ON INTUITIONISTIC FUZZY ABELIAN SUBGROUPS

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Abstract

The notion of intuitionistic fuzzy abelian subgroup and cyclic intuitionistic fuzzy subgroups are proposed, and then their properties and relations are discussed.

1. Introduction

The notion of fuzzy abelian subgroup was pioneered by Bhattacharya and Mukherjee [8]. They define fuzzy abelian subgroup as follows: Let \( \mu \) be a fuzzy subgroup of \( G \). Let \( H = \{ x \in G \mid \mu(x) = \mu(e) \} \). Then \( \mu \) is a fuzzy abelian if \( H \) is an abelian subgroup of Makamba [9] showed that the above definition was too weak. For example, any fuzzy subgroup \( \mu \) satisfying \( \{ x \in G : \mu(x) = \mu(e) \} = \{ e \} \) is necessarily fuzzy abelian even if \( \text{Supp} \ \mu \) is not abelian. The alternative proposed definition to strengthen the above definition is: Let \( \mu \) be a fuzzy subgroup of \( G \). Then \( \mu \) is fuzzy abelian if \( \mu' \) is...
abelian, for all \( t \in (0, \mu(e)] \). The definition given by Bhattachary and Mukherjee was corrected with: \( \mu \) is fuzzy abelian if and only if \( \text{Supp } \mu \) is abelian. Here in this paper, we introduced the notion of intuitionistic fuzzy abelian subgroup and cyclic intuitionistic fuzzy subgroup and studied their properties.

## 2. Preliminaries

We first recall some definition for the sake of completeness of the topic under study.

**Definition 2.1** [2]. Let \( X \) be a fixed non-empty set. An *Intuitionistic Fuzzy Set (IFS)* \( A \) of \( X \) is an object of the following form \( A = \{(x, \mu_A(x), v_A(x)) : x \in X\} \), where \( \mu_A : X \to [0, 1] \) and \( v_A : X \to [0, 1] \) define the degree of membership and degree of non-membership of the element \( x \in X \), respectively, and for any \( x \in X \), we have \( 0 \leq \mu_A(x) + v_A(x) \leq 1 \).

**Remark 2.2** (i). When \( \mu_A(x) + v_A(x) = 1 \), i.e., when \( v_A(x) = 1 - \mu_A(x) = \mu_A^c(x) \). Then \( A \) is called the *fuzzy set*.

(ii) We use the notation \( A = (\mu_A, v_A) \) to denote the IFS \( A \) of \( X \).

**Definition 2.3** [4]. Let \( G \) be a group. An intuitionistic fuzzy subset (IFS) \( A = (\mu_A, v_A) \) of \( G \) is called an *intuitionistic fuzzy subgroup (IFSG)* of \( G \) if

(i) \( \mu_A(xy) \geq \text{Min}\{\mu_A(x), \mu_A(y)\} \)

(ii) \( \mu_A(x^{-1}) = \mu_A(x) \)

(iii) \( v_A(xy) \leq \text{Max}\{v_A(x), v_A(y)\} \)

(iv) \( v_A(x^{-1}) = v_A(x) \), for all \( x, y \in G \)

or equivalently, \( A \) is an IFSG of \( G \) if and only if

\[
\mu_A(xy^{-1}) \geq \text{Min}\{\mu_A(x), \mu_A(y)\} \text{ and } v_A(xy) \leq \text{Max}\{v_A(x), v_A(y)\}.
\]
Definition 2.4 [4]. An IFSG $A = (\mu_A, \nu_A)$ of a group $G$ is said to be intuitionistic fuzzy normal subgroup of $G$ (in short IFNSG) of $G$ if

(i) $\mu_A(xy) = \mu_A(yx)$

(ii) $\nu_A(xy) = \nu_A(yx)$, for all $x, y \in G$

or equivalently, $A$ is an IFNSG of a group $G$ is normal if and only if

$\mu_A(y^{-1}xy) = \mu_A(x)$ and $\nu_A(y^{-1}xy) = \nu_A(x)$, for all $x, y \in G$.

Definition 2.5 [4]. Let $A$ be an intuitionistic fuzzy set of a universe set $X$. Then $(\alpha, \beta)$-cut of $A$ is a crisp subset $C_{\alpha, \beta}(A)$ of the IFS $A$ is given by

$C_{\alpha, \beta}(A) = \{x : x \in X \text{ such that } \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$,

where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$.

Theorem 2.6 [4, 6]. If $A$ is an IFS of a group $G$, then $A$ is an IFSG (IFNSG) of $G$ if and only if $C_{\alpha, \beta}(A)$ is a subgroup (normal) of group $G$, for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$.

Definition 2.7. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFS’s of $X$ and $Y$, respectively. Then the Cartesian product of $A$ and $B$ is denoted by $A \times B$ and is defined as

$A \times B = \{(x, y), \mu_{A \times B}(x, y), \nu_{A \times B}(x, y)) : x \in X \text{ and } y \in Y\}$,

where

$\mu_{A \times B}(x, y) = \text{Min}\{\mu_A(x), \mu_B(y)\}$ and $\nu_{A \times B}(x, y) = \text{Max}\{\nu_A(x), \nu_B(y)\}$.

Proposition 2.8 [6]. If $A$ and $B$ are two IFS of $X$ and $Y$, respectively, then

$C_{\alpha, \beta}(A \times B) = C_{\alpha, \beta}(A) \times C_{\alpha, \beta}(B)$,

for all $\alpha, \beta \in [0, 1]$ with $0 \leq \alpha + \beta \leq 1$.

Theorem 2.9 [6]. Let $A$ and $B$ be an IFSG (IFNSG) of groups $G_1$ and $G_2$, respectively. Then $A \times B$ is also an IFSG (IFNSG) of group $G_1 \times G_2$. 
Definition 2.10 [5]. Let $X$ and $Y$ be two non-empty sets and $f : X \rightarrow Y$ be a mapping. Let $A$ and $B$ be IFS’s of $X$ and $Y$, respectively. Then the image of $A$ under the map $f$ is denoted by $f(A)$ and is defined as

$$f(A)(y) = (\mu_{f(A)}(y), \nu_{f(A)}(y)),$$

where

$$\mu_{f(A)}(y) = \begin{cases} \text{Max} \{\mu_A(x) : x \in f^{-1}(y)\}, & \text{otherwise} \\ 0, & \nu_{f(A)}(y) = \begin{cases} \text{Min} \{\nu_A(x) : x \in f^{-1}(y)\}, & \text{otherwise} \\ 1, & \end{cases} \end{cases}$$

Also, the pre-image of $B$ under $f$ is denoted by $f^{-1}(B)$ and is defined as

$$f^{-1}(B)(x) = (\mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)),$$

where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$, i.e., $f^{-1}(B)(x) = (\mu_B(f(x)), \nu_B(f(x)))$.

Remark 2.11. Note that for any $x \in X$, we have $\mu_{f(A)}(f(x)) \geq \mu_A(x)$ and $\nu_{f(A)}(f(x)) \leq \nu_A(x)$.

Proposition 2.12 [5]. Let $f : X \rightarrow Y$ be a mapping. Then the following holds:

(i) $f(C_{\alpha, \beta}(A)) \subseteq C_{\alpha, \beta}(f(A))$, $\forall A \in IFS(X)$

(ii) $f^{-1}(C_{\alpha, \beta}(B)) = C_{\alpha, \beta}(f^{-1}(B))$, $\forall B \in IFS(Y)$.

3. Intuitionistic Fuzzy Abelian Subgroups and their Properties

Here we introduce the notion of intuitionistic fuzzy abelian subgroup of a group and study their properties.
**Definition 3.1.** Let $G$ be a group and $A$ be an IFSG of $G$. Let

$$N(A) = \{ a \in G : \mu_A(a^{-1}xa) = \mu_A(x) \text{ and } \nu_A(a^{-1}xa) = \nu_A(x) \text{, for all } x \in G \}.$$ 

Then $N(A)$ is called the intuitionistic fuzzy normalizer of $A$ in $G$.

**Theorem 3.2.** Let $A$ be an IFSG of a group $G$. Then

(i) $N(A)$ is a subgroup of $G$.

(ii) $A$ is an IFNSG of $G$ if and only if $N(A) = G$.

(iii) $A$ is an IFNSG of the group $N(A)$.

**Proof.** (i) Let $a, b \in N(A)$ be any two elements. Then we have

$$\mu_A(a^{-1}xa) = \mu_A(x) \text{ and } \nu_A(a^{-1}xa) = \nu_A(x), \text{ for all } x \in G,$$

$$\mu_A(b^{-1}xb) = \mu_A(y) \text{ and } \nu_A(b^{-1}yb) = \nu_A(y), \text{ for all } y \in G. \tag{ii}$$

Put $y = a^{-1}xa$ in (ii) and using (i), we get

$$\mu_A(b^{-1}a^{-1}xab) = \mu_A(a^{-1}xa) = \mu_A(x)$$

and

$$\nu_A(b^{-1}a^{-1}xab) = \nu_A(a^{-1}xa) = \nu_A(x),$$

i.e., $\mu_A((ab)^{-1}x(ab)) = \mu_A(x)$ and $\nu_A((ab)^{-1}x(ab)) = \nu_A(x)$.

Thus $ab \in N(A)$. Next, change $x$ to $x^{-1}$ in (i), we get

$$\mu_A(a^{-1}x^{-1}a) = \mu_A(x^{-1}) = \mu_A(x) \text{ and } \nu_A(a^{-1}x^{-1}a) = \nu_A(x^{-1}) = \nu_A(x),$$

i.e., $\mu_A((axa^{-1})^{-1}) = \mu_A(axa^{-1}) = \mu_A(x)$ and $\nu_A((axa^{-1})^{-1}) = \nu_A(axa^{-1})$

$= \nu_A(x)$, i.e., $\mu_A((a^{-1})^{-1}x(a^{-1})) = \mu_A(x)$ and $\nu_A((a^{-1})^{-1}x(a^{-1})) = \nu_A(x)$

$\Rightarrow a^{-1} \in N(A)$. Hence $N(A)$ is a subgroup of $G$. 
(ii) Obviously, when \( N(A) = G \), then
\[
\mu_A(a^{-1}xa) = \mu_A(x) \text{ and } \nu_A(a^{-1}xa) = \nu_A(x), \text{ for all } x, a \in G.
\]
Hence \( A \) is an IFNSG of group \( G \).

Conversely, when \( A \) is an IFNSG of group \( G \), then
\[
\mu_A(a^{-1}xa) = \mu_A(x) \text{ and } \nu_A(a^{-1}xa) = \nu_A(x), \text{ for all } x, a \in G,
\]
i.e., the set
\[
\{ a \in G : \mu_A(a^{-1}xa) = \mu_A(x) \text{ and } \nu_A(a^{-1}xa) = \nu_A(x), \text{ for all } x \in G \} = G,
\]
i.e., \( N(A) = G \).

(iii) Let \( a, b \in N(A) \) be any two elements. Then
\[
\mu_A(a^{-1}xa) = \mu_A(x) \text{ and } \nu_A(a^{-1}xa) = \nu_A(x), \text{ for all } x \in G.
\]
Putting \( x = ab \), we get
\[
\mu_A(ab) = \mu_A(a^{-1}aba) = \mu_A(ba) \text{ and } \nu_A(ab) = \nu_A(a^{-1}aba) = \nu_A(ba).
\]
Hence \( A \) is an IFNSG of \( N(A) \).

**Definition 3.3.** Let \( G \) be a group and \( A \) be an IFSG of \( G \). Let
\[
C(A) = \{ a \in G : \mu_A([a, x]) = \mu_A(e) \text{ and } \nu_A([a, x]) = \nu_A(e), \text{ for all } x \in G \}.
\]
Then \( C(A) \) is called the intuitionistic fuzzy centralizer of \( A \) in \( G \), where \([x, y]\) is the commutator of the two elements \( x \) and \( y \) in \( G \), i.e., \([x, y] = x^{-1}y^{-1}xy\).

**Theorem 3.4.** Let \( A \) be an IFSG of a group \( G \). Then

(i) \( C(A) \) is a subgroup of \( G \).

(ii) \( C(A) \) is a normal subgroup of \( N(A) \).
Proof. (i) Clearly, \( C(A) \neq \emptyset \), for \( e \in C(A) \). Let \( a, b \in C(A) \). Then 
\[
\mu_A([a, x]) = \mu_A(e) \quad \text{and} \quad \nu_A([a, x]) = \nu_A(e) \quad \text{and} \quad \mu_A([b, y]) = \mu_A(e) \quad \text{and} \quad \nu_A([b, y]) = \nu_A(e)
\]
hold, for all \( x, y \in G \), i.e.,
\[
\mu_A(a^{-1}x^{-1}ax) = \mu_A(e) \quad \text{and} \quad \nu_A(a^{-1}x^{-1}ax) = \nu_A(e), \quad (*)
\]
\[
\mu_A(b^{-1}y^{-1}by) = \mu_A(e) \quad \text{and} \quad \nu_A(b^{-1}y^{-1}by) = \nu_A(e). \quad (**) \]

Putting \( y = a^{-1}za \) in (**) we get
\[
\mu_A(b^{-1}a^{-1}z^{-1}aba^{-1}za) = \mu_A(e) \quad \text{and} \quad \nu_A(b^{-1}a^{-1}z^{-1}aba^{-1}za) = \nu_A(e)
\]
\[
\Rightarrow \mu_A((ab)^{-1}z^{-1}(ab)z)(z^{-1}a^{-1}za) = \mu_A(e) \quad \text{and}
\]
\[
\nu_A((ab)^{-1}z^{-1}(ab)z)(z^{-1}a^{-1}za)) = \nu_A(e)
\]
\[
\Rightarrow \mu_A((ab)^{-1}z^{-1}(ab)z) = \mu_A(e) \quad \text{and} \quad \nu_A((ab)^{-1}z^{-1}(ab)z) = \nu_A(e)
\]
(\text{using } (*) )
\[
\Rightarrow ab \in C(A).
\]

Also, from (*), we have
\[
\mu_A(e) = \mu_A(a^{-1}x^{-1}ax) = \mu_A((a^{-1}x^{-1}ax)^{-1}) = \mu_A(x^{-1}a^{-1}xa),
\]
i.e., \( \mu_A(x^{-1}a^{-1}xa) = \mu_A(e) \). Similarly, we have \( \nu_A(x^{-1}a^{-1}xa) = \nu_A(e) \).

Putting \( x = ta^{-1} \), we get
\[
\mu_A(at^{-1}a^{-1}ta^{-1}a) = \mu_A(at^{-1}a^{-1}t) = \mu_A(e)
\]
and
\[
\nu_A(at^{-1}a^{-1}ta^{-1}a) = \nu_A(at^{-1}a^{-1}t) = \nu_A(e).
\]

Therefore, \( a^{-1} \in C(A) \). Hence \( C(A) \) is a subgroup of \( G \).
(ii) Let \( a \in C(A) \) and \( b \in N(A) \) be any elements. We show that \( b^{-1}ab \in C(A) \). Now
\[
\mu_A(a^{-1}x^{-1}ax) = \mu_A(e) \quad \text{and} \quad \nu_A(a^{-1}x^{-1}ax) = \nu_A(e) \quad \text{hold, for all } x \in G, \quad (\ast)
\]
\[
\mu_A(b^{-1}yb) = \mu_A(y) \quad \text{and} \quad \nu_A(b^{-1}yb) = \nu_A(y) \quad \text{hold, for all } y \in G. \quad (\ast\ast)
\]

Put \( y = a^{-1}x^{-1}ax \) in (\ast\ast) and using (\ast), we get
\[
\mu_A(b^{-1}a^{-1}x^{-1}axb) = \mu_A(a^{-1}x^{-1}ax) = \mu_A(e)
\]
and
\[
\nu_A(b^{-1}a^{-1}x^{-1}axb) = \nu_A(a^{-1}x^{-1}ax) = \nu_A(e).
\]

Again putting \( x = bz b^{-1} \) above, we get
\[
\mu_A(b^{-1}a^{-1}bz^{-1}b^{-1}abz b^{-1}b) = \mu_A(e)
\]
and
\[
\nu_A(b^{-1}a^{-1}bz^{-1}b^{-1}abz b^{-1}b) = \nu_A(e),
\]
i.e.,
\[
\mu_A(b^{-1}a^{-1}bz^{-1}b^{-1}abz) = \mu_A(e) \quad \text{and} \quad \nu_A(b^{-1}a^{-1}bz^{-1}b^{-1}abz) = \nu_A(e),
\]
i.e.,
\[
\mu_A((b^{-1}ab)^{-1}z^{-1}(b^{-1}ab)z) = \mu_A(e)
\]
and
\[
\nu_A((b^{-1}ab)^{-1}z^{-1}(b^{-1}ab)z) = \nu_A(e),
\]
i.e., \( b^{-1}ab \in C(A) \). Hence \( C(A) \) is a normal subgroup of \( N(A) \).

**Proposition 3.5.** Let \( A \) be an IFNSG of a group \( G \). Let
\[
N = \{ a \in G : \mu_A(a) = \mu_A(e) \quad \text{and} \quad \nu_A(a) = \nu_A(e) \}.
\]
Then \( N \subseteq C(A) \).
Proof. Let $A$ be an IFNSG of group $G$. Therefore, $\mu_A(y^{-1}xy) = \mu_A(x)$ and $\nu_A(y^{-1}xy) = \nu_A(x)$, for all $x, y \in G$.

Let $a \in N$. Then $\mu_A(a) = \mu_A(e)$ and $\nu_A(a) = \nu_A(e)$.

Now

$$\mu_A([a, x]) = \mu_A(a^{-1}x^{-1}ax)$$

$$\geq \mu_A(a^{-1}) \land \mu_A(x^{-1}ax)$$

$$= \mu_A(a) \land \mu_A(a)$$

$$= \mu_A(e) \land \mu_A(e)$$

$$= \mu_A(e).$$

Thus $\mu_A([a, x]) = \mu_A(e)$, similarly we can show that $\nu_A([a, x]) = \nu_A(e)$, i.e., $a \in C(A)$. Hence $N \subseteq C(A)$.

Definition 3.6. Let $A$ be an IFSG of a group $G$. Then $A$ is called an intuitionistic fuzzy abelian subgroup (IFASG) of $G$ if and only if $C_{\alpha, \beta}(A)$ is an abelian subgroup of $G$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$.

Remark 3.7. If $G$ is an abelian group, then every IFSG of $G$ is an IFASG of $G$, but the converse need not be true.

Example 3.8. Let $G$ be an abelian group and $A$ be any IFSG of $G$. Then $C_{\alpha, \beta}(A)$ being subgroup of $G$ is also abelian, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$. Thus $A$ is an IFASG of $G$. But the converse is not true.

Consider $G = S_3 = \{i, (12), (13), (23), (123), (132)\}$ be the symmetric group. Consider the IFS $A$ of $G$ defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = i, \\ 0, & \text{if } x^2 = i, \text{ and } \\ 0.6, & \text{if } x^3 = i \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0, & \text{if } x = i, \\ 0.5, & \text{if } x^2 = i, \\ 0.6, & \text{if } x^3 = i. \end{cases}$$
Clearly, $A$ is an IFSG of group $G$. Moreover, $C_{\alpha, \beta}(A) = \{i\}$ or
\{1, (123), (132)\} are abelian subgroups of $G$, for all $\alpha, \beta \in (0, 1]$ with
$0 < \alpha + \beta \leq 1$.

Hence $A$ is an IFASG of $G$, but $G$ is non-abelian group.

**Theorem 3.9.** Let $A$ be an IFASG of $G$. Then the set
\[ H = \{a \in G : \mu_A(ab) = \mu_A(ba) \text{ and } \nu_A(ab) = \nu_A(ba) \text{ for all } b \in G\} \]
is an abelian subgroup of $G$.

**Proof.** Since $A$ is an IFASG of group $G$, $C_{\alpha, \beta}(A)$ is an abelian subgroup
of $G$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$.

To show that the $H$ is an abelian subgroup of $G$: Clearly, $H \neq \emptyset$, for $e \in H$.

Let $a, b \in H$ be any two elements. Then $\mu_A(ax) = \mu_A(xa)$, $\nu_A(ax) = \nu_A(xa)$ and $\mu_A(ax) = \mu_A(xa)$, $\nu_A(ax) = \nu_A(xa)$, for all $x \in G$.

Now
\[
\mu_A((ab)x) = \mu_A(a(bx)) = \mu_A((bx)a) = \mu_A(b xa)) = \mu_A((xa)b) = \mu_A(x(ab))
\]
and
\[
\nu_A((ab)x) = \nu_A(a(bx)) = \nu_A((bx)a) = \nu_A(b xa)) = \nu_A((xa)b) = \nu_A(x(ab))
\]
hold, for all $x \in G$. Therefore, $ab \in H$.

Also, let $a \in H$. To show that $a^{-1} \in H$:

Now
\[ a \in H \Rightarrow \mu_A(ax) = \mu_A(xa), \nu_A(ax) = \nu_A(xa) \text{ holds, for all } x \in G. \ (\ast) \]
We show that $\mu_A(a^{-1}y) = \mu_A(ya^{-1})$, $\nu_A(a^{-1}y) = \nu_A(ya^{-1})$ hold, for all $y \in G$. Putting $x = y^{-1}$ in (*), we get $\mu_A(ay^{-1}) = \mu_A(y^{-1}a)$, $\nu_A(ay^{-1}) = \nu_A(y^{-1}a)$.

Now
$$\mu_A(a^{-1}y) = \mu_A((a^{-1}y)^{-1}) = \mu_A(y^{-1}a) = \mu_A(ay^{-1})$$
$$= \mu_A((ay^{-1})^{-1}) = \mu_A(ya^{-1}).$$

Similarly, we can show that $\nu_A(a^{-1}y) = \nu_A(ya^{-1})$ holds, for all $y \in G$.

Thus $a^{-1} \in H$. So $H$ is a subgroup of $G$.

Next, we show that $H$ is an abelian subgroup of $G$. Let $a, b \in H$.

Without loss of generality, let $\mu_A(a) = \alpha$, $\nu_A(a) \leq 1 - \alpha$ and $\mu_A(b) = \alpha_1$, $\nu_A(b) \leq 1 - \alpha_1$. Then $a \in C_{\alpha, 1-\alpha}(A)$, $b \in C_{\alpha_1, 1-\alpha_1}(A)$.

Let $\alpha < \alpha_1$. Then $\mu_A(b) = \alpha_1 > \alpha$ and $\nu_A(b) \leq 1 - \alpha_1 < 1 - \alpha \Rightarrow b \in C_{\alpha, 1-\alpha}(A)$. Thus $a, b \in C_{\alpha, 1-\alpha}(A)$ and so $ab = ba$.

Hence $H$ is an abelian subgroup of $G$.

**Remark 3.10.** (i) If $A$ is an IFASG of group $G$, then $A$ is also an IFNSG of $G$.

(ii) The sets $H$ and $C(A)$ are same, i.e., $C(A) = H$.

**Proof.**

$C(A) = \{a \in G : \mu_A([a, x]) = \mu_A(e) \text{ and } \nu_A([a, x]) = \nu_A(e), \text{ for all } x \in G\}$

$= \{a \in G : \mu_A(a^{-1}x^{-1}ax) = \mu_A(e) \text{ and } \nu_A(a^{-1}x^{-1}ax) = \nu_A(e),$

for all $x \in G\}$

$= \{a \in G : \mu_A((xa)^{-1}ax) = \mu_A(e) \text{ and } \nu_A((xa)^{-1}ax) = \nu_A(e),$

for all $x \in G\}$
Theorem 3.11. Let $A$ be an IFASG of $G$. Then $C(A)$ is an abelian subgroup of $G$.

Theorem 3.12. Let $A$ and $B$ be two IFSG’s of a group $G_1$ and $G_2$, respectively. Then $A \times B$ is an IFASG of $G_1 \times G_2$ if and only if both $A$ and $B$ are IFASG’s of $G_1$ and $G_2$, respectively.

Proof. First, let $A$ and $B$ be IFASG’s of $G_1$ and $G_2$, respectively.

Then $C_{\alpha, \beta}(A)$ are abelian subgroups of $G_1$ and $G_2$, respectively, for all $\alpha, \beta \in (0, 1]$ with

$$0 < \alpha + \beta \leq 1 \implies C_{\alpha, \beta}(A) \times C_{\alpha, \beta}(B)$$

is an abelian subgroup of $G_1 \times G_2$.

But $C_{\alpha, \beta}(A \times B) = C_{\alpha, \beta}(A) \times C_{\alpha, \beta}(B)$ (by Proposition 2.8)).

Therefore, $C_{\alpha, \beta}(A \times B)$ is an abelian subgroup of $G_1 \times G_2$, for all $\alpha, \beta \in (0, 1]$ with

$$0 < \alpha + \beta \leq 1 \implies A \times B$$

is an IFASG of $G_1 \times G_2$.

Conversely, let $A \times B$ is an IFASG of $G_1 \times G_2$. Then $C_{\alpha, \beta}(A \times B)$ is an abelian subgroup of $G_1 \times G_2$, i.e., $C_{\alpha, \beta}(A) \times C_{\alpha, \beta}(B)$ is an abelian subgroup of $G_1 \times G_2 \implies C_{\alpha, \beta}(A)$ and $C_{\alpha, \beta}(B)$ are abelian subgroups of $G_1$ and $G_2$, respectively $\implies A$ and $B$ are IFASG’s of $G_1$ and $G_2$, respectively.

Definition 3.13. Let $A$ be an IFSG of a group $G$. Then $A$ is called cyclic intuitionistic fuzzy subgroup (CIFSG) of group $G$, if $C_{\alpha, \beta}(A)$ is a cyclic subgroup of $G$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$.

Remark 3.14. (i) If $G$ be a cyclic group, then every IFSG of $G$ is CIFSG of $G$, but converse need not be true.
Proof. Let $G = \langle x \rangle$ be cyclic group and let $A$ be any IFSG of $G$. Then we know that $\mu_A(x^n) \geq \mu_A(x^{n-1}) \geq \mu_A(x^{n-2}) \geq \cdots \geq \mu_A(x^2) \geq \mu_A(x)$ and $\nu_A(x^n) \leq \nu_A(x^{n-1}) \leq \nu_A(x^{n-2}) \leq \cdots \leq \nu_A(x^2) \leq \nu_A(x)$ holds, for all $n \in N$.

Therefore, if $x^m \in C_{\alpha, \beta}(A)$, for some $m \in N$, then $x^m, x^{m+1}, x^{m+2}, \ldots \in C_{\alpha, \beta}(A)$, i.e., $C_{\alpha, \beta}(A) = \langle x^{-1} \rangle$, which is a cyclic subgroup of $G$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$. Hence $A$ is CIFSG of $G$.

Converse need not be true: For example, see Example 3.8. $A$ is a cyclic intuitionistic fuzzy subgroup of $G$, but $G$ is not cyclic.

(ii) Every CIFSG of a group $G$ is IFASG, but converse need not be true.

Proof. Trivial.

4. Homomorphism of Intuitionistic Fuzzy Abelian Groups

Theorem 4.1. Let $f : G_1 \to G_2$ be homomorphism of group $G_1$ into a group $G_2$. Let $B$ be an IFASG of group $G_2$. Then $f^{-1}(B)$ is an IFASG of group $G_1$.

Proof. Let $B$ be an IFASG of group $G_2$. Therefore, $C_{\alpha, \beta}(B)$ is an abelian subgroup of $G_2$, for all $\alpha, \beta \in (0, 1]$ with $0 < \alpha + \beta \leq 1$.

By Proposition 2.12, we have

$$C_{\alpha, \beta}(f^{-1}(B)) = f^{-1}(C_{\alpha, \beta}(B)) = \{x \in G_1 : f(x) \in C_{\alpha, \beta}(B)\}.$$

Let $x_1, x_2 \in C_{\alpha, \beta}(f^{-1}(B))$ be any two points. Then $f(x_1), f(x_2) \in C_{\alpha, \beta}(B)$ as $C_{\alpha, \beta}(B)$ is an abelian subgroup of $G_2$. Therefore, we have $f(x_1)f(x_2) = f(x_2)f(x_1)$ and so $\mu_B(f(x_1x_2)) = \mu_B(f(x_2x_1))$ and $\nu_B(f(x_1x_2)) = \nu_B(f(x_2x_1))$.

By Proposition 2.12, we have

$$\nu_{f^{-1}(B)}(x_1x_2) = \nu_{f^{-1}(B)}(x_2x_1) \Rightarrow x_1x_2 = x_2x_1.$$
Thus \( C_{\alpha, \beta}(f^{-1}(B)) \) is an abelian subgroup of \( G_1 \), for all \( \alpha, \beta \in (0, 1] \) with \( 0 < \alpha + \beta \leq 1 \).

Hence \( f^{-1}(B) \) is an IFASG of group \( G_1 \).

**Theorem 4.2.** Let \( f : G_1 \rightarrow G_2 \) be surjective homomorphism and \( A \) be an IFASG of group \( G_1 \). Then \( f(A) \) is an IFASG of group \( G_2 \).

**Proof.** Since \( A \) is an IFASG of group \( G_1 \), \( C_{\alpha, \beta}(A) \) is an abelian subgroup of \( G_1 \), for all \( \alpha, \beta \in (0, 1] \) with \( 0 < \alpha + \beta \leq 1 \). To show that \( f(A) \) is an IFASG of \( G_2 \):

For this, we show that \( C_{\alpha, \beta}(f(A)) \) is an abelian subgroup of \( G_2 \).

Let \( y_1, y_2 \in C_{\alpha, \beta}(f(A)) \). Then \( \exists x_1, x_2 \in G_1 \) such that \( f(x_1) = y_1 \), \( f(x_2) = y_2 \).

\[
\because f(x_1), f(x_2) \in C_{\alpha, \beta}(f(A)) \text{ as } C_{\alpha, \beta}(A) \text{ is an abelian subgroup of } G_1. 
\therefore \exists C_{\delta, \theta}(A) \text{ such that } x_1, x_2 \in C_{\delta, \theta}(A), \text{ where } \delta, \theta \in (0, 1] \text{ and } 0 < \delta + \theta \leq 1.
\]

But \( C_{\delta, \theta}(A) \) is an abelian group. Therefore,

\[
x_1x_2 = x_2x_1 \Rightarrow f(x_1x_2) = f(x_2x_1) \Rightarrow f(x_1)f(x_2) = f(x_2)f(x_1),
\]
i.e., \( y_1y_2 = y_2y_1 \).

Thus \( C_{\alpha, \beta}(f(A)) \) is an abelian subgroup of \( G_2 \). Hence \( f(A) \) is an IFASG of group \( G_2 \).

**Theorem 4.3.** Let \( f : G_1 \rightarrow G_2 \) be homomorphism of group \( G_1 \) into a group \( G_2 \). Let \( B \) be CIFSG of group \( G_2 \). Then \( f^{-1}(B) \) is CIFSG of group \( G_1 \).
On Intuitionistic Fuzzy Abelian Subgroups

Proof. Since \( B \) is CIFSG of group \( G_2 \), \( C_{\alpha, \beta}(B) \) is a cyclic subgroup of \( G_2 \), for all \( \alpha, \beta \in (0, 1] \) with \( 0 < \alpha + \beta \leq 1 \). Let \( C_{\alpha, \beta}(B) = \langle g_2 \rangle \), for some \( g_2 \in G_2 \). Now for \( g_2 \in G_2 \), \( \exists \ g_1 \in G_1 \) such that \( f(g_1) = g_2 \). Thus \( C_{\alpha, \beta}(B) = \langle f(g_1) \rangle \).

And so \( f^{-1}(C_{\alpha, \beta}(B)) = C_{\alpha, \beta}(f^{-1}(B)) = \langle g_1 \rangle \).

Hence \( f^{-1}(B) \) is CIFSG of group \( G_1 \).

**Theorem 4.4.** Let \( f : G_1 \rightarrow G_2 \) be surjective homomorphism and \( A \) be CIFSG of group \( G_1 \). Then \( f(A) \) is CIFSG of group \( G_2 \).

Proof. Let \( A \) be CIFSG of group \( G_1 \). Therefore, \( C_{\alpha, \beta}(A) \) is a cyclic subgroup of \( G_1 \), for all \( \alpha, \beta \in (0, 1] \) with \( 0 < \alpha + \beta \leq 1 \).

To show that \( f(A) \) is also CIFSG of \( G_2 \): Let \( g \in C_{\alpha, \beta}(f(A)) \) be any element. As \( f \) is surjective, therefore let \( g = f(g') \), for some \( g' \in G_1 \).

As \( g' \in G_1 \), we can find one \( C_{\delta, 0}(A) \) which exists, for all \( g' \in G_1 \) (and hence for all \( g \in C_{\alpha, \beta}(f(A)) \) such that \( g' \in C_{\delta, 0}(A) \)). But \( C_{\delta, 0}(A) \) is a cyclic subgroup of \( G_1 \). Let \( C_{\delta, 0}(A) = \langle g_1 \rangle \). So \( g' = (g_1)^n \). Thus \( g = f(g') = f((g_1)^n) = (f(g_1))^n \), i.e., \( C_{\alpha, \beta}(f(A)) \) is a cyclic subgroup of \( G_2 \). Hence \( f(A) \) is CIFSG of \( G_2 \).

**Conclusion**

The motivation to study intuitionistic fuzzy abelian subgroups is that abelian groups play an important role in the development of Group Theory. The classification of finitely generated abelian groups is so precise, the interest is in discerning the structure of large abelian groups, and thus this area overlaps to a degree with Set Theory. Certain classes of abelian groups are more amenable to study, those which are torsion, or torsion free, for
example, or those which have some additional structure such as an ordering or a topology, this leads to the introduction of homological methods. Thus studying intuitionistic fuzzy abelian subgroups and intuitionistic fuzzy cyclic groups will develop the theory of intuitionistic fuzzy groups.

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References


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