NOTE

A Note on Fisher’s Inequality

Douglas R. Woodall

Department of Mathematics, University of Nottingham, Nottingham NG7 2RD, England

Communicated by Laci Babai

Received May 3, 1995

A new proof is given of the nonuniform version of Fisher’s inequality, first proved by Majumdar. The proof is “elementary,” in the sense of being purely combinatorial and not using ideas from linear algebra. However, no nonalgebraic proof of the $n$-dimensional analogue of this result (Theorem 3 herein) seems to be known.

$\text{A design } D \text{ consists of a family } B_1, ..., B_b \text{ of subsets, called blocks, of a finite set } S = \{P_1, ..., P_v\} \text{ whose elements are called points or varieties. } D \text{ is balanced or } \lambda \text{-linked if every pair of points is contained in exactly } \lambda \text{ blocks. If, in addition, } \lambda > 0 \text{ and no block contains all the points, then } D \text{ is non-trivial, and if every block has the same cardinality } k \text{ then } D \text{ is a balanced incomplete-block design or } BIBD.$

Fisher [5] proved that if $D$ is a BIBD, then $b \geq v$. Bose [3] gave a neat short proof of this result using a determinant. Majumdar [8] provided an easy modification of Bose’s method that extends the result to arbitrary non-trivial $\lambda$-linked designs, which one can think of as a nonuniform version of Fisher’s inequality. (The case $\lambda = 1$ of Majumdar’s result had been proved earlier by de Bruijn and Erdős [4].)

My attention has recently been drawn to the statement of Babai [1] that no proof of Majumdar’s inequality appears to be known that does not use some form of linear algebra trick. Accepting the challenge, I offer the proof below (Theorem 1). Fisher’s proof relies on the fact that the variance of the quantities $|B_i \cap B_j| (i \neq j)$, being a sum of squares, is nonnegative, and his proof shows that (when $D$ is a BIBD) these quantities are all equal if and only if $b = v$. The key to the proof below (which involved a fair amount of hindsight) was to discover a similar sum of squares in the nonuniform case, which is equal to zero if and only if $b = v$. 

171
Ryser [11] and Woodall [12] considered the case of equality in Majumdar’s theorem and independently made the same conjecture, which is still open, and is usually referred to in Ryser’s terminology as the $\lambda$-design conjecture. (A $\lambda$-design is what one gets by taking a non-trivial $\lambda$-linked design with $b = v$ that is not a BIBD, and dualizing it, that is, interchanging the roles of points and blocks. Most recent authors have followed Ryser in writing in this dual terminology, but I shall keep to the original formulation of Fisher, Bose and Majumdar.) Sadly if unsurprisingly, the following proof of Theorem 1 seems to give no extra information about the cases of equality that might help in proving the $\lambda$-design conjecture; as we shall see in Theorem 2, the equations obtained seem identical to those obtained by the use of linear algebra in [11] and [12].

**Theorem 1.** If $D$ is a non-trivial $\lambda$-linked design with $v$ points and $b$ blocks, then $b \geq v$.

**Proof.** For each point $P_a$ in $S$, let $r_a$ be the number of blocks containing $P_a$ and let $\rho_a := (r_a - \lambda)^{-1}$, called the residue of $P_a$. As in [12], we define

$$R' := \sum_{P_a \in S} \rho_a,$$
$$R_i := \sum_{P_a \in B_i} \rho_a,$$
$$\hat{R}_j := \sum_{P_a \in B_i \cap B_j} \rho_a$$

for $i, j \in \{1, \ldots, b\}$, and

$$R := R' + \lambda^{-1}. \tag{2}$$

(Note that $R_a = R_i$.) Since $r_a \rho_a = 1 + \lambda \rho_a$, and counting the number of times $\rho_a$ is involved in each sum, (1) gives

$$\sum_{i=1}^b R_i = \sum_{P_a \in S} r_a \rho_a = v + \lambda R' \tag{3}$$

and

$$\sum_{i=1}^b \hat{R}_j = \sum_{P_a \in B_i} r_a \rho_a = |B_j| + \lambda \hat{R}_j. \tag{4}$$

In a similar way,

$$\sum_{i=1}^b R_i^2 = \sum_{i=1}^b \left( \sum_{P_a \in B_i} \rho_a^2 + \sum_{P_a \in B_i} \rho_a \rho_B \right)$$

$$= \sum_{a} r_a \rho_a^2 + \sum_{a \neq B} \lambda \rho_a \rho_B$$

$$= \sum_{a} (r_a - \lambda) \rho_a^2 + \lambda \left( \sum_{a} \rho_a \right)^2 = R' + \lambda R^2 = \lambda RR'. \tag{5}$$
\begin{align*}
\sum_{i=1}^{b} R_{ij}^2 &= \sum_{P_r \in B_j} r_r \rho_r^2 + \sum_{P_r \neq P_eta \in B_j} \lambda \rho_r \rho_eta = R_j + \lambda R_j^2 \\
\text{and} \\
\sum_{i=1}^{b} R_j(R_i - R_j) &= \sum_{P_r \neq \rho_\beta \in B_j} \lambda \rho_r \rho_\beta = \lambda R_j(R' - R_j).
\end{align*}

By (5), (6) and (7),

\begin{align*}
0 \leq \sum_{i \neq j} (RR_{ij} - R_j R_i)^2 \\
= \sum_{i=1}^{b} \left[ (R^2 - 2 RR_j) R_{ij}^2 - 2 RR_j R_i(R_i - R_j) + R_i^2 R_j^2 \right] - (RR_j - R_j R_i)^2 \\
= (R^2 - 2 RR_j)(R_j + \lambda R_j^2) - 2 \lambda RR_j(R' - R_j) + \lambda RR' R_j^2 - (RR_j - R_j R_i)^2 \\
= R_j(R - R_j)(R - RR_j + R_j^2)
\end{align*}

since

\[(R^2 - 2 RR_j)(R_j + \lambda R_j^2) = R^2 R_j + \lambda RR'R_j^2 - RR_j^2 - 2 \lambda RR_j^3\]

by (2). Since $R_i > 0$ and $R - R_j > 0$, if follows from (9) that

\[R - RR_j + R_j^2 \geq 0\]  \hspace{1cm} (10)

for each $j$ ($1 \leq j \leq b$). Summing (10) over all $j$ and using (3) and (5) we obtain

\[bR - R(v + \lambda R') + \lambda RR' = R(b - v) \geq 0.\]

Since $R > 0$, this gives $b \geq v$ as required. \[\square\]

The above proof gives exactly the same information as the algebraic proof \cite{11, 12} about the cases of equality in Theorem 1:

**Theorem 2.** If $D$ is a non-trivial $\lambda$-linked design with $v$ points and $b$ blocks, where $b = v$, and $D$ is not a BIBD, then $D$ has blocks of exactly two distinct sizes $k_1$ and $k_2$, where $k_1 + k_2 = v + 1$. Moreover, if we define

\[S_1 = \frac{v - 1}{k_2 - 1} \quad \text{and} \quad S_2 = \frac{v - 1}{k_1 - 1},\]  \hspace{1cm} (11)
then $R_i = S_1$ or $S_2$ according as $|B_i| = k_1$ or $k_2$, and

$$
R_{ij} = \begin{cases} 
S_1 - 1 & \text{if } |B_i| = |B_j| = k_1, \\
S_2 - 1 & \text{if } |B_i| = |B_j| = k_2, \\
1 & \text{if } |B_i| \neq |B_j|.
\end{cases}
$$

(12)

Proof. If $b = v$, then all the inequalities in the proof of Theorem 1 are equalities, and (10) and (8) give

$$
R - RR_j + R_j^2 = 0
$$

(13)

for each $j$ and

$$
RR_{ij} - R_i R_j = 0
$$

(14)

whenever $i \neq j$ ($i, j \in \{1, ..., b\}$). By (13), there are at most two possible values for $R_j$, and if (anticipating somewhat) we denote these by $S_1$ and $S_2$ then

$$
S_1 + S_2 = S_1 S_2 = R.
$$

(15)

(13) and (14) can be combined as

$$
R(\delta_{ij} - R_{ij}) + R_i R_j = 0
$$

(16)

where $\delta_{ij}$ is the Kronecker delta, and summing (16) over all $i$, and using (2), (3) and (4), we find

$$
R(1 - |B_i| - i R_j) + R_j(v - 1 + i R) = 0,
$$

that is,

$$
|B_i| = 1 + R_j(v - 1)/R = k_1 \quad \text{or} \quad k_2,
$$

where

$$
k_1 = 1 + S_1(v - 1)/R, \quad k_2 = 1 + S_2(v - 1)/R
$$

(17)

and

$$
k_1 + k_2 = 2 + (S_1 + S_2)(v - 1)/R = 2 + v - 1 = v + 1
$$
by (15). Also, (15) and (17) give (11). Finally, if \(|B_i| = |B_j| = k_1\), then (14) and (13) give

\[ R_{ij} = \frac{S^2_i}{R} = \frac{R S_1 - R}{R} = S_1 - 1, \]

and the rest of (12) follows similarly and from (14) and (15).

In [13] I gave a short algebraic proof of the following \(n\)-dimensional analogue of Theorem 1.

**Theorem 3.** Suppose we are given a finite set \(S = \{P_1, \ldots, P_v\}\), positive integers \(n\) and \(\lambda_2, \ldots, \lambda_n\), and \(n\) families of proper subsets of \(S\) called \(t\)-blocks \((t = 1, 2, \ldots, n)\), such that

(i) the 1-blocks are precisely the singletons \(\{P_i\}\) of \(S\), and

(ii) for each \(t \geq 2\), for each \((t-1)\)-block \(B\) and each \(P_x\) in \(S\setminus B\) there are exactly \(\lambda_t\) \(t\)-blocks containing \(B\cup\{P_x\}\).

Then for each \(t\) \((1 \leq t \leq n)\), the number \(b_t\) of \(t\)-blocks satisfies \(b_t \geq v\).

The case \(n = 2\) of this result is Theorem 1. The case in which all the \(\lambda_s\) equal 1 is also well known, being the combinatorial analogue of Motzkin’s hyperplane inequality [10]. This asserts that if \(v\) points in a Euclidean or projective space do not all lie in the same hyperplane (= affine or projective subspace of codimension 1), then they determine at least \(v\) distinct hyperplanes. The combinatorial generalization of this is the analogous statements about matroids (that the number of hyperplanes is at least as large as the number of atoms); it follows easily from Motzkin’s work (see Mason [9]), and was proved directly and independently by Basterfield and Kelly [2], Greene [6] and Heron [7]. Although this special case can be proved without using ideas from linear algebra, I do not know of any non-algebraic proof of Theorem 3 itself.

**REFERENCES**


