Topologies Associated with Rough Automata

S.P. Tiwari  
Department of Applied Mathematics  
Indian School of Mines  
Dhanbad-826004, India  
sptiwarimaths@gmail.com

Shambhu Sharan  
Department of Applied Mathematics  
Indian School of Mines  
Dhanbad-826004, India  
shambhupuremaths@gmail.com

Abstract: It is shown that the concepts of definite (possible) source and definite (possible) successor give rise to two topologies on the state-set of a rough automaton. Further, we show that the connectivity and separation properties of such rough automaton can be discussed in terms of these topologies.

Keywords: rough automaton; subautomata; source; successor; separability; retrievability.

I. INTRODUCTION

In [3] and [4], Shukla and Srivastava were demonstrated that several known topological concepts and ideas can often be used in automata theory to obtain certain results therein, pertaining particularly to their connectivity and separation properties. In [5] and [6], Srivastava and Tiwari mainly use these ideas for the study of fuzzy automata and intuitionistic fuzzy automata. Following the introduction of rough sets by Pawlak [2], Basu [1] has recently introduced the concept of rough finite state automaton and introduced the concept of product of two rough finite state automata. In [8], Tiwari and Sharan introduced the concept of approximation space, where \( X \) is a nonempty set and \( (X,R) \) is called an approximation space and \( [x]_R \) be the equivalence class of \( x \). Let \( A \subseteq X \) be an approximation space and \( [x]_R \) be the equivalence class of \( x \) under \( R \). Then lower approximation and upper approximation of \( A \subseteq X \) are, respectively, defined to be the sets

\[
A = \{ x \in X : [x]_R \subseteq A \}, \quad \overline{A} = \{ x \in X : [x]_R \cap A \neq \emptyset \}.
\]

For an approximation space \((X,R)\), \( A \subseteq X \) is called a definable set if it is an union of equivalence classes under \( R \) and a pair \((L,U)\) of definable sets is called a rough set in \((X,R)\), if \( L \subseteq U \), and if any equivalence class of \( x \) is a singleton set \([x]\) such that \([x] \in U\), then \([x] \in L\).

Definition 2.3: [1] A rough automaton is a 4-tuple \( M = (Q,R,X,\delta) \), where \( Q \) is a nonempty set (the set of states of \( M \)), \( R \) is a given equivalence relation on \( Q \), \( X \) is a nonempty finite set (the set of inputs) and \( \delta \) is a map \( \delta : Q \times X \rightarrow D \times D \). \( D \) is the collection of all definable sets in the approximation space \((Q,R)\), if \( \delta(q,a) = (D_1,D_2) \), where \( q \in Q \) and \( a \in X \), then \((D_1,D_2)\) is a rough set with \( D_1 = A \) and \( D_2 = \overline{A} \), for some \( A \subseteq Q \).

If \( \delta(q,a) = (D_1,D_2) \) with \( D_1 = A \) and \( D_2 = \overline{A} \), then by the abuse of notation, we identify \( A \) as \( \delta(q,a) \); thus \( D_1 = \delta(q,a) \) and \( D_2 = \overline{\delta(q,a)} \), i.e., \( \delta(q,a) = (\delta(q,a),\overline{\delta(q,a)}) \).

Definition 2.4: [1] For an equivalence class (or block) \( B \) of \( R \) and \( a \in X \), \( \delta(B,a) = (\delta(B,a),\overline{\delta(B,a)}) \) is called block transition, where \( \delta(B,a) = \bigcup \{ \delta(q,a) : q \in B \} \), \( \overline{\delta(B,a)} = \bigcup \{ \overline{\delta(q,a)} : q \in B \} \) and for a definable set \( D \), \( \delta(D,a) = \bigcup \{ \delta(B,a) : B \text{ is a block of } R \text{ and } B \subseteq D \} \).

Let \( X^* \) be the set of all words on \( X \) (i.e., finite strings of elements of \( X \), which form a monoid under concatenation of
strings) including the empty word (which we shall denote by \( e \)).

**Definition 2.5:** [1] Let \( M = (Q, R, X, \delta) \) be a rough automaton. Define \( \delta^* : Q \times X' \rightarrow D \times D \) as follows:

i) \( \forall q \in Q, \delta(q,e) = ([q]_R, [q]_R) \) if \( [q]_R \) is the equivalence class of \( q \) under \( R \), and \( \forall x \in X' \setminus \{ e \} \}, \delta(q,x) = \delta^*([q]_R, x), e \) being the identity of \( X' \}

ii) \( \forall q \in Q, \forall x \in X' \), \( \delta(q,xa) = (\delta(q,x), \delta(q,xa)) \), where \( \delta(q,x) = \delta^*([q]_R, x), a \)}\(\in\{\}, \delta(q,xa) = \delta^*([q]_R, xa)\) = \(\cup\{\delta(B,a) : B \text{ is a block of } R \text{ and } B \subseteq \delta^*(q,x)\}

and \( \delta(q,xa) = \delta^*([q]_R, xa) = \cup\{\delta(B,a) : B \text{ is a block of } R \text{ and } B \subseteq \delta^* (q,x)\}. \)

**Definition 2.6:** [7] For an equivalence class (or block) \( B \) of \( R \) and \( x \in X' \), \( \delta(B,x) = (\delta^*(B,x), \delta(B,x)) \), where \( \delta^*(B,x) = \cup\{\delta^*(q,x) : q \in B\} \), \( \delta(B,x) = \cup\{\delta^*(q,x) : q \in B\} \) and for a definable set \( D \), \( \delta^*(D,x) = \cup\{\delta^*(B,x) : B \text{ is a block of } R \text{ and } B \subseteq D\} \), \( \delta^*(D,x) = \cup\{\delta^*(B,x) : B \text{ is a block of } R \text{ and } B \subseteq D\} \).

In [1], it is shown that the transition function \( \delta \) of a rough automaton \( (Q, R, X, \delta) \) can be recursively extended to a function \( \delta^* : Q \times X' \rightarrow D \times D \) as follows:

i) \( \forall q \in Q, \delta(q,e) = ([q]_R, [q]_R) \) where \( [q]_R \) is the equivalence class of \( q \) under \( R \), and \( \forall x \in X' \setminus \{ e \} \}, \delta(q,x) = \delta^*([q]_R, x), e \) being the identity of \( X' \} \)

ii) \( \forall q \in Q, \forall x \in X' \), \( \delta(q,xy) = (\delta^*(q,x), \delta(q,xy)) \), where \( \delta(q,x) = \delta^*([q]_R, x), \delta(q,xy) = \cup\{\delta^*(B,y) : B \text{ is a block under } R \text{ and } B \subseteq \delta^* (q,x)\}

and \( \delta(q,xy) = \delta^*([q]_R, xy) = \cup\{\delta^*(B,y) : B \text{ is a block under } R \text{ and } B \subseteq \delta^* (q,x)\}. \)

The following concept of rough automata resembles the concept of rough finite state automata, as given in [1], but is actually different in two respects. First, we permit the set of states to be infinite and secondly, we take the input set to be a monoid rather than just a set.

**Definition 2.7:** [7] A rough automaton is a 4-tuple \( M = (Q, R, X, \delta) \), where \( Q \) is a nonempty set (the set of states of \( M \)), \( R \) is a given equivalence relation on \( Q \), \( X \) is a monoid (whose elements are the input symbol) and \( \delta : Q \times X \rightarrow D \times D \), where \( D \) is the class of all definable sets in \( (Q, R) \) such that

i) \( \forall q \in Q, \delta(q,e) = ([q]_R, [q]_R), \) where \( [q]_R \) is the equivalence class of \( q \) under \( R \), and \( \forall x \in X \setminus \{ e \} \}, \delta(q,x) = \delta^*([q]_R, x), e \) being the identity of \( X \}

ii) \( \forall q \in Q \) and \( \forall x, y \in X \), \( \delta(q,xy) = (\delta(q,xy), \delta(q,xy)) \), where \( \delta(q,xy) = \delta^*([q]_R, xy) = \cup\{\delta(B,y) : B \text{ is a block under } R \text{ and } B \subseteq \delta^* (q,xy)\}

and \( \delta(q,xy) = \delta^*([q]_R, xy) = \cup\{\delta(B,y) : B \text{ is a block under } R \text{ and } B \subseteq \delta^* (q,xy)\}. \)

**Definition 2.8:** [1] Let \( (Q, R, X, \delta) \) be a rough automaton and \( A \subseteq Q \). Then the definite successor and possible successor of \( A \) are respectively the sets \( D(A) = \{ p \in Q : p \in \delta(q,x) \text{, for some } (q,x) \in A \times X \} \) and \( P(A) = \{ p \in Q : p \in \delta(q,x) \text{, for some } (q,x) \in A \times X \} \).

**Definition 2.9:** [7] Let \( (Q, R, X, \delta) \) be a rough automaton and \( A \subseteq Q \). Then the definite source and possible source of \( A \) are respectively the sets \( D'(A) = \{ p \in Q : \delta(p,x) \text{, for some } (q,x) \in A \times X \} \) and \( P'(A) = \{ p \in Q : \delta(p,x) \text{, for some } (q,x) \in A \times X \} \).

**Proposition 2.1:** [7] Let \( (Q, R, X, \delta) \) be a rough automaton and \( A, B \subseteq Q \). Then

i) \( D(\phi) = \phi, P(\phi) = \phi \) and \( D'(\phi) = \phi, P'(\phi) = \phi \);

ii) \( A \subseteq D(A) \subseteq P(A) \text{ and } A \subseteq D'(A) \subseteq P'(A) \);

iii) \( D(A \cup B) = D(A) \cup D(B) \text{ and } D'(A \cup B) = D'(A) \cup D'(B) \);

iv) \( P(A \cup B) = P(A) \cup P(B) \text{ and } D'(A \cup B) = D'(A) \cup D'(B) \);

v) \( D(A \cap B) \subseteq D(A) \cap D(B) \text{ and } D'(A \cap B) \subseteq D'(A) \cap D'(B) \);

vi) \( P(A \cap B) \subseteq P(A) \cap P(B) \text{ and } D'(A \cap B) \subseteq P'(A) \cap P'(B) \);

vii) \( D(D(A)) = D(A) \text{ and } D'(D'(A)) = D'(A) \);

viii) \( P(P(A)) = P(A) \text{ and } P'(P'(A)) = P'(A) \).

**Definition 2.10:** [7] A rough automaton \( (Q', R', X, \delta') \) is called definite rough subautomaton (resp. possible rough subautomaton) of a rough automaton \( (Q, R, X, \delta) \) if \( Q' \subseteq Q \), \( D(Q') = Q' \), \( R' = R \mid_{Q'} \) and \( \delta' = \delta \mid_{Q' \times X} \).
Definition 2.11: [7] A definite rough subautomaton (resp. possible rough subautomaton) \((Q', R', X, \delta')\) of \((Q, R, X, \delta)\) is called definite separated subautomaton (resp. possible separated subautomaton) if \(D(Q \setminus Q') \cap Q' = \emptyset\) (resp. \(P(Q \setminus Q') \cap Q' = \emptyset\)).

Definition 2.12: [7] A rough automaton \((Q, R, X, \delta)\) is called

1) definite strongly connected (resp. possible strongly connected) if \(\forall p, q \in Q, p \in D(q)\) (resp. \(p \in P(q)\)).
2) definite connected (resp. possible connected) if \(M\) has no proper definite separated (resp. possible separated) rough subautomaton.
3) definite retrievable (resp. possible retrievable) if \(\forall p, q \in Q, p \in D(q)\), then \(q \in D(p)\) (resp. if \(p \in P(q)\), then \(q \in P(p)\)).

Proposition 2.2: [7] A rough automaton \(M = (Q, R, X, \delta)\) is definite (possible) strongly connected if and only if \(M\) has no proper definite (possible) rough subautomaton.

III. TOPOLOGIES ON THE STATE-SETS OF ROUGH AUTOMATON

We begin by observing that the concepts of definite source (resp. possible source) and definite successor (resp. possible successor) give rise to two topologies on the state-set of rough automaton. Further, we show that the connectivity and separation properties of a rough automaton can be discussed in terms of these topologies. In this section, we consider the case of possible source and possible successor. The same for definite source and definite successor can be given analogously.

Proposition 3.1: Let \((Q, R, X, \delta)\) be a rough automaton.

(i) The possible source and possible successor, viewed as functions \(P: 2^Q \to 2^Q\) and \(P: 2^Q \to 2^Q\), turn out to be Kuratowski closure operators on \(Q\), inducing two topologies on \(Q\) (which we shall respectively denote as \(T(Q)\) and \(T'(Q)\)).

(ii) Both the topologies \(T(Q)\) and \(T'(Q)\) are ‘saturated’ (in the sense that they are closed under arbitrary intersections also).

(iii) The topologies \(T(Q)\) and \(T'(Q)\) are dual in the sense that each \(A \subseteq Q\) is \(T(Q)\)-open if and only if \(A\) is \(T'(Q)\)-closed.

Proof: (i) Follows from Proposition 2.1.

(ii) Let \([A_j : j \in J]\) be a family of \(T(Q)\)-closed subset of \(Q\) and let \(A = \bigcup [A_j : j \in J]\). Now, \(r \in P'(A) \Rightarrow s \in \overline{d(r, y)}\), for some \((s, y) \in A \times X\). As \(s \in A\), \(s \in A_j\), for some \(j \in J\). But \(s \in P'(A_j) = A_j \Rightarrow P'(s) \subseteq P'(P'(A_j)) = P'(A_j)\). Thus \(r \in P'(A_j) \Rightarrow s \subseteq A_j\), whereby \(P'(A) \subseteq A\), with the fact that \(A \subseteq P'(A)\), showing that \(A\) is \(T(Q)\)-closed. The saturatedness of \(T'(Q)\) can be given similarly.

(iii) Let \(A \subseteq Q\) be \(T(Q)\)-open. Then \(P'(Q - A) = Q - A\), whereby \(A = Q - P'(Q - A)\). Now, \(q \in P(A) \Rightarrow q \in \overline{d(p, x)}\), for some \((p, x) \in A \times X\). But \(p \in A \Rightarrow p \in P'(Q - A)\), whereby \(q \in P'(Q - A)\) (since \(q \in \overline{d(p, x)}\)), i.e., \(q \in A\). Thus \(P(A) \subseteq A\), whereby \(P(A) = A\), implying that \(A\) is \(T'(Q)\)-closed. Conversely, let \(A \subseteq Q\) be \(T'(Q)\)-closed. Then \(P(A) = A\). Also, \(q \in P'(Q - A) \Rightarrow p \in \overline{d(p, x)}\), for some \((p, x) \in (Q - A) \times X\). But \(P \in Q - A \Rightarrow p \in P'(Q - A) = P(A) \Rightarrow p \in \overline{d(r, y)}\), \(\forall (r, y) \in A \times X \Rightarrow q \notin A\), whereby \(q \in Q - A\). Thus \(P'(Q - A) \subseteq Q - A\) and so \(P'(Q - A) = Q - A\), implying that \(A\) is \(T(Q)\)-open.

Proposition 3.2: Let \(M = (Q, R, X, \delta)\) and \(N = (Q', R', X, \delta')\) be rough automata. Then

(i) \(N\) is possible rough subautomaton of \(M\) if and only if \(Q'\) is \(T(Q)\)-open.

(ii) \(N\) is a possible separated subautomaton of \(M\) if and only if \(Q'\) is \(T'(Q)\)-clopen (i.e., \(T(Q)\)-open as well as \(T'(Q)\)-closed).

Proof: (i) This follows from Definition 2.10 of possible rough subautomaton and Proposition 3.1.

(ii) \(N\) is a possible separated subautomaton of \(M\) if and only if \(P(Q') = Q'\) and \(P(\overline{Q'}) = \overline{Q'}\) if and only if \(Q'\) and \(\overline{Q'}\) are \(T(Q)\)-open if and only if \(Q'\) is \(T'(Q)\)-open and \(T(Q)\)-closed if and only if \(Q'\) is \(T(Q)\)-clopen.

Before stating the next proposition, we first recall the following from [9].

Definition 3.1: A topological space \((X, T)\) is called \(R_0\) if \(\forall x, y \in X, x \in cl(y) \Rightarrow y \in cl(x)\).

Proposition 3.3: Let \(M = (Q, R, X, \delta)\) be a rough automaton. Then

(i) \(M\) is possible strongly connected if and only if \(T(Q)\) is an indiscrete topology.

(ii) \(M\) is possible connected if and only if \(T(Q)\) is a connected topology.
iii) \( M \) is possible retrievable if and only if \( T(Q) \) is an \( R_0 \)-topology.

Proof: (i) As \( \forall q \in Q, P(q) = Q \). Thus \( T(Q) \) must be an indiscrete and conversely.

(ii) \( M \) is possible connected if and only if \( M \) has no possible separated proper subautomaton if and only if \( Q \) and \( \emptyset \) are the only \( T(Q) \)-clopen subsets of \( Q \) if and only if \( T(Q) \) is a connected topology.

(iii) It follows from the fact that \( \forall q \in Q, P(q) = \{ q \} \) just the closure of \( \{ q \} \) under the topology \( T(Q) \).

**Proposition 3.4:** Let \( M = (Q, R, X, \delta) \) be a rough automaton. Then

\[ \begin{align*}
  & i) \text{ } M \text{ is possible connected if and only if for every possible rough subautomaton } N = (Q', R', X, \delta') \text{ of } M, \exists r \in Q' \text{ and } q \in (Q - A), \text{ with } P(r) \cap P(q) \neq \emptyset. \\
  & ii) \text{ } M \text{ is possible retrievable if and only if for every possible rough subautomaton } N = (Q', R', X, \delta') \text{ of } M, P(Q') = Q'. \\
  & iii) \text{ } M \text{ is possible strongly connected if and only if } M \text{ has no proper possible rough subautomaton.}
\end{align*} \]

Proof: Let \( M \) be possible connected and \( N = (Q', R', X, \delta') \) be any proper possible rough subautomaton of \( M \). Then the topology \( T(Q) \) is connected and \( Q' \) is a proper open subset of \( Q \). Now as \( T(Q) \) is connected topology, \( Q' \) cannot be a closed subset of \( Q \) (as it is already open), i.e., \( P(Q - Q') \neq Q - Q' \), or that \( \exists p \in P(Q - Q') \) such that \( p \neq Q - Q' \). Thus \( \exists p \in P(Q - Q') \) such that \( p \in Q' = P(Q') \), whereby \( P(Q') \cap P(Q - Q') \neq \emptyset \), or that \( \exists r \in Q' \) and \( q \in Q - Q' \), with \( P(r) \cap P(q) \neq \emptyset \). Conversely, let \( N = (Q', R', X, \delta') \) be any proper possible rough subautomaton of \( M \) such that \( \exists r \in Q' \) and \( q \in Q - Q' \), with \( P(r) \cap P(q) \neq \emptyset \). We need to show that the topology \( T(Q) \) is connected. For this it suffices to show that \( Q' \) is not a closed subset of \( Q \), i.e., \( P(Q - Q') \neq Q - Q' \) (as \( Q' \) is a proper open subset of \( Q \)). Now as \( \exists r \in Q' \) and \( q \in Q - Q' \) such that \( P(r) \cap P(q) \neq \emptyset \), \( P(Q') \cap P(Q - Q') \neq \emptyset \). Thus \( \exists p \in P(Q - Q') \) such that \( p \in P(Q') = Q' \), or that \( p \in Q - Q' \). Hence \( P(Q - Q') \neq Q - Q' \).

(ii) Let \( M \) be possible retrievable. Then \( T(Q) \) is an \( R_0 \)-topology. Thus for any open subset \( Q' \) of \( Q \), \( P'(Q') \subseteq Q' \). Hence \( P'(Q') = Q' \). Conversely, let for any possible rough subautomaton \( N = (Q', R', X, \delta') \) of \( M \) such that \( P'(Q') = Q' \). Then as \( Q' \) is \( T(Q) \)-open and contains the closure of its each points, the topology \( T(Q) \) is \( R_0 \). Thus \( M \) is possible retrievable.

(iii) Let \( M \) be possible strongly connected. Then \( T(Q) \) is an indiscrete topology, i.e., \( Q \) has no proper open subsets. Thus \( M \) has no proper possible rough subautomaton. Conversely, let \( M \) has no proper possible rough subautomaton, i.e., \( Q \) has no proper open subsets. Then \( T(Q) \) is an indiscrete topology. Thus \( M \) is possibly strongly connected.

**Proposition 3.5:** Let \( M = (Q, R, X, \delta) \) be a rough automaton. Then the following statements are equivalent:

\[ \begin{align*}
  & i) \text{ } M \text{ is possible strongly connected.} \\
  & ii) \text{ } M \text{ is possible connected and possible retrievable.} \\
  & iii) \text{ } Every possible rough subautomaton of } M \text{ is possible strongly connected.}
\end{align*} \]

Proof: (i) \( \Rightarrow (ii) \) Let \( M \) be possible strongly connected so that the topology \( T(Q) \) is an indiscrete topology. As an indiscrete topology is obviously connected and \( R_0 \), \( M \) is possible connected and possible retrievable.

\[ \begin{align*}
  & (ii) \Rightarrow (iii) : \text{ Let } M \text{ be possible connected and possible retrievable. Then the topology } T(Q) \text{ is possible connected and } R_0. \text{ Also, let } A \text{ be any nonempty } T(Q) \text{-open subset of } Q. \text{ Then as } T(Q) = R_0, A \text{ contains the closure of its each point. Hence } A \text{ is closed also. Thus } A \text{ is } T(Q) \text{-clopen subset of } Q. \text{ But as } T(Q) \text{ is possible connected, } A = Q. \text{ Thus } Q \text{ is its only nonempty open subset, whereby the subspace topology on each nonempty open subset of } Q \text{ is indiscrete. Hence every possible rough subautomaton of } M \text{ is possible strongly connected.}
\end{align*} \]

\[ \begin{align*}
  & (iii) \Rightarrow (i) : \text{ Let every possible rough subautomaton of } M \text{ be possible strongly connected. Then each open subset of } Q \text{ has indiscrete topology. Since, } Q \text{ itself is open, its topology is also indiscrete. Hence } M \text{ is possible strongly connected.}
\end{align*} \]

IV. Conclusions

Chiefly inspired from \([3, 4, 5]\), we tried to introduce the topologies on the state-set of a rough automaton and discussed its some properties in terms of these topologies. We have tried to introduce the topologies by using possible source and possible successor only. While the concept of definite source and definite successor also leads us to two topologies. It will be interesting to see the relationship between topologies introduce by possible and definite source (possible and definite successor).

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