Abstract

Let $G$ be a $K_4$-minor-free graph with maximum degree $\Delta$. It is known that if $\Delta \in \{2, 3\}$ then $G^2$ is $(\Delta + 2)$-degenerate, so that $\chi(G^2) \leq \chi(G) \leq \Delta + 3$. It is also known that if $\Delta \geq 4$ then $G^2$ is $(\lfloor \frac{3}{2} \Delta \rfloor + 1)$-degenerate and $\chi(G^2) \leq \lfloor \frac{3}{2} \Delta \rfloor + 1$. It is proved here that if $\Delta \geq 4$ then $G^2$ is $\lceil \frac{3}{2} \Delta \rceil$-degenerate and $\chi(G^2) \leq \lfloor \frac{3}{2} \Delta \rfloor + 1$. These results are sharp.

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1. Introduction

We use standard terminology, as defined in the references: for example [2,5]. The square $G^2$ of a graph $G$ has the same vertex-set as $G$, and two vertices are adjacent in $G^2$ if they are within distance two of each other in $G$.

There is great interest in discovering classes of graphs $G$ for which the choosability or list chromatic number $\chi(G)$ is equal to the chromatic number $\chi(G)$. The list-square-colouring conjecture (LSCC) [2] is that, for every graph $G$, $\chi(G^2) = \chi(G^2)$. It is clear that this conjecture holds when the maximum degree $\Delta(G)$ of $G$ is 0 or 1. For $\Delta(G) = 2$, it can be deduced from the results of [4]. Specifically, we can state the following, in which we say that a graph $G$ is cycle-$k$-divisible if every cycle in $G$ has length divisible by $k$.

**Theorem 1.** If $G$ is a graph with maximum degree 2, then

$$\chi(G^2) = \chi(G^2) = \begin{cases} 3 & \text{if } G \text{ is cycle-3-divisible}, \\ 5 & \text{if } G \text{ has } C_5 \text{ as a component}, \\ 4 & \text{otherwise}. \end{cases}$$

For a $K_4$-minor-free graph with maximum degree $\Delta \geq 3$ we cannot prove that $\chi(G^2) = \chi(G^2)$, but we can prove the same sharp upper bound for $\chi(G^2)$ as for $\chi(G^2)$. Specifically, the purpose of this paper is to prove the following result, in which $\text{degenc}eracy}(G)$ is the smallest integer $k$ such that $G$ is $k$-degenerate, that is, every subgraph of $G$ contains a vertex with degree at most $k$.

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**Theorem 2.** Let \( G \) be a \( K_4 \)-minor-free graph with maximum degree \( \Delta \). Then
\[
\text{ch}(G^2) \leq \begin{cases} 
\Delta + 3 & \text{if } \Delta = 2 \text{ or } 3, \\
\lfloor \frac{3}{2} \Delta \rfloor + 1 & \text{if } \Delta \geq 4, 
\end{cases}
\]
and
\[
\text{degeneracy}(G^2) \leq \begin{cases} 
\Delta + 2 & \text{if } \Delta = 2 \text{ or } 3, \\
\lceil \frac{3}{2} \Delta \rceil & \text{if } \Delta \geq 4.
\end{cases}
\]

Lih et al. [3] obtained the same upper bounds as in (1) but for \( \chi(G^2) \) rather than \( \text{ch}(G^2) \), and they gave examples to show that these bounds are sharp (see Fig. 1 for the cases \( 3 \leq \Delta \leq 7 \)). Their examples all have the property that \( G^2 \) is a complete graph. We strongly suspect that these bounds are only attained when \( G \) has a block \( B \), of the order given in the bound, such that \( B^2 \) is complete. Theorem 1 shows that this is true when \( \Delta = 2 \) (since \( C_5 \cong K_5 \)), but we cannot prove it in general.

In [3] the authors proved also the weaker form of (2) with \( \lfloor \frac{3}{2} \Delta \rfloor + 1 \) in place of \( \lceil \frac{3}{2} \Delta \rceil \), and they gave examples claiming to show that it is sharp; but their examples for even values of \( \Delta \) are wrong. However, their examples for odd \( \Delta \) are correct, and can easily be modified to show that the bound in (2) is sharp even when \( \Delta \) is even. To be specific, let \( k \geq 2 \), and let \( G_{2k} \) be formed from two nonadjacent edges \( uv \) and \( wx \) by adding \( k - 1 \) paths of length 2 between \( u \) and \( v \), and between \( w \) and \( x \), and adding \( k \) paths of length 2 between \( u \) and \( w \), and between \( v \) and \( x \). Then \( G_{2k} \) is \( K_4 \)-minor-free, and has maximum degree \( 2k \), and the minimum degree of \( G_{2k}^2 \) is \( 3k \). (For the examples when \( \Delta \) is odd, given in [3], form \( G_{2k+1} \) from \( G_{2k} \) by adding a further path of length 2 between \( u \) and \( w \), and another between \( v \) and \( x \). Then \( G_{2k+1} \) has maximum degree \( \Delta = 2k + 1 \), and the minimum degree of \( G_{2k+1}^2 \) is \( 3k + 2 = \lceil \frac{3}{2} \Delta \rceil \).)

In proving Theorem 2 we will make use of the following result of Dirac [1].

**Theorem 3 (Dirac [1]).** Every \( K_4 \)-minor-free graph has a vertex with degree at most 2.

If \( G \) is a graph such that \( \Delta(G) \geq 3 \), then \( G_1 \) will denote the graph whose vertices are the vertices that have degree at least 3 in \( G \), where two vertices are adjacent in \( G_1 \) if and only if they are connected in \( G \) by an edge or by a path whose internal vertices all have degree 2 in \( G \). So \( G_1 \) exists if and only if \( \Delta(G) \geq 3 \). Clearly \( G_1 \) is a minor of \( G \). The following result is not difficult to see.

**Theorem 4.** If \( G \) is a graph that does not contain a vertex with degree 0 or 1 or two adjacent vertices with degree 2, then \( G_1 \) exists and has no vertex with degree 0. If, in addition, \( G \) does not contain a 4-cycle \( xuwy \) such that \( u \) and \( v \) both have degree 2 in \( G \), then \( G_1 \) has no vertex with degree 1.

We will denote \( (G_1)_1 \) by \( G_2 \). As usual, \( N(v) = N_G(v) \) will denote the set, and \( d(v) = d_G(v) \) will denote the number, of vertices adjacent to \( v \) in the graph \( G \).

**2. Proof of Theorem 2**

The rest of this paper is devoted to a proof of Theorem 2. Lih et al. [3] proved that if \( G \) is a \( K_4 \)-minor-free graph such that \( \Delta(G) = 2 \) or 3, then \( G^2 \) is \( (\Delta(G) + 2) \)-degenerate, and it follows immediately from this that \( \text{ch}(G^2) \leq \Delta(G) + 3 \). Thus to prove Theorem 2 it suffices to prove the result for \( \Delta(G) \geq 4 \), which we restate as follows.
Theorem 5. Let $G$ be a $K_4$-minor-free graph with maximum degree $\Delta \geq 4$. Then $\chi(G^2) \leq \lceil \frac{3}{2} \Delta \rceil + 1$ and $G^2$ is $\lceil \frac{3}{2} \Delta \rceil$-degenerate.

Proof. Fix the value of $\Delta \geq 4$, and note that $\Delta + 2 \leq \lceil \frac{3}{2} \Delta \rceil$ and $\Delta + 3 \leq \lceil \frac{3}{2} \Delta \rceil + 1$. Suppose if possible that $G_c$ and $G_d$ are $K_4$-minor-free graphs with maximum degree at most $\Delta$ and as few vertices as possible such that $\chi(G_c^2) > \lceil \frac{3}{2} \Delta \rceil + 1$ and $G_3^d$ is not $\lceil \frac{3}{2} \Delta \rceil$-degenerate. Then

$$A + 3 \leq \lceil \frac{3}{2} A \rceil + 1 \leq \lceil \frac{3}{2} A \rceil + 1 \leq \delta(G_3^d).$$

(3)

Assume that every vertex $v$ of $G_c$ is given a list $L(v)$ of $\lceil \frac{3}{2} A \rceil + 1$ colours in such a way that $G_c^2$ has no proper colouring from these lists. Let $G$ denote $G_c$ or $G_d$. We will prove various statements about $G$. Clearly $G$ is connected.

Claim 1. $G$ does not contain a vertex of degree 1, or two adjacent vertices of degree 2.

Proof. Suppose $G$ contains a vertex $u$ of degree 1, or two adjacent vertices $v, w$ of degree 2. Then

$$(G - u)^2 = G^2 - u, \quad (G - \{v, w\})^2 = G^2 - \{v, w\},$$

(4)

$$d_{G^2}(u) \leq \Delta \quad \text{and} \quad d_{G^2}(v), d_{G^2}(w) \leq \Delta + 2 \leq \lceil \frac{3}{2} A \rceil + 1 \leq \delta(G_c^2)$$

(5)

by (3). This is a contradiction if $G = G_d$, and so we may suppose that $G = G_c$. By the minimality of $G_c$ there is a colouring of $(G - u)^2$ or $(G - \{v, w\})^2$ from its lists, and this colouring can be extended to $G^2$ by (4) and (5). This contradiction shows that $G$ contains no such vertex $u$ or vertices $v, w$. \hfill \Box

Claim 2. The graph $G_1$ (defined before Theorem 4) exists, and has no vertex with degree 0 or 1, and at least one vertex with degree 2.

Proof. By Theorem 4 and Claim 1, $G_1$ exists and has no vertex with degree 0. Suppose $G_1$ has a vertex $u$ with exactly one neighbour $x$ in $G_1$. Then $x$ may or may not be a $G$-neighbour of $u$, but every $G$-neighbour of $u$ different from $x$ is a vertex of degree 2 that is adjacent to $x$. Thus $(G - u)^2 = G^2 - u$ and $d_{G^2}(u) \leq d_G(x) + 1 \leq \Delta + 1 < \delta(G_c^2)$, and if $G = G_c$ then a colouring of $G^2 - u$ from its lists can be extended to $G^2$. This contradiction shows that $G_1$ has no vertex with degree 1. Since $G_1$ is a minor of $G$ and so is $K_4$-minor-free, it follows from Theorem 3 that $G_1$ must have a vertex with degree 2. This completes the proof of Claim 2. \hfill \Box

Before considering a vertex with degree 2 in $G_1$, we will consider an arbitrary vertex $w$ with degree 2 in $G$. If the neighbours of $w$ are $u, v$, say, let $M_{uv}$ be the set, and $m_{uv}$ the number, of vertices of degree 2 in $G$ with the same neighbours $u, v$ as $w$ (so that $w \in M_{uv}$), and suppose there are $m_{uv}'$ vertices of degree greater than 2 in $G$ that are adjacent to both $u$ and $v$. Let $H := G - w$ if $uv \in E(G)$ and $H := (G - w) + uv$ otherwise, so that $G^2 - w \subseteq H^2$. By (3), and since a colouring of $H^2$ can be extended to $G^2$ if $d_{G^2}(v) \leq \lceil \frac{3}{2} A \rceil$, we may assume that

$$d_{G^2}(w) \geq \lceil \frac{3}{2} A \rceil + 1 \geq A + 3.$$  

(6)

However,

$$d_{G^2}(w) \leq d_G(u) + d_G(v) - m_{uv} - m_{uv}' + 1 - 2e_{uv},$$

(7)

where $e_{uv} = 1$ if $u, v$ are adjacent in $G$ and 0 otherwise. We will use this terminology in what follows.

Claim 3. $A$ is odd, say $A = 2k + 1$, where $k \geq 2$. Also, every vertex of degree 2 in $G_1$ looks in $G$ like vertex $u$ of Fig. 2, where $x$ and $y$ are nonadjacent and are the only vertices in Fig. 2 with neighbours that are not shown, and $d_G(x) = d_G(u) = d_G(y) = A = 2k + 1$ and $d_G(z) = \lceil \frac{3}{2} A \rceil = 2k + 2$ for every vertex $z \in M_{ux} \cup M_{uy}$.

Proof. It follows from Claim 2 that there is a vertex with degree 2 in $G_1$. Let $u$ be any such vertex, with neighbours $x, y$ in $G_1$, so that

$$d_G(u) = m_{ux} + m_{uy} + e_{ux} + e_{uy}. $$

(8)
By the definition of $G_1$, $d_G(u) \geq 3$, and so $m_{ux}$ and $m_{uy}$ are not both zero. If $m_{ux} \neq 0$ and $w \in M_{ux}$, then (7) and (8) give
\[
d_{G^2}(w) \leq m_{ux} + m_{uy} + e_{ux} + e_{uy} + d_G(x) - m_{ux} - m'_{ux} + 1 - 2e_{ux} \\
\leq A + 1 + m_{ay} - e_{ux} + e_{uy}.
\]
(9)

If $m_{ux} = 0$ and $w \in M_{ux}$, then (9) gives $d_{G^2}(w) \leq A + 2$, which contradicts (6); and the same holds by symmetry if $m_{uy} = 0$ and $w \in M_{uy}$. Thus $m_{ux}$ and $m_{uy}$ are both nonzero. Let $w \in M_{ux}$ and $w' \in M_{uy}$. Then, by analogy with (9),
\[
d_{G^2}(w') \leq A + 1 + m_{ux} + e_{ux} - e_{uy}.
\]
(10)

Therefore
\[
\min\{d_{G^2}(w), d_{G^2}(w')\} \leq A + 1 + \frac{1}{2}(m_{ux} + m_{uy}) \\
= A + 1 + \frac{1}{2}(d_G(u) - e_{ux} - e_{uy})
\]
(11)

by (8). It follows that $\min\{d_{G^2}(w), d_{G^2}(w')\} \leq \frac{3}{2}A + 1$.

Suppose first that $e_{ux} = e_{uy} = 0$. Then
\[
d_{G^2}(u) \leq d_G(u) + 2 \leq A + 2 < \delta(G^2_1)
\]
by (3). This is a contradiction if $G = G_d$; so suppose $G = G_c$, and suppose w.l.o.g. $d_{G^2}(w) \leq d_{G^2}(w')$. Then we can colour $G^2$ from its lists by first colouring $G^2 - w$, which is possible by the minimality of $G_c$, then uncolouring $u$, then colouring $w$, and finally colouring $u$. This contradiction shows that $e_{ux} + e_{uy} \geq 1$.

If $A$ is even, then it follows from (11) that $\min\{d_{G^2}(w), d_{G^2}(w')\} \leq \frac{3}{2}A$, which contradicts (6). So $A$ must be odd, say $A = 2k + 1$ and $\lceil \frac{3}{2}A \rceil + 1 = 3k + 2$, where $k \geq 2$ since $A \geq 4$ by the hypothesis of the theorem. In order to avoid the contradiction $\min\{d_{G^2}(w), d_{G^2}(w')\} \leq \lceil \frac{3}{2}A \rceil = 3k + 1$, necessarily $d_G(u) = A$, $e_{ux} + e_{uy} = 1$, and equality holds in (11). Therefore equality holds in (9) and (10), and $d_{G^2}(w) = d_{G^2}(w') = 3k + 2 = \lceil \frac{3}{2}A \rceil$. Assuming w.l.o.g. that $e_{ux} = 0$ and $e_{uy} = 1$, (9) and (10) give $m_{uy} = k - 1$ and $m_{ux} = k + 1$. Moreover, for equality to hold in (9) and (10), necessarily $d_G(x) = d_G(y) = A$ and $m'_{ux} = m'_{uy} = 0$. In particular, since $m'_{ux} = 0$, there is no edge $xy$ in $G$. This completes the proof of Claim 3.

Since Claim 3 contradicts (3) if $G = G_d$, this completes the proof that $G^2$ is $\lceil \frac{3}{2}A \rceil$-degenerate. So from now on we will assume that $G = G_c$, and that every vertex of $G$ has a list of $\lceil \frac{3}{2}A \rceil + 1 = 3k + 2$ colours.

**Claim 4.** $G_1$ does not contain two adjacent vertices with degree 2.

**Proof.** Suppose it does. Then, in $G$, these vertices occur as $u$ and $v$ in Fig. 3(a) or (b), where $x$ and $y$ are the only vertices with neighbours that are not shown, and $x \neq y$, since the maximum degree $\Delta(G) = 2k + 1$ would be exceeded if $x = y$ in Fig. 3(a), and by Claim 3, $x$ and $y$ must not be adjacent in Fig. 3(b). Possibly $x$ and $y$ are adjacent, in which case $x$ counts as one of the $k + 1$ ‘unshown’ neighbours of $y$, and vice versa; this does not affect the following argument. Note that $M_{ux} \neq \emptyset$ since $k \geq 2$. If $w \in M_{ux}$ then $G^2 - w = (G - w)^2$. Let us colour $G^2 - w$ from its lists, and then uncolour all the vertices in $M_{ux} \cup M_{uy} \cup M_{vy}$. For each uncoloured vertex $z$, let $L'(z)$ denote the ‘residual list’ of colours in $L(z)$ that are not used on any $G^2$-neighbour of $z$ and so are still available for use on $z$. At this point every vertex $z \in M_{ux} \cup M_{vy}$ has $k + 3$ coloured neighbours in $G^2$ and so $|L'(z)| \geq 2k - 1$. Note that all the uncoloured
vertices have degree \( \lceil \frac{3}{2} \Delta \rceil + 1 = 3k + 2 \) in \( G^2 \), and so if we try to recolour first the vertices in \( M_{ux} \) and then those in \( M_{uv} \cup M_{vy} \), it is only at the last vertex to be coloured that we may fail.

Let \( w \in M_{ux} \), \( w' \in M_{uv} \) and \( w'' \in M_{vy} \). Then \( w'' \) has \( k + 3 \) coloured \( G^2 \)-neighbours and \( 2k - 1 \) uncoloured \( G^2 \)-neighbours, and \( |L'(w'')| \geq 2k - 1 \). In Fig. 3(a), \( u \) has only two coloured \( G^2 \)-neighbours, and is not used on any other \( G^2 \)-neighbour of \( w'' \), then we can change the colour of \( u \) to make \( |L'(w'')| = 2k \); then we can recolour all the vertices in \( M_{ux} \), then \( M_{uv} \), then \( M_{uy} \), ending with \( w'' \). This contradiction shows that \( u, v \) must be as in Fig. 3(b). Then \( w' \) has four coloured \( G^2 \)-neighbours, and so \( |L'(w')| \geq 3k - 2 \). If \( L'(w) \cap L'(w'') = \emptyset \), then we can give \( w \) and \( w'' \) the same colour, then recolour all remaining vertices in \( M_{ux} \cup M_{vy} \), and then recolour those in \( M_{uv} \), which is possible since every vertex in \( M_{uv} \) has two \( G^2 \)-neighbours with the same colour. So we may suppose that \( L'(w) \cap L'(w'') = \emptyset \), so that \( |L'(w) \cup L'(w'')| \geq 4k - 2 \). Thus either \( |L'(w)| \geq 4k - 2 > 3k - 2 \), or else \( w \) or \( w'' \) can be given a colour not in \( L'(w') \). In either case, the remaining vertices can now be coloured, with \( w' \) being coloured last. This contradiction completes the proof of Claim 4. □

**Claim 5.** \( G_1 \) does not contain a 4-cycle \( xuyvx \) in which \( u \) and \( v \) both have degree 2.

**Proof.** Suppose it does. Then, by Claim 3, \( u \) would contribute \( k \) to the degree of one of \( x, y \) in \( G \) and \( k + 1 \) to the degree of the other in \( G \), and so would \( v \), so that (since \( \Delta(G) = 2k + 1 \)) \( x \) and \( y \) could have no other neighbours. Thus \( x, u, y \) and \( v \) would all have degree 2 in \( G_1 \), and this would contradict Claim 4. □

**Claim 6.** The graph \( G_2 = (G_1)_2 \) exists, and has no vertex with degree 0 or 1, and at least one vertex with degree 2.

**Proof.** It follows from Theorem 4 and Claims 2, 4 and 5 that the graph \( G_2 \) exists and has no vertex with degree 0 or 1. Since \( G_2 \) is a minor of \( G_1 \) and hence of \( G \), it follows from Theorem 3 that \( G_2 \) must have a vertex with degree 2. □

We will now establish a contradiction by proving the following.

**Claim 7.** \( G_2 \) contains no vertex with degree 2.

**Proof.** Let \( y \) be a vertex of degree 2 in \( G_2 \), with neighbours \( x, y' \). Since \( y \) has degree at least 3 in \( G_1 \), it follows from Claims 4 and 5 that \( y \) appears in \( G_1 \) as in Fig. 4(a) or (b), where \( x \) and \( y' \) are the only vertices with neighbours that are
not shown. (Note that if \( y \) appears as in Fig. 4(b) but with the edge \( yy' \) missing, then it also appears as in Fig. 4(a).)

However, by Claim 3, each of \( u \) and \( u' \) will contribute at least \( k \) edges towards the degree of \( y \) in \( G \), and so if \( y \) is as in Fig. 4(b) in \( G_1 \) then \( d_G(y) \geq 2k + 2 > \Delta \), a contradiction. So \( y \) occurs in \( G_1 \) as in Fig. 4(a), and so by Claim 3 it occurs in \( G \) as in Fig. 5(a) or (b), where \( x \) and \( y' \) are the only vertices with neighbours that are not shown, and \( e_{yy'} = 1 \) if there is an edge \( yy' \) in \( G \) and 0 otherwise. (Possibly one of the edges from \( x \) to an ‘unshown’ neighbour actually goes to \( y' \); this does not affect the following argument.) There is at least one edge of \( G \) from \( x \) to an unshown vertex (or to \( y' \)), since otherwise \( u \) and \( x \) are adjacent vertices of degree 2 in \( G_1 \), contrary to Claim 4. Thus \( m_{xy} \leq k - 1 \) in Fig. 5(a), and the same is true in Fig. 5(b) since otherwise \( d_G(y) \) would be at least \( 2k + 2 > \Delta \) in view of the edge between \( y \) and \( y' \) in \( G_1 \). This implies that \( M_{yy'} \neq \emptyset \) in Fig. 5(a) (but possibly \( M_{yy} = \emptyset \) in Fig. 5(b)). Since we know that \( x \) and \( y \) are adjacent in \( G_1 \) by Fig. 4(a) but not in \( G \) by Claim 3, there must be at least one vertex of degree 2 between them in \( G \).

Thus \( 1 \leq m_{xy} \leq k - 1 \).

Let \( w \in M_{ux} \), \( w' \in M_{uy} \) and \( w'' \in M_{xy} \), and in case (a) let \( w''' \in M_{yy'} \). Colour \( G^2 - w' \) from its lists, and then uncolour all the vertices in

\[
\{u, y\} \cup M_{ux} \cup M_{uy} \cup M_{xy} \cup M_{yy'},
\]

leaving \( x \) and \( y' \) coloured. For each uncoloured vertex \( z \), let \( L'(z) \) denote the residual list of colours that can be used on \( z \). Since \( |L(z)| = 3k + 2 = d_G(z) \) for every vertex \( z \in M_{ux} \cup M_{uy} \), there is no loss of generality in assuming that \( |L'(z)| \) is equal to the number of uncoloured \( G^2 \)-neighbours of \( z \), for all such \( z \). In particular, since \( x \) is a \( G^2 \)-neighbour of \( w' \) in case (b) but not in case (a), we may assume that

\[
|L'(w')| = \begin{cases} 3k + 2 - e_{yy'} & \text{in case (a),} \\ 3k + 1 - e_{yy'} & \text{in case (b).} \end{cases}
\]

We are going to recolour all the uncoloured vertices and then colour \( w' \) last. To do this, we will colour two \( G^2 \)-neighbours of \( w' \) (\( w \) and \( w'' \) in case (a), and \( w \) and \( y \) in case (b)) so that either they have the same colour, or one of them has a colour not in \( L'(w') \). Note that, since \( w \) can be given any colour not used on \( x \) or a \( G \)-neighbour of \( x \),

\[
|L'(w)| \geq \begin{cases} (3k + 2) - (k + 1 - m_{xy}) = 2k + 1 + m_{xy} & \text{in case (a),} \\ (3k + 2) - (k + 2 - m_{xy}) = 2k + m_{xy} & \text{in case (b).} \end{cases}
\]

In case (a), \( w''' \in N_{G^2}(w') \setminus N_{G^2}(w) \). Now, \( y' \in N_{G^2}(w'') \), and \( y' \) has \( k + 1 - m_{xy} \) uncoloured \( G \)-neighbours (including \( y \) if \( e_{yy'} = 1 \), hence at most \( \Delta - (k + 1 - m_{xy}) \) \( G \)-neighbours that are already coloured. Therefore

\[
|L'(w''')| \geq (3k + 2) - 1 - \Delta + (k + 1 - m_{xy}) = 2k + 1 - m_{xy},
\]

so that \( |L'(w)| + |L'(w'')| \geq 4k + 2 > |L'(w')| \).

In case (b), \( y \in N_{G^2}(w') \setminus N_{G^2}(w) \). Now, \( x, y' \in N_{G^2}(y) \), and \( y' \) has at most \( \Delta - (k - m_{xy}) \) \( G \)-neighbours that are already coloured, and so the number of coloured \( G \)-neighbours of \( y' \) that are in \( N_{G^2}(y) \setminus \{x\} \) is at most

\[
e_{yy'}[\Delta - (k - m_{xy})] \leq (e_{yy'} - 1) + \Delta - (k - m_{xy}).
\]
Thus
\[ |L'(y)| \geq (3k + 2) - 2 - (e_{y'y'} - 1) - A + (k - m_{xy}) = 2k - e_{y'y'} - m_{xy}, \]
so that \(|L'(w)| + |L'(y)| \geq 4k - e_{y'y'} > |L'(w')|\).

Let \(z := w'''\) in case (a) and \(z := y\) in case (b), so that \(|L'(w)| + |L'(z)| > |L'(w')|\) in each case. Colour \(w\) and \(z\) so that either they have the same colour or one of them has a colour that is not in \(L'(w')\). In case (a), we can now recolour all remaining vertices of \(M_{yy}\) and then \(y\), which is adjacent in \(G^2\) to at most \(\Delta + 2\) coloured vertices, namely \(w, x, y'\), and at most \(\Delta - 1\) coloured \(G\)-neighbours of \(y'\). In case (b), we can recolour all vertices (if any) in \(M_{yy'}\), the last of which to be recoloured is adjacent in \(G^2\) to at most \(\Delta + 1\) coloured vertices, namely \(y, y'\), and at most \(\Delta - 1\) coloured \(G\)-neighbours of \(y'\).

We can now recolour all vertices in \(M_{xy} \cup \{u\}\), since the last of these to be coloured will be adjacent in \(G^2\) to at most \(3k + 1\) coloured vertices, namely \(w, x, y\), and vertices that are adjacent to \(x\) and \(y\) in \(G\), a total of at most \(3 + (2k + 1) - m_{xy} \leq 2k + 3 \leq 3k + 1\). Finally, we can recolour all the vertices in \(M_{ux} \cup M_{uy}\) that are still uncoloured, ending with \(w'\), since each of these vertices has degree \(3k + 2\) in \(G^2\), and \(w'\) either has two \(G^2\)-neighbours with the same colour or has a \(G^2\)-neighbour \(w\) or \(z\) with a colour not in \(L'(w')\). Thus all vertices of \(G^2\) can be coloured from their lists, and this contradiction completes the proof of Claim 7. □

Claim 7 contradicts Claim 6, and this contradiction completes the proof of Theorem 5. □

**References**