A MARKOVIAN MODEL OF CONSUMER BUYING BEHAVIOR AND OPTIMAL ADVERTISING PULSING POLICY

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Scope and Purpose—Achievement of the advertising manager's goal to improve consumer attitudes toward a product or service requires efficient allocation of a limited advertising budget over a finite planning horizon. An important option is to employ advertising pulsing policy (APP), which alternates between low and high levels of spending with finite frequency, as opposed to a strategy of a constant advertising level. In addition to the firm's advertising policy there are a multitude of social-psychological and cultural-environmental factors affecting purchase behavior, some unknown to the advertiser. A descriptive model to aid in this decision must include consumer buying behavior under this uncertainty. This paper presents a set of analytical, statistical and numerical procedures to support the scheduling APP decisions under the uncertainty. These modelling tools enable us to develop a prescriptive model which provides maximum profit as a performance criterion for analyzing APP strategy. Since the advertising manager is concerned with the perceived economic risk of his decision alternatives, the profit function includes the variation in sales, which for APP could be much larger than under a constant advertising level. Operational issues such as modelling self-validation and estimation of parameters are treated using the discrete-time records of attitudes and sales.

Abstract—This paper deals with the problem of scheduling optimal advertising policy for a very general class of consumer buying behavior models. To avoid analysis of a complex multitude of social-psychological and cultural-environmental factors affecting the consumer's decision we construct a stochastic model. Because of the diminishing effect of even advertising policy, we consider advertising pulsing policy (APP) as a means to increase advertising effectiveness. By reducing the probabilistic evolution of sales and consumers' attitude over time to their means, multivariate linear least-square regression is used to estimate the market parameters and validate the model. The prescribed strategy scheduling maximizes the discounted profit function, which includes uncertainty in sales over a finite campaign duration. The superiority of APP over an even advertising policy is illustrated numerically.

I. INTRODUCTION

Existing mathematical advertising models can be grouped into two broad classes: (1) models based on the concept of "selling" with some assumed advertising/sales response (ASR) functions; and (2) models based on "marketing" using the theory of consumer buying behavior.

Selling models focus on the need of the seller to convert the product into cash. In such models the seller thinks mainly in terms of his sales and is concerned whether sales are going up or going down. The best selling models are based on the Vidale and Wolfe [1] ASR model, which is one of the most intensively analyzed models [2]. These models require a very strong assumption (possibly correct) of the specific shape of the ASR function. Many researchers who follow the ASR approach readily admit difficulties in the assumption of the shape of the ASR function [e.g. 3, 4]. In short, this approach, known as the "macro approach", deals with modelling the aggregate behavior of the market in response to change in advertising stimulus.

The consumer behavior models focus upon the marketing manager's concern for delivering product or service benefit, changing brand attitudes and influencing consumer perceptions [e.g.}
The customer-driven organizations realize that individuals and highly varied people are involved in buying and using their product. These models presume that advertising plans must be based on the psychological, environmental and social forces that condition the consumer, i.e. the "micro approach". Modern business firms plan their advertising campaigns using the marketing approach rather than the selling approach. Managers want to know how they can influence consumers and which decision variables have more impact than others. They want to sharpen their ability to plan, allocate and forecast.

Christner [6] summarizes the rationale as follows: in order to sell something the marketer must know what the potential buyer wants and/or wants to hear. The considerable discussion in the marketing literature about "consumer behavior" centers around the need for marketing to be consumer-oriented and to focus on the whole cluster of factors associated with creation, delivery and consumption of the product to satisfy consumer need. This is made possible, in part, by the availability of group work techniques such as "brain storming", the Delphi method, consumer collaborative technology (consumer involvement in all phases from design to selecting packaging of the product) and GDSS, to construct models of the consumer decision process [e.g. 7]. A popular example of GDSS is in the airline industry where competing carriers, travel agents and consumers are all interconnected through a network of airline reservation systems [e.g. 8]. The products or services that a firm provides to its customers are, from the consumers' perspective, supporting resources for their needs. To buy them, the consumer goes through a resource life cycle (RLC).

Prominent consumer behavior deterministic models based upon a set of social and psychological factors include those in Refs [10–12]. The essential classes of variables used in these models to measure behavior are: communication (stimuli); level of purchase; and attitude.

The two major drawbacks of these descriptive models of consumer buying behavior are the assumption that the advertising expenditure rate is constant over time and the assumption of infinite horizon [e.g. 13]. It is clear that the return on constant advertising rate diminishes with time and therefore, further expenditures will not increase sales revenues. On the other hand the need for reinforcement is a major issue in learning theory [e.g. 14, 15] with clear application to advertising. The dilemma of providing reinforcement in the face of diminishing effect is addressed by advertising pulsing policy (APP), a policy with periods of high constant advertising intensity alternating with periods of no advertising (see, for example, Ref. [16] for some empirical evidence). Further evidence is provided in Refs [17–28] for ASR, Ref. [1] or its modified models.

The infinite horizon assumption decreases the usefulness of consumer buying behavior models, since budget planning for advertising expenditures seldom has an infinite horizon. Moreover, it is not clear how to generate the sales response function in these models when advertising is shut off.

There are a multitude of social—psychological and environmental factors affecting purchase, some of which cannot be captured or explained by any moderate-size deterministic model, so we will consider a stochastic model. Markovian assumptions concerning a consumer's attitude and purchase level are made. By reducing the probabilistic evolution of sales and attitude over time to their means, multivariate linear least-square (MLLS) regression is used to estimate the market parameters and validate the model. We will consider the problem of designing an optimal APP for a finite campaign duration. The prescribed APP strategy is to maximize the discounted profit function based upon the mean and variance of sales. We include variation in sales since marketing managers are concerned with the economic and perceived risk of their decision alternatives. The superiority of APP over an even policy is illustrated numerically.

The remainder of this article is divided into six sections. Section 2 presents a conceptual model of consumer buying behavior. In Section 3 the model behavior is formulated for both parts of the APP cycle, with advertising off as well as on. Discussion of parameter, identification, estimation and an indication of how the proposed model could be implemented in a real-life situation is given in Section 4. When consistent parameter estimates are obtained, Section 5 designs the optimal APP strategy. This is followed by an application with numerical analysis. The final section provides a summary of the findings.
A CONCEPTUAL MODEL OF CONSUMER BUYING BEHAVIOR

A consumer buying behavior model is based upon many factors. For example, Noriss [29] found that consumers tend to perceive heavily advertised brands to be of higher quality. However, the marketing literature provides strong evidence that consumers use rules of their own rather than information about product quality, perceived value and price. The lower search and evaluation costs associated with such rules may more than offset the monetary or quality losses [e.g. 14, 30, 31].

F. Nietzsche, the first psychologist [e.g. 32], discovered that human beings are "attitudinal beings" who evaluate just about everything they come into contact with through "revision of values". Consider the question "How do you feel about this particular brand?" Surely, the answer will indicate the extent to which you like or dislike, value or disvalue, the brand. Thus, the crux of consumer behavior modeling is that the marketer attempts to recognize the consumer as an attitudinal being who constantly revises all values, while developing his belief confidence toward a particular brand [e.g. 33, 34]. Once the goal-directed behavior is manifested, the consumer experiences the consequences of his behavior. He uses this experience as a source of learning in which he revises his total attitude about the product or service. The consumer then has to make decisions about whether and what to buy, how much and at what cost, and when and where. There are also various usage or consumption activities and responses which generate post-usage feelings of satisfaction and change attitudes [35]. However, feelings of satisfaction may change with a purchase decision over time [36]. Moreover, statistics show that it is 5 times cheaper to retain a customer than obtain a new one, and that a dissatisfied customer tells an average of 12 other people about his dissatisfaction [37].

Several marketing researchers have stated the fact that attitude alone can determine subsequent behavior (see, for example, Refs [38–43] and references therein). In this section, the components of the model are briefly outlined. The reader is directed to these references for a more exhaustive elaboration of the underlying theory and test of this theoretical position.

Consider a flow chart of a simple model, shown in Fig. 1, which is based upon the three main factors common to existing consumer buying behavior models. The structure of the decision process of a typical consumer concerning a specific brand X, contains three functional values:

\[ B(t) = \text{the buying behavior, i.e. amount of purchase (units) at time } t. \]
\[ A(t) = \text{the consumers' attitude (scalar) toward the brand which results from some} \]

\[ \text{Firm's Strategies} \]

\[ \text{External Influence Communication Ct) } \]

\[ \text{Consumer's Pre-purchase Needs} \]

\[ \text{Evaluation} \]

\[ \text{Experience Intention to buy} \]

\[ \text{Consumption Final Decision & Action} \]

\[ \text{Purchase B(t)} \]

\[ \text{Attitude A(t)} \]

Fig. 1. A simplified description of the theory of consumer buying behavior with three main variables (A, B, C).
A variety of complex interactions of various factors, some of which are indicated in Fig. 1. 

\[ C(t) = \text{communication rate, i.e. the advertising campaign of the business firm. This may be any stimuli for a particular brand.} \]

A successful marketing strategy develops product and promotional stimuli that consumers will perceive as relevant to their needs. Consumer needs are also influenced by factors such as past experience [e.g. 44] and social influences [e.g. 45]. The consumer may become aware of the product by the firm's communication, develop a preference for it, purchase or react negatively to it. This information is sought by the firm. In an internal flow of communication, the firm makes plans for future actions on the basis of current and past information. The physical flows are the movement of products or services to consumers and the simultaneous movement of sales dollars to the firm.

For computer-controlled experimentation in consumer decision making, see Ref. [46].

Because of the diminishing effect of advertising [e.g. 47], and other factors such as learning and forgetting effects [e.g. 48], we consider \( C(t) \) to be a pulse function, as opposed to an even advertising policy. An advertising pulsing policy APP with a cycle of length \( \tau \), has advertising over a period \( t_1 \), (\( t_1 \leq \tau \)) followed by no advertising from \( t_1 \) until the next cycle starts at \( \tau \):

\[
C(t) = \begin{cases} 
C & 0 \leq t \leq t_1 \\
0 & t_1 < t \leq \tau 
\end{cases}
\]

Under this APP model we distinguish, in the following section, two periods: advertising with high intensity and little or no advertising.

3. STOCHASTIC CONSUMER BEHAVIOR UNDER ADVERTISING PULSING POLICY

Consider a cycle of length \( \tau \), we distinguish the following two periods.

3.1. Advertising with high intensity: \( C(t) = c, 0 \leq t \leq t_1 \)

Let \( A(t) \) and \( B(t) \) be the purchase and attitude at time \( t \), and

\[ P[A, B; t] = Pr[A(t) = A, B(t) = B; t]. \]

We consider a first-order Markovian two-dimensional discrete-space \([A, B; t]\), continuous-time process. Assume that in a small interval of time \( \delta t \) only one of the events shown in Fig. 2 would occur. We postulate a bivariate birth-death Markov process with the following (conditional) transition probability structure:

\[
P[A, B; t + \delta t | A - 1, B; t] = a_1 B(t) \cdot \delta t + O(\delta t), \quad A > 1, B > 0, \quad (1)
\]

\[
P[A, B; t + \delta t | A, B + 1; t] = b_1 [B(t) + 1] \cdot \delta t + O(\delta t), \quad A > 0, B \geq 0, \quad (2)
\]

\[
P[A, B; t + \delta t | A - 1, B; t] = a_2 B(t) \cdot \delta t + C \cdot \delta t + O(\delta t), \quad A > 1, B > 0, \quad (3)
\]

\[
P[A, B; t + \delta t | A, B - 1; t] = b_2 [A(t) + 1] \cdot \delta t + O(\delta t), \quad A \geq 0, B > 0. \quad (4)
\]

The first two equations state that the probability of a change in purchase in a time interval \( (t, t + \delta t) \) by one unit is a linear function of the level of consumer's attitude \( A(t) \) and level of purchase \( B(t) \) at time \( t \). Equation (3) states that the probability of increasing attitude by one unit depends on the previous purchase level and the communization level. However, the probability of losing a unit in consumer's attitude is proportional to the attitude level at time \( t \).

Upon constructing the probability generating function for the joint distribution function \( P[B(t), A(t); t] \) one obtains the mean and variance of \( B(t) \) denoted by \( (M_B, V_B) \) and that of \( A(t) \) by \( (M_A, V_A) \) together with covariance of \( [A(t), B(t)] \) denoted by \( W_{AB} \). Because mathematically rigorous treatment of these computations is readily available in marketing literature [e.g. 49, 50], we present the following results, where \( (') \) denotes derivative with respect to time:

\[
M'_A = a_2 M_B - b_1 M_A + C; \quad M_A(0) = A_0, \quad 0 \leq t \leq t_1, \quad (5)
\]

\[
M'_B = -b_2 M_B + a_1 M_A; \quad M_B(0) = B_0, \quad 0 \leq t \leq t_1, \quad (6)
\]
The first two equations represent the evaluation of the mean purchase level \( B(t) \) and attitude \( A(t) \). The remaining equations describe the evaluation of variance and covariance of \( B(t) \) and \( A(t) \). The model's dynamic mean state equations are described by the following two linear differential equations:

\[
M_b' = -b_1 M_b + a_1 M_A; \quad M_b(0) = B_0
\]

and

\[
M_A' = a_2 M_b - b_2 M_A + C; \quad M_A(0) = A_0.
\]

The first term in each equation postulates that both mean sales and attitude are positively correlated. Both sales and attitude are proportional to the other variable. Furthermore, the mean function must satisfy the known initial conditions: \( M_b(0) = B_0 \) and \( M_A(0) = A_0 \), which are a residual influence from past advertising efforts.

The "market" parameters of the model are \( a_1, a_2, b_1, \) and \( b_2 \) which have dimension \( (1/time) \). The meaning of these coefficients can be determined by considering the equilibrium (i.e. steady state)
conditions as follows:

(1) In the equilibrium condition, the first equation become

\[ M_B' - a_1 [M_A - (b_1/a_1)M_B] = 0. \]

This suggests \( M_B = (a_1/b_1)M_A \). The coefficient \( (b_1/a_1) \) summarizes the social-psychological mechanisms involved in the act of buying. The coefficient \( a_1 \) determines how rapidly the consumer will resolve the conflict \( A - (b_1/a_1)M_B \), i.e., the difference between perceived need and current buying level.

(2) In the equilibrium condition, the second equation becomes

\[ M_A' = a_2 [M_B - (b_2/a_2)M_A] + C = 0. \]

In the absence of advertising \( (C = 0) \) the coefficient \( b_2/a_2 \), which is dimensionless, establishes \( M_A = (a_2/b_2)M_B \). Thus, the coefficient \( (b_2/a_2) \) measures the attitude level that will be generated by a purchase. The coefficient \( a_2 \) determines how rapidly the consumer will resolve the conflict \( M_B - (b_2/a_2)M_A \), i.e., the difference between \( (b_2/a_2)M_A \) and the purchase level. \( C \) is communication rate \( (1/time) \).

Upon estimating the parameters, this system of first-order differential equations can be solved numerically, for example, using DVERK subroutines in the IMSL [51] package [52], or analytically as done in our numerical example in Section 6.

3.2. Advertising off: \( C(t) = 0, t_f < t < \tau \)

As before, let \( A(t) \) and \( B(t) \) be the purchase and attitude at time \( t \), and \( P[A, B; t] = Pr[A(t) = A, B(t) = B; t] \). By assuming Markovian property, we obtain a pure death process of discrete space with the following (conditional) transition probability structure:

\[ P[A, B; t + \delta t | A, B + 1; t] = \beta_1 [B(t) + 1] \cdot \delta t + O(\delta t), \quad A \geq 0, B \geq 0 \] (10)

and

\[ P[A, B; t + \delta t | A + 1, B; t] = \beta_2 [A(t) + 1] \cdot \delta t + O(\delta t), \quad A \geq 0, B \geq 0. \] (11)

The first equation states that the probability of a change in purchase in a time interval \( (t, t + \delta t) \) by one unit is a linear function of the level of consumer's attitude \( A(t) \) and level of purchase at time \( t \). The last equation states that the probability of losing a unit in consumer's attitude is proportional to the attitude level at time \( t \).

As before, upon constructing the probability generating function for the joint distribution function \( P[B(t), A(t); t] \) one can obtain the mean and variance of \( B(t) \), denoted by \( (M_{B_1}, V_{B_1}) \), and \( A(t) \), by \( (M_{A_1}, V_{A_1}) \), together with the covariance of \( [A(t), B(t)] \), denoted by \( W_{AB} \). The results are:

\[ M_{A_1} = -\beta_2 M_{A_1}; \quad M_{A_1}(t_1) = M_{A_1}(t_1), \quad t_1 \leq t \leq \tau, \] (12)

\[ M_{B_1} = -\beta_1 M_{B_1}; \quad M_{B_1}(t_1) = M_{B_1}(t_1), \quad t_1 \leq t \leq \tau, \] (13)

\[ V_{A_1} = \beta_2 M_{A_1} - 2\beta_2 V_{A_1}; \quad V_{A_1}(t_1) = V_{A_1}(t_1), \quad t_1 \leq t \leq \tau. \] (14)

\[ V_{B_1} = -2\beta_1 V_{B_1}; \quad V_{B_1}(t_1) = V_{B_1}(t_1), \quad t_1 \leq t \leq \tau. \] (15)

and

\[ W_{AB_1} = -\beta_1 W_{AB_1}; \quad W_{AB_1}(t_1) = W_{AB_1}(t_1), \quad t_1 \leq t \leq \tau. \] (16)

The first two equations represent the means of purchase level \( B(t) \) and attitude level \( A(t) \). The remaining equations describe the evaluation of variance and covariance of \( B(t) \) and \( A(t) \). Notice that the initial conditions are already known from the end of the period when the advertising was at high intensity.

Benjamin et al. [53] empirically verified the much assumed hypothesis that in the absence of advertising sales decay exponentially, as is implied by result (13). This decline in sales is caused by factors such as memory loss or competitors efforts. This decreasing sales assumption is also consistent with our model, even when advertising is of high intensity. That is, if \( a_2 \cdot M_B > b_1 \cdot M_A \), then the time rate of change of \( M_B (M_B) \) becomes negative and thus \( M_B \) decreases. Notice that, in
the absence of a constant advertising rate, the sales function is not defined in the existing consumer behavior models [e.g. 54].

4. ESTIMATION AND VALIDATION

Once we estimate the parameters, this model can find the optimal advertising policy over a finite horizon campaign duration and observe the subsequent impact on the mean and variance of sales rate function $B$. To perform these experiments, however, one must first be able to estimate the model's parameters in an operational manner. In practice, data concerning attitude $A$ (i.e. realization of $A$), are available as a result of survey methodologies. See Ref. [55] for a detailed description of several general methods of measuring the attitude. For dynamic aspects of attitude see Ref. [56]. Specifically, for marketing applications, [57] devised an operational procedure to measure the attitude for a given brand.

Assuming differentiability the following second-order differential equation can be obtained by eliminating $M_A$ in equations (5) and (6) and solving for $M_B$. The result is

$$M_B'(t) + (b_1 + b_2)M_B(t) + (b_1b_2 - a_1a_2)M_B(t) = a_1C$$

or

$$M_B'(t) + d_1M_B(t) + d_2M_B(t) = d_3C,$$

where

$$d_1 = b_1 + b_2, \quad d_2 = b_1b_2 - a_1a_2 \quad \text{and} \quad d_3 = a_1. \quad (19)$$

Similarly, a second order differential equation can be obtained by eliminating $M_B$ in equations (5) and (6) and solving for $M_A$. The result is

$$M_A'(t) + (b_1 + b_2)M_A(t) + (b_1b_2 - a_1a_2)M_A(t) = b_1C$$

or

$$M_A'(t) + d_1M_A(t) + d_2M_A(t) = d_4C,$$

where

$$d_4 = b_1. \quad (20)$$

Thus, given discrete-time records of attitudes and sales (i.e. realization of both) when the advertising is with high intensity, MLLS regression techniques can be satisfactorily employed by using the discrete version of equations (18) and (21). More specifically, we observe the sales behavior at discrete, equally spaced time $t_i, i = 1, 2, 3, \ldots$ over a specific duration while $C(t) = C$. By constructing a new table with entries $(B_i, \delta B_i, \delta^2 B_i)$ where $\delta B_i = B_{i+1} - B_i$ and $\delta^2 B_i = \delta B_{i+1} - \delta B_i$, we use MLLS to estimate $d_1, d_2, d_3$ as a result of the MLLS fit of the form: $\delta^2 B_i - d_3 C - d_2(\delta B_i) - d_1(\delta B_i)$. Similarly, the MLLS technique can be used to estimate the coefficient of equation (21). Upon estimating the parameter $d_i, i = 1, 2, 3, 4$, one can find $a_1, a_2, b_1, b_2$ using one-to-one relationships:

$$d_1 = b_1 + b_2, \quad d_2 = b_1b_2 - a_1a_2, \quad d_3 = a_1 \quad \text{and} \quad d_4 = b_1. \quad (22)$$

Similarly, given the discrete-time record of the sales and attitude record when the advertising is off, a log-linear regression equation fit can estimate the decay constants $\beta_1$ and $\beta_2$.

Notice that the two planes given by equations (18) and (21) are parallel in principle. This characteristic of attitude and sales means functions in the model could be used as a modeling self-validation tool. More specifically, parallel regression [58] can be used to test whether or not the model is consistent with the data set containing $(A_i, B_i)$. Ross [59] provides a computer code which is designed to test the parallelism hypothesis in MLLS by measuring the $R$-statistic (see Ref. [60] for details). Clearly, upon estimating the market parameters one can estimate the dynamic behavior of the model by solving equations (5)–(9) and then equations (12)–(16). An outline of how to obtain the general solution for these equations is given in the Appendix. A typical purchase behavior under APP is shown in Fig. 3.
5. OPTIMAL STRATEGY UNDER APP

For a marketing manager the key question is, given a certain advertising budget how should the APP be scheduled? In this section, we consider a firm facing the problem of designing an APP with the following decision variables: advertising rate \( C(t) = C \), with duration \( t_1 \), cycle length \( \tau \) and number of pulse exposures \( n \), given the campaign duration \( T \) and the total advertising budget \( I \) to maximize a profit function. The choice of an optimal advertising strategy scheduling denoted by \((C^*, t^*_1, \tau^*, n^*)\) must reflect both the firm's attitude toward profit maximization, as well as its attitude toward risk as measured by the purchase variation. It is noteworthy that advertising policies which call for maximization of a profit function without taking account of risk have already been recognized as inappropriate (see, for example, Ref. [61] and references therein). Moreover, if the firm is not going out of business, the impact of advertising beyond the planning horizon must be incorporated in the objective function. In dealing with the ASR model, a number of authors [23, 27, 28, 62–65] have incorporated the impact of advertising beyond the planning horizon in various forms such as "residual awareness" and "salvage value" at time \( T \). Following Sweeney et al. [65] and Sasieni [28], we specify a salvage value term \( \Theta M_{\phi}(T) \), where \( \Theta \) is the value the firm attaches to having a certain mean sales rate at the end of planning horizon. Along these lines, we want to solve the following discounted optimal control problem:

Problem P: Maximize \( J \),

\[
J = \Theta [M_{\phi}(T)]_0 \exp (-iT) + \sum_{j=1}^{n} \exp [-i(j-1)\tau] \left\{ \int_{0}^{\tau} \left[ pM_{\phi}(t) - r_1 V_{\phi} - qC \right] \exp (-it) dt \right. \\
+ \left. \int_{t_1}^{\tau} \left[ pM_{\phi_1}(t) - r_2 V_{\phi_1} \right] \exp (-it) dt \right\}
\]

subject to \( nC t_1 = I \), \( n \tau = T \), \( 0 \leq t_1 \leq \tau \leq T \), where \( p \) and \( q \) are the net revenue per unit, product sold per unit of time and cost of communication respectively, and are assumed to be non-negative constants over \( t \in [0, T] \). The two factors \( r_1 \) and \( r_2 \) (both $/time) are assumed known positive
constants and define the firm's attitude toward risk, expressed as variation in sales for both periods when the advertising is off and when it is on. The larger \( r_1 \) and \( r_2 \) the greater the cost of risk. In particular, they reflect the price the firm is willing to pay to "manage" its sales uncertainty [64]. In designing an APP, the economic as well as risk implications can now be considered in determining an optimal APP. The firm will seek an optimal APP which can also reduce the sales variation.

Notice that upon estimating the parameters, \( d_1, d_2, d_3, d_4, \beta_1 \) and \( \beta_2 \), one can find the optimal advertising pulsing policy strategy \((C^*, \tau^*, \tau^*, n^*)\), as is done in the next section.

6. NUMERICAL ANALYSIS AND APPLICATION

Consider a new product \((B_0 = 0)\) with \( A_0 = 1 \) (for a range 0, 100). A hypothetical record of sales and attitude over a period of (0, 15) is given in Table 1.

Applying two linear regressions to \( A_i \) and \( B_i \) and then testing for the parallel hypothesis gives:

\[
d_1 = 1.82, \quad d_2 = 0.12, \quad d_3 = 1.20 \quad \text{and} \quad d_4 = 0.32, \quad \text{with} \quad R = 0.89 \quad \text{(large enough)}.
\]

In terms of the original market parameters, we obtain \( a_1 = 1.2, \quad b_1 = 0.32, \quad a_2 = 0.5 \quad \text{and} \quad b_2 = 1.5 \). The means and variances functions used in the profit function (24) are derived in the Appendix. Let \( p = \Theta = 0.75, \quad q = 0.15, \quad r_1 = r_2 = 0.01 \quad \text{and} \quad \delta = 0.10 \), we wish to design the optimal advertising pulsing policy over a horizon of \( T = 6 \), given the total available budget \( I = 1.5 \). The optimal strategy \((C^*, \tau^*, \tau^*, n^*)\) is equivalent to the solution of an optimal (mixed-integer) control problem (23).

To utilize the existing efficient and robust numerical methods of nonlinear programming, we must convert Problem P into a nonlinear program. The most common approach is to discretize the problem. This allows us to solve the resulting problems by any nonlinear programming code [e.g. 66], while iterating on \( n \). However, algorithms based on Lagrangian quadratic programming are strongly recommended [67]. Powell [68] proposed such an algorithm with quadratic convergence rate.

Using the Kraft method coupled with the Powell algorithm, the optimal strategy is found to be \((C^* = 0.801, \quad \tau^* = 0.301, \quad \tau^* = 1.010, \quad n^* = 6)\), with the associated optimum profit \( J^* = 2.751 \). If we set \( n = 1 \) we would obtain the optimal “blitz” strategy—all advertising at the start of the planning period [23, 28] with \((C^* = 1.231, \quad \tau^* = 1.219, \quad \text{and} \quad \tau^* = 4.781)\) with optimal value \( J^* = 1.935 \). Since all existing consumer behaviour models assume a constant advertising rate over the entire planning horizon, in the following we report our results under such strategy.

Constant advertising level

A uniform advertising policy, known also as "even policy" applied over the entire interval \((0, 6)\), using the same market parameter and objective function

\[
J = \Theta M_{\beta_1}(T)\exp(-iT) + \int_0^T \left[pM_{\beta_2}(t) - r_1 J_B - qC\right]\exp(-it)dt,
\]

(24)
gives \( C^* = 1.470 \) with profit \( J^* = 0.793 \), which is substantially lower than the both optimal profit under pulsing policy and blitz strategy. It is clear that the return on constant advertising diminishes with time and hence additional expenditures on advertising will not bring about any increase in sales revenues. This indicates that substantial savings in advertising budget may be achieved by using an optimal APP strategy. As always, one must be careful when generalizing about this result, which was obtained from a numerical experiment on a specific market. Since all advertising is inherently discrete in a fine enough time frame we need to develop an empirically based definition as to what is constant and what is pulsing [69]. This paper does not pretend to solve the complexity of using continuous time but its main contribution is in the analysis of APP for consumer buying behavior models.

Table 1. Hypothetical purchase and attitude over a period of 15 units

<table>
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<tr>
<th>t</th>
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<tbody>
<tr>
<td>A_i</td>
<td>75</td>
<td>60</td>
<td>65</td>
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<tr>
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<td>72</td>
<td>117</td>
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<td>158</td>
<td>131</td>
<td>91</td>
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7. CONCLUDING REMARKS

In contrast to the ASR models which start with the assumption of specified sales response functions, consumer buying behavior models do not require limiting assumptions on the shape of the ASR functions. The fundamental difference in approach is the sense that advertising causes sales rather than that advertising strategies are determined by sales. We describe a consumer behavior model with a well-defined operational concept, based on the attitude of a given (statistically) homogenous market. The model presented in this study can be considered as both an extension and unification of the deterministic models. Since there is a complex multitude of social–psychological and cultural–environmental factors affecting purchase which cannot be captured or explained by any moderate-size deterministic model, we considered a stochastic model. We were able to obtain the mean and variance functions of sales generated under an APP. In the course of this analysis we developed an optimal strategy when following an APP.

Although the empirical studies support the notion of APP, they do not provide any prescriptive guidelines for scheduling such policy. The exact optimal APP strategy depends upon the value of four market parameters used in the descriptive model and few managerial factors defining the profit function. By having time records (i.e. realizations) of attitudes and sales, the market parameters can be estimated and applied in model validation using MLLS regression methods. These tools enable us to study whether or not the proposed model is consistent with the regularities of buyer behavior. Moreover, this paper provides the needed analytical and computational tools to determine the optimal scheduling.

Our model includes the large variation in sales caused by APP compared with smaller variation under a constant advertising rate; thus we have included this uncertainty in the profit function. In practice, some additional cost may occur under APP when advertising is stopped frequently [e.g. 25, 27]. Clearly, the proposed model serves as an aid to management judgment based on variables over which the manager has a measure of control, therefore the marketing manager’s experience must be incorporated in our proposed prescriptive model.

Our approach permits us to determine whether an APP is superior to a constant rate policy in a given market. Furthermore, if superior, allows us to find an optimal level of the decision variables to schedule such policy. Within the present paper, we have not been able to explore sensitivity of optimal solution with respect to the managerial parameters. To broach this issue we might start from empirical studies which use actual data, rather than artificial data used in this paper, to identify the model, estimate various model’s parameters and then simulate the effect of varying our key managerial parameters, such as risk factors. We did not consider pulsation in media, i.e. optimal policy over media [e.g. 70].

Our analysis assumed a stationary market in which the market parameters are constant over the duration of the campaign, a viable assumption for a short campaign. If a longer planning period is contemplated, then one may have to consider a nonstationary market, as well as the lagged effect of advertising. An adaptation of the scheme suggested by Erickson [71, 72] may be used. The proposed model and the derived optimal strategy were functions of force decay which may be caused by competitors. A more realistic approach assumes that competition would react to the firm’s action, possibly by a counter pulsation strategy such that competitors take turns at advertising.

A future area of the research would be the development of a model under pulsing/maintenance policy—a pulsing of high and low level of advertising, usually a maintenance level [e.g. 73]. This would more accurately reflect the world of advertising. Future work would include a study of our model behavior in a competitive environment, similar to approaches suggested by Park and Hahn [20] and Villas-Boas [74] who studied the behavior of the Lanchester ASR model [e.g. 75] in competitive markets. However, for operational purposes one must include an effective measure such as the profitability barrier [e.g. 76] as a strategy determinant for entry/exit into the competitive market. An immediate extension would be a design of an expert support system for implementation purposes similar to the works by Burke et al. [77] and Vask et al. [78].

Acknowledgements—The author is most appreciative of the comments and suggestions of the referees. Thanks are also due to the University of Baltimore for granting a sabbatical during which this work was completed and my colleague and friend Dr F. Niederman. A Robert G. Merrick School of Business grant allowed the author to start this research.
REFERENCES


APPENDIX

Outline Deviation of the Solution to Differential Equations (5)–(9) and (12)–(16)

Rewriting equations (18) and (21) we have

\[ M_1'(t) + d_1 M_1'(t) + d_2 M_1(t) = d_3 C \]  
(A1)

and

\[ M_2'(t) + d_1 M_2'(t) + d_2 M_2(t) = d_4 C. \]  
(A2)

The general solution of the second-order ordinary differential equations (A1) and (A2) under varying conditions are tabulated below:

<table>
<thead>
<tr>
<th>Conditions</th>
<th>( M_g )</th>
<th>( M_A )</th>
</tr>
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<tbody>
<tr>
<td>( d_1 \neq d_2, d_2 \neq 0 )</td>
<td>( R \exp(X_1 t) + S \exp(X_2 t) + K_g )</td>
<td>( R \exp(X_1 t) + S \exp(X_2 t) + K_A )</td>
</tr>
<tr>
<td>( d_1 \neq d_2, d_2 = 0 )</td>
<td>( P \exp(-d_1 t) + Q_g + d_3 C t/d_1 )</td>
<td>( P \exp(-d_1 t) + Q_A + d_4 C t/d_1 )</td>
</tr>
<tr>
<td>( d_1 = 4d_2 )</td>
<td>( H \exp(-X_1 t) + L \exp(X_1 t) + K_g )</td>
<td>( H \exp(-X_1 t) + L \exp(X_1 t) + K_A )</td>
</tr>
</tbody>
</table>

where

\[ X_1 = -d_1/2 + [(d_1^2 - 4d_2)/4]^{1/2}, \quad X_2 = -d_1/2 - [(d_1^2 - 4d_2)/4]^{1/2}, \quad K_g = d_3 C/d_2, \]

\[ S_g = (B_9 X_1 - B_9 - K_A X_1)(X_1 - X_2), \quad B_9 = a_1 A_0 - b_1 B_0, \quad B_9 = K_A - S_9 + B_9, \quad P_9 = d_3 C/d_1 - B_9/d_1, \]

\[ Q_g = B_9 P_9, \quad H_g = B_9 - K_A, \quad L_g = B_9 - X_1 H_g, \quad B_9 = a_1 A_0 - b_1 B_0, \quad K_g = d_4 C/d_2, \]

\[ S_A = (A_0 X_1 - A_0 - K_A X_1)(X_1 - X_2), \quad A_0 = a_2 A_0 - b_2 A_0, \quad R_A = K_A - S_A + A_0, \quad P_A = \frac{d_4 C}{d_1} - A_0/d_1, \]

Having \( M_g \) and \( M_A \) at our disposal, we can solve the differential equations (7)–(9), which are rewritten as a nonhomogenous system of differential equations:

\[
\begin{bmatrix}
V_g \\
V_A \\
W_{AB}
\end{bmatrix} =
\begin{bmatrix}
-2b_1 & 0 & 2a_1 \\
0 & -2b_2 & 2a_2 \\
a_2 & a_1 - b_1 & 0
\end{bmatrix}
\begin{bmatrix}
V_g \\
V_A \\
W_{AB}
\end{bmatrix} + 
\begin{bmatrix}
a_1 M_g \\
0
\end{bmatrix}.
\]

We need to find a particular solution of this system and add it to the general solution of the associated homogeneous system.
system. What is needed is the solution to the following characteristics (i.e. auxiliary) equation of the homogeneous part,

$$\mu^3 - (a_1 - 3b_1 - 2b_2)\mu^2 - (-2b_1^2 - 6b_1b_2 + 2a_1b_1 + 4a_1a_2)\mu + 4b_1b_2 - 2b_1b_2a_1 - 4a_1a_2b_2 - 4a_1a_2b_1 = 0, \quad (A3)$$

to obtain the eigenvalues. The nature of the solution of the homogeneous part depends on whether the eigenvalues are real and distinct, complex, or repeated. For details see, for example, Ref. [79]. Let $z_2 = -(a_1 - 3b_1 - 2b_2)$, $z_1 = (-2b_1^2 - 6b_1b_2 + 2a_1b_1 + 4a_1a_2)$ and $z_0 = 4b_1b_2 - 2b_1b_2a_1 - 4a_1a_2b_2 - 4a_1a_2b_1$. Furthermore, let $\delta = (z_1/z_2 - 3z_1/9)^2 + (z_2 - z_0)^2/27$ then if

- $\delta > 0$ one real root and a pair of complex conjugate roots,
- $\delta = 0$ all real, at least two are equal,
- $\delta < 0$ all roots real.

Upon finding the eigenvalues the "general solution" can be obtained. The "particular solution" part depends on the forms of $M_A$ and $M_B$. For details see Ref. [79]. Similarly, one can solve the system of differential equations (12)–(16). the results are:

$$M_A(t) = M_A(t_1)\exp[-\beta_1(t - t_1)], \quad t_1 \leq t \leq t, \quad (A4)$$

$$M_B(t) = M_B(t_1)\exp[-\beta_2(t - t_1)], \quad t_1 \leq t \leq t, \quad (A5)$$

$$V_{h1}(t) = V_{h1}(t_1)\exp[-2\beta_1(t - t_1)], \quad t_1 \leq t \leq t, \quad (A6)$$

$$W_{h1}(t) = W_{h1}(t_1)\exp[-\beta_1(t - t_1)], \quad t_1 \leq t \leq t, \quad (A7)$$

and

$$V_{h1} = h\exp[-2\beta_1(t - t_1)] + V_{h1}, \quad t_1 \leq t \leq t, \quad (A8)$$

where $h$ is a constant which can be found by using the initial condition $V_{h1}(t_1) = V_{h1}(t_1)$ and $V_{h1}$ is a particular solution of the nonhomogeneous equation (14), which depends on a particular form of $M_A(t)$ as well as whether $-2\beta_2 = X_1$ and/or $-2\beta_2 = X_2$; for details see Ref. [79]. For clarity of our discussions we present a numerical example.

**A Numerical Example**

Let $d_1 = 0.49$, $d_2 = -2.42$, $d_3 = 1.20$ and $d_4 = -1.01$. In terms of the original market parameters, we obtain $a_1 = 1.2$, $b_1 = 0.32$, $a_2 = 0.5$ and $b_2 = 1.5$. This gives

$$M_A = (2.5794C + 0.9673)\exp(0.0637t) + (0.0873C + 0.0327)\exp(-1.883) - 2.6667C \quad (A9)$$

and

$$M_B = (10.3272C + 0.6164)\exp(0.0637t) - (0.3272C + 0.6164)\exp(-1.883) - 10C \quad (A10)$$

The characteristic equation (A3) becomes

$$\mu^3 + 2.760\mu^2 - 3.683\mu - 6.058 = 0. \quad (A11)$$

Since $\delta = -8.614 < 0$, therefore all roots real. The three roots are $\mu_1 = 1.6594$, $\mu_2 = -1.0996$ and $\mu_3 = -3.3198$, which can be verified by inspection. Eigenvectors for these eigenvalues are $[1, 0.2056, 0.9581]^T$, $[1, -0.1008, -0.1915]^T$ and $[1, 3.4916, -1.1166]^T$, respectively, where $T$ means transpose. The general solution of the homogeneous system is

$$V_A = g_1[1.0206\exp(1.6594t) + 0.9581\exp(-1.0996t) + 1.0115\exp(-3.3198t)] \quad (A12)$$

We now seek a solution to the nonhomogeneous equation. We will use the variation of parameters technique to find the particular solution. The particular solution is equation (A12) with the following $g_1(t)$, $g_2(t)$ and $g_3(t)$:

$$g_1(t) = (-0.3131 - 2.5611C)\exp(-1.5957t) + (0.0523 - 0.0370C)\exp(-3.5424t) + 7.5070C\exp(-1.6594t),$$

$$g_2(t) = (-0.5269 - 0.4005C)\exp(1.1633t) + (0.1325 + 0.0239C)\exp(-0.7834t) - 11.9469C\exp(1.0996t)$$

and

$$g_3(t) = (0.1383 + 0.7429C)\exp(3.3835t) + (-0.0537 + 0.0569C)\exp(1.4368t) - 3.7370C\exp(3.3198t) \quad (A13)$$

Using the initial conditions $V_A = V_A = W_{h1} = 0$ we obtain: $g_1 = -3.9990C + 0.3425$, $g_2 = 11.4912C - 0.3186$ and $g_3 = 2.8596C + 0.0169$. Finally, by adding this to the general solution (A5), the solution to the nonhomogeneous system is:

$$V_A = t(3.3990C + 0.3425)\exp(1.6594t) + (11.4912C - 0.3186)\exp(-1.0996t) + (2.8596C + 0.0169)\exp(-3.3198t)$$

- (0.3131 + 2.5611C)\exp(-1.5957t) + (0.0523 - 0.0370C)\exp(-3.5424t) + 7.5070C\exp(-1.6594t),$$

$$- (0.5169 + 0.4005C)\exp(1.1633t) + (0.1325 + 0.0239C)\exp(-0.7834t) - 11.9469C\exp(1.0996t) + (0.1383 + 0.7429C)\exp(3.3835t) + (-0.0537 + 0.0569C)\exp(1.4368t) - 3.7370C\exp(3.3198t) \quad (A13)$$

$$V_A = (-8.2221C + 0.0704)\exp(1.6594t) - (1.1583C - 0.0321)\exp(-1.0996t) + (9.9846C + 0.0590)\exp(-3.3198t)$$

- (0.0643 + 0.5266C)\exp(-1.5957t) + (0.0108 - 0.0076C)\exp(-3.5424t) + 1.5434C\exp(-1.6594t)$$

+ (0.0521 + 0.0404C)\exp(1.1633t) - (0.0134 + 0.0024C)\exp(-0.7834t) + 1.2042C\exp(1.0996t)$$

+ (0.4829 + 2.5939C)\exp(3.3835t) + (-0.1875 + 0.1987C)\exp(1.4368t) - 13.048C\exp(3.3198t) \quad (A14)$$

CAOR 20:1-0
and

\[ W_{st} = ( -3.8314C + 0.3281 )\exp(1.6594t) - (2.2006C - 0.0610)\exp(-1.0996t) - (3.1930C + 0.0189)\exp(-3.3198t) \\
+ (0.3000 + 2.4539C)\exp(-1.5957t) + (0.0501 - 0.0061C)\exp(-1.0996t) \\
+ (0.0990 + 0.0767C)\exp(1.1633t) - (0.0254 + 0.0046C)\exp(-0.7834t) + 2.2878C\exp(1.0996t) \\
- (0.1544 + 0.8295C)\exp(3.3835t) - (0.0600 + 0.0635C)\exp(1.4368t) + 4.1727C\exp(3.3198t). \quad (A15) \]

Assuming the sales and attitude decay (when \( C = 0 \)) factors to be \( \beta_1 = 0.06 \) and \( \beta_2 = 0.04 \), respectively, then from equations (A4)-(A7), we have:

\[
M_{\alpha_1}(t) = M_{\alpha_1}(t_1)\exp[-0.04(t - t_1)], \quad t_1 < t < \tau, \quad (A16)
\]

\[
M_{\beta_1}(t) = M_{\beta_1}(t_1)\exp[-0.06(t - t_1)], \quad t_1 < t < \tau, \quad (A17)
\]

\[
V_{\alpha_1}(t) = V_{\alpha_1}(t_1)\exp[-0.12(t - t_1)], \quad t_1 < t < \tau \quad (A18)
\]

and

\[
V_{\alpha_1} = h\exp(-2\beta_2t) + V_{\alpha_0}, \quad t_1 < t < \tau, \quad (A19)
\]

where

\[
h = V_{\alpha_1}(t_1)\exp(0.08t_1) - (0.7180C + 0.2693)\exp(0.1437t_1) + (0.0019C + 0.0007)\exp(-1.803t_1) + 1.3334C\exp(0.08t_1) \]

and, finally,

\[
V_{\alpha_1}(t) = V_{\alpha_1}(t_1)\exp[-0.08(t - t_1)] - (0.7180C + 0.2693)\exp(-1.803t_1) - \exp(0.06371t) \\
+ (0.0019C + 0.0007)\exp(-0.08t - 1.803t_1) - \exp(-1.883t_1) \quad t_1 < t < \tau. \quad (A20)
\]