot{U} \) will be negative definite provided the conditions,
\[
[m_0r C_r + m_1(\alpha C_r^* - \pi \nu C_a) ]^2 < \frac{q}{4C_r^*}m_0(\delta_0 + \delta) \quad (B.2)
\]
\[
m_2[\alpha_1 C_p1]^2 < \frac{q}{2C_r^*}m_0(\delta_1 + \alpha_1 C_r^*) \quad (B.3)
\]
\[
m_3[\alpha_2 C_p2]^2 < \frac{q}{C_r^*}m_0(\delta_2 + \alpha_2 C_r^*) \quad (B.4)
\]
\[
m_4[(\alpha C - \nu C_a)]^2 < \frac{q}{3C_r^*}m_0(k + \nu C_r^*) \quad (B.5)
\]
\[ m_2 \delta^2 < \frac{1}{2} m_1 (\delta_0 + \delta + \alpha C_r)(\delta_1 + \alpha_1 C_r^a) \]  
(B.6)

\[ [m_1 (k \theta + \pi v C_r^a)]^2 < \frac{1}{3} m_1 m_4 (\delta_0 + \delta) (k + v C_r^a) \]  
(B.7)

\[ [m_4 \alpha C_r^a]^2 < \frac{1}{3} m_1 m_4 (\delta_0 + \delta + \alpha C_r) (k + v C_r^a). \]  
(B.8)

After choosing \( m_0 = m_1 = 1, m_2 \) \(< \min \{ \frac{q (\delta_1 + \alpha_1 C_r^a)}{2 C_r^a (\frac{\alpha_1 \delta_0}{\delta_1 \delta_0})}, (\delta_0 + \delta + \alpha C_r) \} \) \( m_3 \) \(< \frac{q (\delta_2 + \alpha_2 C_r^a)}{C_r^a (\frac{\alpha_2 \delta_0}{\delta_2 \delta_0})} \), Eqs. (B.2)–(B.8) reduce to

\[ \left[ \frac{r q}{r_0} + \alpha C_r^a + \pi v \frac{Q_0}{\delta_m} \right]^2 < \frac{q}{4 C_r^a (\delta_0 + \delta)} \]  
(B.9)

\[ \frac{9(\theta k + \pi v C_r^a)^2}{(\delta_0 + \delta)} < (k + v C_r^a)^2 \min \left\{ \frac{q}{C_r^a (\alpha + v)^2 (\frac{Q_0}{\delta_m})}, \frac{(\delta_0 + \delta)}{(\alpha C_r^a)^2} \right\}. \]  
(B.10)

Under conditions (B.9) and (B.10), \( \dot{U} \) will be negative definite showing that \( U \) is a Lyapunov’s function and hence the theorem.

References


