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Extended Lagrange interpolation in weighted uniform norm

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A B S T R A C T

The author studies the uniform convergence of extended Lagrange interpolation processes based on the zeros of Generalized Laguerre polynomials.

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1. Introduction

Let \( \rho \) and \( \sigma \) be two weight functions, both of them supported in \(-\infty \leq a < b \leq +\infty\). Denote by \( \{p_m(\rho)\}_m, \{p_m(\sigma)\}_m \), the corresponding sequences of orthonormal polynomials with positive leading coefficients. If the polynomial \( Q_{m+n} = p_m(\rho)p_n(\sigma) \) has simple zeros \( \zeta_i, i = 1, 2, \ldots, m + n \), we define the extended interpolation polynomial \( \mathcal{L}_{m+n}(\rho, \sigma, f) \) as the Lagrange polynomial interpolating any continuous function \( f \) at the zeros of \( Q_{m+n} \).

\[
\mathcal{L}_{m+n}(\rho, \sigma, f; \zeta_i) = f(\zeta_i), \quad i = 1, 2, \ldots, m + n.
\]

Usually, to construct an interpolant of degree \( m + n - 1 \) based on orthogonal polynomials, we need the zeros of an \( (m + n) \) degree polynomial, and for \( m + n \) “large” the computation of orthogonal polynomials zeros presents well-known difficulties. The polynomial \( \mathcal{L}_{m+n}(\rho, \sigma, f) \) has degree \( m + n - 1 \), but it can be constructed by means of two Lagrange polynomials of degree \( m - 1 \) and \( n - 1 \) separately, since

\[
\mathcal{L}_{m+n}(\rho, \sigma, f) = p_m(\rho)\mathcal{L}_{m}(\sigma, \frac{f}{p_m(\rho)}) + p_n(\sigma)\mathcal{L}_{n}(\rho, \frac{f}{p_n(\sigma)}).
\]

Moreover, the polynomial \( \mathcal{L}_{m+n}(\rho, \sigma, f) \) can be used to extend a previous interpolation polynomial \( \mathcal{L}_{n}(\sigma, f) \), reusing the previous \( n \) function evaluations.

Several authors studied extended interpolation processes for functions defined in \([-1, 1]\), giving the estimate of the error in different norms (see for instance [2–4,6,9,10,5,21,22]). Here, we study extended interpolation processes on the positive real semi-axis. Consider two Generalized Laguerre weights \( w(x) = e^{-x}\alpha x^\beta, \alpha > 0, \beta > \frac{1}{2} \) and \( w(x) = xw(x) \). Since the polynomial \( Q_{2m+1} := p_{m+1}(w)p_m(w) \) has simple zeros (see Proposition 2.1), we consider the extended Lagrange polynomial based at its zeros. Following then an idea introduced in [15], the main goal of the paper is the construction of a new truncated interpolation process based essentially at the zeros of \( Q_{2m+1} \) and interpolating only a finite part of \( f \). The last approach affords the considerable advantage of a reduced computational effort. Moreover, by neglecting some of the greatest zeros, the possible overflow will be avoided whenever \( f \) grows exponentially. Our study starts from the distance between consecutive zeros of \( Q_{2m+1} \). Indeed, in view of Proposition 2.2, a “good distance” between consecutive interpolation abscissas is an important
ingredient to obtain “optimal” interpolation processes, i.e. with Lebesgue constants behaving like log m (see also [26]). The zeros of $Q_{2m-1}$ are “good candidates” in this sense, but the “good” distance is not enough (see Proposition 2.3). Therefore, following an idea of Szabados [24] (see also [16]), consider the Lagrange polynomial $L_{2m-2}(w.f)$ based on the zeros of $Q_{2m-1}(x)$ and on an extra knot $a_{m+1}$, connected with the space of functions in which $f$ varies and such that it approximates a finite section of $f$. For this process, in suitable functional spaces, we find necessary and sufficient conditions under which the sequence $\{L_{2m-2}(w.f)\}_{m\geq 1}$, $u$ being a weight function, approximates $f$ like the best approximation sequence, except the $\log m$ factor. By using this procedure, for $f$ belonging to a given space, we can select suitable interpolation knots such that the corresponding Lebesgue constants sequence has a logarithmic behaviour. Vice versa, for a fixed interpolation matrix of zeros $Q$ in the extended interpolation. In Section 3 the extended processes are introduced, and their convergence is studied. Section 4 contains some numerical examples and in Section 5 the proofs of the main results are given.

2. Notations and preliminary results

In the sequel $\varrho$ will denote any positive constant which can be different in different formulas. Moreover, $\varrho \neq \varrho(a, b, \ldots)$ will be used to mean the that constant $\varrho$ is independent of $a, b, \ldots$. The notation $A \sim B$, where $A$ and $B$ are positive quantities depending on some parameters, will be used if and only if $(A/B)^{1/\varrho} \leq \varrho$, with $\varrho$ positive constant independent of the above parameters.

Throughout the paper $\theta$ will denote a fixed real number, with $0 < \theta < 1$, which can be different in different formulas. Denote by $P_m$ the space of all algebraic polynomials of degree at most $m$.

Consider the weight $w(x) = e^{-x^2}, x > -1, \beta > 1/2$, let $\{p_m(w)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients

$$p_m(w,x) = \gamma_m(w)x^m + \text{terms of lower degree}, \quad \gamma_m(w) > 0$$

and denote by $\{x_k\}_{k=1}^{m}$ the zeros of $p_{m-1}(w)$ in increasing order, i.e.

$$x_k < x_{k+1}, \quad k = 1, \ldots, m.$$ 

In a similar way introduced the weight function $\bar{w}(x) = xw(x)$, let $\{y_k\}_k$ the zeros of the corresponding $m$th orthonormal polynomial $p_m(w)$. Denoted by $a_m(w) =: a_m$ the Mhaskar–Rachmanoff–Saff number [20] (in the sequel we say shortly M–R–S number) w.r.t. $w$, in [13, p. 112] it was proved

$$\bar{a}_m = \varrho(\beta, x) m^{1/2} \sim m^{1/2}. \quad (1)$$

Let $z_i, i = 1, 2, \ldots, 2m + 1$ be the zeros of $Q_{2m-1} := p_{m-1}(w)p_m(w)$. Denoted by $a_{m-1}(w) =: a_{m-1}(w)$ the M–R–S number w.r.t. $w$ define

$$z_i = z_{2m} = \min \{z_k : x_k \geq \theta a_{m-1}, \quad k = 1, 2, \ldots, 2m + 1\}.$$ 

The next proposition is one of the basic tools for constructing the extended Lagrange polynomials.

Proposition 2.1. The zeros of $p_{m-1}(w)$ interlace with those of $p_m(w)$, i.e.

$$x_k < y_k < x_{k+1}, \quad k = 1, 2, \ldots, m.$$ 

(2)

Setting $z_{2i-1} = x_i, i = 1, 2, \ldots, m + 1, z_{2i} = y_i, i = 1, 2, \ldots, m$, and denoted by $z_j$ the knot defined in (1), we have

$$\Delta z_k = z_{k+1} - z_k \sim \frac{\sqrt{a_{m-1}}}{m} \sqrt{z_{k+1}}, \quad k = 1, 2, \ldots, j$$ 

(3)

uniformly in $m \in \mathbb{N}$.

In the special case $\beta = 1$, the interlacing property was proved in [8].

Note that (3) is comparable with the distance between two consecutive zeros of $p_m(w)$ (see [12,11])

$$\Delta y_k = y_{k+1} - y_k \sim \varrho \frac{\sqrt{a_m}}{m} \sqrt{y_k}, \quad y_k \leq \theta a_m.$$ 

(4)

Now we show how this distance seems to be a relevant “ingredient” in order to obtain good interpolation processes. To this end, we introduce the space of functions

$$C(u_j) = \left\{ f : f \in C(\mathbb{R}^+), \lim_{x \to 0} \frac{|f(x)|}{u_j(x)} = 0 = \lim_{x \to \infty} \frac{|f(x)|}{u_j(x)} \right\},$$

where $u_j(x) = e^{-x^2} x^\gamma, \gamma \geq 0, \beta > 1/2$ with the norm $\|f\|_{C(u_j)} = \sup_{x > 0} |f(x)|u_j(x)$. 

Setting \( \mathcal{X} = \{ \xi_m, i = 1, 2, \ldots, m, m \in \mathcal{X} \} \), let \( \mathcal{P}_m(\mathcal{X}, g) \) be the Lagrange polynomial interpolating \( g \in C(\mathcal{U}_m) \) at the elements of the \( m \)th row of \( (\mathcal{X}) \). The \( m \)th Lebesgue constant is defined as

\[
\| \mathcal{P}_m(\mathcal{X}) \|_{\mathcal{C}(\mathcal{U}_m)} = \sup_{[u_1, \ldots, u_{10}]} \| \mathcal{P}_m(\mathcal{X}, g)u_1 \|, \quad m = 1, 2, \ldots
\]

(5)

The following proposition holds:

**Proposition 2.2.** If for \( m \) sufficiently large (say \( m > m_0 \)), there exists \( k := k(m) \) s.t.

\[
\Delta_q \leq \sqrt{\frac{a_m}{m}} \eta_{q+1} \quad \sqrt{\frac{a_m}{m}}, \quad \eta > 0,
\]

(6)

where \( a_m = a_m(u_i) \) is the M–R–S number w.r.t. the weight \( u_i \), then

\[
\| \mathcal{P}_m(\mathcal{X}) \|_{\mathcal{C}(\mathcal{U}_m)} \geq \Omega \left( \frac{m}{\sqrt{a_m}} \right)^{\eta},
\]

(7)

where \( \Omega \neq \Omega(m) \).

In \([-1, 1] \) an analogous result was proved in [25].

Obviously, a system of knots satisfying only property (3) does not guarantee Lebesgue constants behaving like \( O(\log m) \). Indeed, denoting by \( \mathcal{P}_{2m+1}(w, f) \), the Lagrange polynomial interpolating a given \( f \) at the zeros of \( Q_{2m+1} \), i.e.

\[
\mathcal{P}_{2m+1}(w, f; z_i) = f(z_i), i = 1, 2, \ldots, 2m + 1,
\]

the following proposition holds:

**Proposition 2.3.** For any choice of \( \alpha, \gamma \geq 0 \) and \( \beta > \frac{1}{2} \) there exits a positive \( \tau \) s.t.

\[
\| \mathcal{P}_{2m+1}(w, f) \|_{\mathcal{C}(w)} = \sup_{[u_1, \ldots, u_{10}]} \| \mathcal{P}_{2m+1}(w, f)u_1 \|_{\infty} \geq \Omega m^\tau
\]

with \( 0 < \Omega \neq \Omega(m) \).

Nevertheless, the system of knots made up of the zeros of \( \{ Q_n \} \) can be proposed in order to obtain optimal Lebesgue constants too. This goal will be achieved in the next section by the extended processes based on the zeros of \( Q_{2m+1} \) and on some additional knots.

In the following, since by \( a_{m-1}(u) \sim a_{m-1}(w) \sim a_m(w) \) results, we employ the same symbol \( a_m \) to denote one of them.

### 3. Extended interpolating polynomials

First we give an expression for the polynomial \( \mathcal{P}_{2m+1}(w, f) \)

\[
\mathcal{P}_{2m+1}(w, f; x) = \sum_{k=1}^{2m+1} \ell_{2m+1,k}(x)f(z_k), \quad \ell_{2m+1,k}(x) = \frac{Q_{2m+1}(x)}{Q_{2m+1}(z_k)(x - z_k)}.
\]

(8)

Consider now the Lagrange polynomial \( L_{2m+2}(w, f) \) interpolating \( f \) at the zeros of \( Q_{2m+1}(x) \) and at the special knot \( a_m \), i.e.

\[
L_{2m+2}(w, f; z_i) = f(z_i), i = 1, 2, \ldots, 2m + 1, \quad L_{2m+2}(w, f; a_m) = f(a_m).
\]

The polynomial \( L_{2m+2}(w, f) \) takes the expression

\[
L_{2m+2}(w, f; x) = \sum_{k=1}^{2m+1} \ell_{2m+2,k}(x)f(z_k) + \ell_{2m+2,2m+2}(x)f(a_m).
\]

(9)

where

\[
\ell_{2m+2,k}(x) = \ell_{2m+1,k}(x) \frac{(a_m - x)}{(a_m - z_k)}, \quad k = 1, 2, \ldots, 2m + 1,
\]

\[
\ell_{2m+2,2m+2}(x) = \frac{Q_{2m+1}(x)}{Q_{2m+1}(a_m)}.
\]

Denoting by \( \chi_j \) the characteristic function of the segment \( (0, z_j) \) where \( z_j \) is defined in (1), let us introduce the Lagrange polynomial

\[
L_{2m+2}^\star(w, f) := L_{2m+2}(w, f \chi_j),
\]

i.e.

\[
L_{2m+2}^\star(w, f; x) = \sum_{k=1}^{j} \ell_{2m+2,k}(x)f(z_k).
\]

(11)
Obviously, \( L_{2m+2}(w, w, f) \) is a polynomial of degree \( 2m + 1 \) such that \( L_{2m+2}(w, w, f; a_m) = 0 = L_{2m+2}(w, w, f; z_k) \), for \( k > j \).

The following result holds:

**Theorem 3.1.** For any function \( f \in C(u_j) \), with \( \gamma > 0 \),

\[
\|f - L_{2m+2}(w, w, f)u_j\|_\infty \leq \mathcal{O}\left\{ E_m(f)_{u_j} \log m + e^{-\lambda m}\|f_{u_j}\|_\infty \right\},
\]  

(12)

with \( 0 < \mathcal{O} \neq \mathcal{O}(m, f) \), if and only if \( 1 \leq \gamma - \alpha < 2 \).

Moreover,

\[
\|f - L_{2m+2}(w, w, f)u_j\|_\infty \leq \mathcal{O}\left\{ E_m(f)_{u_j} \log m + e^{-\lambda m}\|f_{u_j}\|_\infty \right\},
\]  

(14)

with \( M = (2m(\alpha_0)^{\frac{1}{16}})^4 \sim m \), \( 0 < \mathcal{O} \neq \mathcal{O}(m, f) \), \( 0 < A \neq A(m, f) \).

The Lagrange operator \( L_{2m+2}(w, w) \) projects \( C(u_j) \) on \( P_{2m+1} \), while \( L_{2m+2}(w, w) \) does not. However, letting

\[
\beta_{2m+1}^\gamma = \{ q \in P_{2m+1} : q(z_i) = q(a_m) = 0, z_i > z_j \} \subset P_{2m+1}
\]  

with \( z_j \) defined in (1), we have \( L_{2m+2}(w, w) \) which is a projector of \( C(u_j) \) on \( \beta_{2m+1}^\gamma \). Moreover, \( \cup_m \beta_m^\gamma \) is dense in \( C(u_j) \). In fact, denoting by

\[
E_m(f)_{u_j} = \inf_{P \in \beta_m^\gamma} \|f - P\|u_j\|_\infty,
\]

the error of the best approximation of \( f \) in \( C(u_j) \) and setting

\[
\tilde{E}_{2m+1}(f)_{u_j} := \inf_{Q \in \beta_m^\gamma} \|f - Q\|u_j\|_\infty,
\]  

from a more general in [17], next estimate follows:

**Lemma 3.1.** For any function \( f \in C(u_j) \),

\[
\tilde{E}_{2m}(f)_{u_j} \leq \mathcal{O}\left\{ E_m(f)_{u_j} + e^{-\lambda m}\|f_{u_j}\|_\infty \right\},
\]  

(15)

where \( M = (2m(\alpha_0)^{\frac{1}{16}})^4 \) and the constants \( 0 < A \neq A(m, f) \), \( 0 < \mathcal{O} \neq \mathcal{O}(m, f) \).

In view of (15), \( \tilde{E}_{2m}(f)_{u_j} \) can be estimated by the best approximation error \( E_m(f)_{u_j} \), where \( M \) is a proper fraction of 2m. We conclude recalling that in the Sobolev space

\[
W_r(u_j) = \{ f \in C(u_j) : \|f^{(k)}q^\gamma u_j\|_\infty < \infty \}, \quad \varphi(x) = \sqrt{x}
\]

with \( r > 1 \), the following estimate holds [18]:

\[
E_m(f)_{u_j} \leq \mathcal{O}\left( \frac{\sqrt{A_m}}{m} \right)^r \|f^{(k)}q^\gamma u_j\|_\infty, \quad \mathcal{O} \neq \mathcal{O}(m, f).
\]  

(16)

In a similar way, we could prove that for any function \( f \in C(u_j) \), with \( \gamma > 0 \)

\[
\|L_{2m+2}(w, w, f)u_j\|_\infty \leq \mathcal{O}\|f_{u_j}\|_\infty \log m \iff 1 \leq \gamma - \alpha < 2
\]  

(17)

with \( \mathcal{O} \neq \mathcal{O}(m, f) \). In this case, the following error estimate holds true:

\[
\|f - L_{2m+2}(w, w, f)u_j\|_\infty \leq \mathcal{O}E_{2m+1}(f)_{u_j}\log m,
\]  

(18)

where \( \mathcal{O} \neq \mathcal{O}(m, f) \). Comparing (12) with (17), we deduce that the two sequences \( \{L_{2m+2}(w, w, f)u_j\}_m \) and \( \{L_{2m+2}(w, w, f)u_j\}_j \), under the same conditions, have similar behaviours. Certainly the polynomial \( L_{2m+2}(w, w, f) \) offers some advantages: a reduced computational effort and the possibility of constructing higher degree approximation for functions \( f \) with an exponential growth, avoiding possible overflow drawbacks.

The idea of interpolating a truncation \( f_j \) of the function \( f \) was introduced in [15], where \( f_j \) is obtained as a link between the function \( f \) with zero by a smooth function, \( f_j \) having the same smoothness of \( f \). Successively in [12] the authors proved that this procedure is equivalent to interpolating the truncated function \( f_j \). Another approach consists in using the truncated sequences of Lagrange polynomial interpolating finite sections of \( f \), i.e. \( \{X_jL_m(w, f X_j)\}_m \) (see [19,17,23]).

Also in the extended interpolation, we can prove that the non-polynomial sequence \( \{X_j, \beta_{2m+1}^\gamma(w, w, f X_j)u_j\}_m \), with \( \beta_{2m+1}^\gamma(w, w, f) \) defined in (8), can be used to approximate \( f_{u_j} \) successfully.

\[ [a] denotes the largest integer smaller than or equal to \( a \in \mathbb{R} \). \]
Theorem 3.2. For any function \( f \in C(u_r) \) with \( \gamma > 0 \),
\[
\| L_{2m+1} [w, w, f] u_r \|_\infty \leq \varepsilon \| f u_r \|_\infty \log m
\]  
(19)
with \( 0 < \varepsilon \neq \varphi(m, f) \) if and only if
\[
1 \leq \gamma - \alpha \leq 2.
\]  
(20)
Moreover,
\[
\| f - L_{2m+1} [w, w, f] u_r \|_\infty \leq \varepsilon \left\{ E_m(f) u_r \log m + e^{-\delta m} \| f u_r \|_\infty \right\}
\]
(21)
where \( M = \left[ 2m \left( \frac{\alpha^2}{\gamma^2} \right)^{\gamma} \right] \sim m, 0 < \varepsilon \neq \varphi(m, f), 0 < A \neq A(m, f). \)

We will omit the proof of Theorem 3.2 since it follows similar arguments used to prove Theorem 3.1.
The sequences \( \{ L_{2m+1} [w, w, f] u_r \}_m \) and \( \{ \chi_m, L_{2m+1} [w, w, f] u_r \}_m \) have similar behaviours under the same assumptions. However, \( \{ L_{2m+1} [w, w, f] \}_m \) is a polynomial sequence and this property can be useful in many context (for instance, in the Gaussian quadrature and/or in numerical methods for integral equations).

Finally, we consider the case in which the interpolation knots are fixed for a given function \( f \in C(u_r) \), that is the parameters \( \gamma \) and \( \alpha \) are both fixed and (13) is not satisfied. In this case it can be useful to supply the Lagrange polynomial \( L_{2m+1} [w, w, f] \) making use of the additional nodes method introduced by Szabados and extensively used by several authors in different contexts (see [25] and its bibliography). Let \( t_i = 1, \ldots, s \) be some simple knots added in the range \( [0, x_i] \), for instance, \( t_i = \frac{1}{i} x_i, i = 1, 2, \ldots, s \) and let \( B_i(x) = \prod_{j \neq i}^s (x - t_j) \). Denote by \( L_{2m+2} [w, w, f] \) the Lagrange polynomial interpolating \( f \) at the zeros of \( Q_{2m+1}(x) B_i(x) (a_m - x) \), and define \( L_{2m+2} [w, w, f] = L_{2m+2} [w, w, f] \). An expression for this polynomial is
\[
L_{2m+2} [w, w, f] (x) = \sum_{i=1}^s \frac{Q_{2m+1}(i)(a_m - x) B_i(x)}{Q_{2m+1}(i)(a_m - t_i) B_i(t_i)} f(t_i) + \sum_{k=1}^s \frac{Q_{2m+1}(z_k)(a_m - z_k) B_i(z_k)}{Q_{2m+1}(z_k)(a_m - z_k) B_i(z_k)} f(z_k).
\]  
(22)
The following result holds:

Theorem 3.3. For any function \( f \in C(u_r) \), if there exists an integer \( s \) such that
\[
1 \leq \gamma - \alpha + s \leq 2,
\]  
(23)
then we have
\[
\| L_{2m+2} [w, w, f] u_r \|_\infty \leq \varepsilon \| f u_r \|_\infty \log m.
\]  
(24)
where \( 0 < \varepsilon \neq \varphi(m, f) \). Moreover,
\[
\| f - L_{2m+2} [w, w, f] u_r \|_\infty \leq \varepsilon \left\{ E_m(f) u_r \log m + e^{-\delta m} \| f u_r \|_\infty \right\}
\]
(25)
where \( M = \left[ 2m \left( \frac{\alpha^2}{\gamma^2} \right)^{\gamma} \right] \sim m, 0 < \varepsilon \neq \varphi(m, f), 0 < A \neq A(m, f). \)

In the case \( f \in C(u_r) \) with \( \gamma = 0 \), there does not exist \( \alpha \) such that (13) holds true. Nevertheless, in the last case we can again construct an interpolant sequence with Lebesgue constants going like \( \log m \). Indeed, let us consider the sequence \( \{ L_{2m+2} [w, w, f] u_r \}_m \), with \( t_1 = 0 \), i.e.
\[
L_{2m+2} [w, w, f] (x) = \frac{Q_{2m+1}(x)(a_m - x)}{Q_{2m+1}(0)a_m} f(0) + \sum_{k=1}^s \frac{Q_{2m+1}(z_k)(a_m - z_k)}{Q_{2m+1}(z_k)a_m} f(z_k).
\]  
(26)
In this case, we have the following:

Corollary 3.1. For any function \( f \in C(u_r) \) with \( \gamma = 0 \) and for any \( \alpha \leq 0 \),
\[
\| L_{2m+2} [w, w, f] u_r \|_\infty \leq \varepsilon \| f u_r \|_\infty \log m, \quad 0 < \varepsilon \neq \varphi(m, f)
\]  
(27)
and
\[
\| f - L_{2m+2} [w, w, f] u_r \|_\infty \leq \varepsilon \left\{ E_m(f) u_r \log m + e^{-\delta m} \| f u_r \|_\infty \right\}
\]
(28)
where \( M = \left[ 2m \left( \frac{\alpha^2}{\gamma^2} \right)^{\gamma} \right] \sim m, 0 < \varepsilon \neq \varphi(m, f), 0 < A \neq A(m, f). \)

Similarly we can proceed when \( \alpha > 0 \), with \( t_1 = 0, t_2 = \frac{x_1}{m^2} \) and the sequence \( \{ L_{2m+2} [w, w, f] u_r \}_m \).

4. Numerical examples

Now we propose some examples to show the performance of the interpolation processes introduced in this paper. In case it is possible, we compare the results obtained by the extended Lagrange polynomials with those produced by the Lagrange polynomial based on the zeros of \( p_{2m+1}(\rho, x)(a_m - x) \), where \( \rho(x) = e^{-\delta x^2}, a_m = a_m(u_r) \). More precisely, denote by \( a_i, i = 1, 2, \ldots, 2m + 1 \) the zeros of \( p_{2m+1}(\rho) \), define

---

We choose as interpolation weights $\eta$ and let $L^{2m+2}(\rho, f)$ be the Lagrange polynomial interpolating $f$ at $\{\sigma_i\}_{i=1}^{2m+2}$, being $\sigma_{2m+2} = a_m$. Setting $L^{2m+2}_m(\rho, f) := L^{2m+2}_m(\rho, f \chi_R)$, where $\chi_R$ is the characteristic function of the segment $(0, \sigma_R)$, in [14] it was proved that for any $f \in C(u)$

$$\frac{1}{4} < \gamma^2 \leq \frac{5}{2} \Rightarrow \| L^{2m+2}_m(\rho, f) u_i \|_\infty < \varepsilon \| f u_i \|_\infty \log m, \quad 0 < \varepsilon \neq \varepsilon(m, f).$$

(30)

Example 1

$$f_1(x) = \sin(x) e^{\frac{x^2}{2}}, \quad u_i(x) = xe^{-x}, \quad f_1 \in W_i(u_i) \quad \forall r \geq 1.$$  

We choose as interpolation weights $\rho(x) = e^{-2x^2}$ and $w(x) = e^{-x^4}$, respectively, so the assumptions in (30) and (13) are both fulfilled.

<table>
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<th>$j$</th>
<th>$| f_1 - L^{2m+2}<em>m(\rho, w, f) u_i |</em>\infty, F_m$</th>
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<td>32</td>
<td>2.5e-6</td>
</tr>
<tr>
<td>42</td>
<td>42</td>
<td>2.6e-8</td>
<td>42</td>
<td>4.5e-8</td>
</tr>
<tr>
<td>52</td>
<td>49</td>
<td>1.5e-10</td>
<td>49</td>
<td>2.6e-10</td>
</tr>
<tr>
<td>62</td>
<td>55</td>
<td>2.0e-12</td>
<td>55</td>
<td>4.1e-12</td>
</tr>
<tr>
<td>72</td>
<td>60</td>
<td>1.7e-14</td>
<td>60</td>
<td>3.0e-14</td>
</tr>
<tr>
<td>82</td>
<td>65</td>
<td>6.6e-16</td>
<td>65</td>
<td>9.8e-16</td>
</tr>
</tbody>
</table>

In this example, the function is smooth and with $2m + 2 = 81$ we obtain the machine precision.

Example 2

$$f_2(x) = |x - 5|^2 e^{x^2/2} \in W_3(u_i), \quad u_i(x) = x^4 e^{-x^2}.$$  

(34)

We choose $\rho(x) = e^{-2x^4}$ and $w(x) = e^{-x^4}$, so that (13) and (30) are both satisfied.

<table>
<thead>
<tr>
<th>$2m + 2$</th>
<th>$h$</th>
<th>$| f_2 - L^{2m+2}<em>m(\rho, f_2) u_i |</em>\infty, D_m$</th>
<th>$j$</th>
<th>$| f_2 - L^{2m+2}<em>m(\rho, w, f_2) u_i |</em>\infty, F_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>16</td>
<td>2.7e-5</td>
<td>16</td>
<td>2.4e-5</td>
</tr>
<tr>
<td>32</td>
<td>27</td>
<td>1.2e-6</td>
<td>28</td>
<td>2.3e-7</td>
</tr>
<tr>
<td>64</td>
<td>51</td>
<td>5.1e-6</td>
<td>51</td>
<td>3.3e-7</td>
</tr>
<tr>
<td>120</td>
<td>90</td>
<td>4.8e-8</td>
<td>90</td>
<td>3.5e-8</td>
</tr>
<tr>
<td>302</td>
<td>178</td>
<td>9.0e-9</td>
<td>178</td>
<td>4.1e-9</td>
</tr>
<tr>
<td>400</td>
<td>236</td>
<td>1.5e-10</td>
<td>236</td>
<td>2.3e-10</td>
</tr>
<tr>
<td>798</td>
<td>–</td>
<td>–</td>
<td>377</td>
<td>3.3e-11</td>
</tr>
</tbody>
</table>
The zeros of \( p_m(x) \) are computable up to \( m = 399 \). So \( L_{400}^q(x) \) is the maximum degree Lagrange polynomial that we can consider. In this case, using the zeros of \( p_{399}(w) \) and \( p_{398}(w) \) we can construct the polynomial \( L_{798}^q(w; x) \) interpolating \( f \) at the zeros of \( p_{399}(w; x)p_{398}(w; x)(a_{399} - x) \).

**Example 3**

\[
\begin{align*}
  f_3(x) &= \log(1 + x) - \log(1 + x^2), & W_5(u_i), & u_i(x) = x^3e^{-x^2}.
\end{align*}
\]

According to (30) and (13) in Theorem 3.1, we choose \( \rho(x) = e^{-2x^2} \) and \( w(x) = e^{x^2} \).

| \( 2m + 2 \) | \( h \) | \( ||f_3 - L_{2m+2}^q(\rho, f_3)|_{\infty, D_m}|| \) | \( j \) | \( ||f_3 - L_{2m+2}^q(w, w, f_3)|_{\infty, F_m}|| \) |
|---|---|---|---|---|
| 8 | 6 | 5.0e-4 | 6 | 4.8e-4 |
| 16 | 12 | 1.8e-5 | 12 | 2.0e-5 |
| 32 | 25 | 2.2e-6 | 26 | 1.1e-6 |
| 48 | 38 | 5.3e-7 | 38 | 3.8e-7 |
| 64 | 51 | 2.0e-7 | 51 | 1.6e-8 |
| 202 | 152 | 5.0e-9 | 162 | 1.2e-9 |
| 302 | 227 | 9.9e-10 | 242 | 2.4e-10 |
| 400 | 381 | 8.1e-10 | 281 | 2.35e-11 |
| 422 | – | – | 390 | 9.6e-12 |
| 502 | – | – | 445 | 7.8e-13 |

Also in this test by using the extended interpolation, we can almost double the maximum degree Lagrange polynomial, since we are able to construct \( L_{502}^q(w; w, f) \) which interpolates \( f \) at the zeros of \( p_{251}(w; x)p_{250}(w, x)(a_{251} - x) \).

**Example 4**

\[
\begin{align*}
  f_4(x) &= \arctan(x)^{2/3} - \arctan(x), & W_5(u_i), & u_i(x) = e^{-x^2}.
\end{align*}
\]

In this case for \( \alpha = 0 \), i.e. \( w(x) = e^{-x^2} \) (13) is not satisfied. Therefore, we can supply by using the Lagrange polynomial with an additional knot defined in (26).

| \( 2m + 2 \) | \( h \) | \( ||f_4 - L_{2m+2}^q(\rho, f_4)|_{\infty, D_m}|| \) | \( j \) | \( ||f_4 - L_{2m+2}^q(w, w, f_4)|_{\infty, F_m}|| \) |
|---|---|---|---|---|
| 8 | 8 | 9.7e-4 | 8 | 1.5e-3 |
| 16 | 16 | 4.1e-6 | 16 | 2.1e-5 |
| 32 | 32 | 1.4e-7 | 31 | 8.1e-8 |
| 50 | 50 | 2.1e-8 | 40 | 5.0e-8 |
| 100 | 88 | 6.5e-10 | 81 | 6.5e-10 |
| 202 | 190 | 5.8e-11 | 164 | 3.5e-11 |
| 302 | 195 | 1.1e-11 | 193 | 6.5e-12 |
| 400 | 257 | 3.3e-12 | 256 | 2.0e-12 |
| 600 | – | – | 330 | 3.2e-13 |

The computation of the zeros of Generalized Laguerre polynomials with parameter \( \beta \neq 1 \) requires a higher computational effort. Indeed, when \( \beta \neq 1 \) the coefficients in the three-term recurrence relation for the polynomials \( \{ p_m(w) \}_m \) are not always known. However, there exists in MATHEMATICA the package "Orthogonalization" [7] to compute these zeros by using "high" variable precision.

**5. The proofs**

Now we collect some polynomial inequalities deduced in [17] by a change of variable in analogous estimates found in [11].

Let \( x \in [x_1, x_m] \) and \( d = d(x) \in \{ 1, \ldots, m \} \) be an index of a zero of \( p_m(w) \) closest to \( x \). Then, for some positive constant \( C \neq C(m, x, d) \), we have
\[
\left( \frac{x - x_d}{x_k - x_{k+1}} \right)^2 \leq p_m^2(x) \exp \left( x + \frac{a_m}{m} \right)^{\frac{1}{2}} \sqrt{|a_m - x| + a_m m^{-\frac{1}{2}}} \leq \frac{1}{x^2} \left( \frac{x - x_d}{x_k - x_{k+1}} \right)^2
\] 
\[(37)\]

and for a fixed real number \(0 < \delta < 1\),
\[
|p_m(x)| \sqrt{w(x)} \leq \frac{1}{\sqrt{x} \sqrt{|a_m - x| + a_m m^{-\frac{1}{2}}}} \cdot a_m \leq a_m(1 + \delta).
\] 
\[(38)\]

In particular, for a fixed \(0 < \theta < 1\)
\[
|p_m(x)| \sqrt{w(x)} \leq \frac{1}{\sqrt{a_m x}} \cdot \frac{a_m}{m^2} \leq x \leq \theta a_m.
\] 
\[(39)\]

Moreover, for \(k = 1, 2, \ldots, m\)
\[
\frac{1}{|p_m(w, x_k)| \sqrt{w(x_k)}} \sim \Delta x_k \sqrt{a_m x_k} \left( 1 - \frac{x_k}{a_m} + m^{-\frac{1}{2}} \right), \quad \Delta x_k = x_{k+1} - x_k.
\] 
\[(40)\]

The Bernstein inequality \([11,18]\), for any polynomial \(P_m \in \mathbb{P}_m\)
\[
\max_{x \geq 0} |p'_m(x)| \sqrt{w(x)} \leq \frac{m}{\sqrt{a_m}} \max_{x \geq 0} |p_m(x)| \sqrt{w(x)} \leq \mathcal{C}(m, P_m)
\] 
\[(41)\]

and the Remez-type inequality \([18]\)
\[
\max_{x \geq 0} |p_m(x)u_i(x)| \leq \mathcal{C}(m, A) \max_{x \geq 0} |p_m(x)u_i(x)|.
\] 
\[(42)\]

Finally, we recall that for any polynomial \(P_{2m} \in \mathbb{P}_{2m}\), the following inequality holds \([18]\):
\[
\max_{x \geq 0} |p_{2m}(x)u_i(x)| \leq \mathcal{C}(m, A) \max_{x \geq 0} |p_{2m}(x)|u_i(x),
\] 
\[(43)\]

where \(\mathcal{C} \neq \mathcal{C}(m, A) \neq A(m)\).

We recall the three-term recurrence relation for the orthogonal polynomials w.r.t. the weight \(w\):
\[
p_{-1}(w, x) = 0, \quad p_0(w, x) = \left( \int_0^\infty w(x)dx \right)^{-\frac{1}{2}},
\]
\[
b_{n+1}p_{n+1}(w, x) = (x - e_n)p_n(w, x) - b_n p_{n-1}(w, x),
\]
\[
b_n = \frac{\gamma_{n-1}(w)}{\gamma_n(w)}, \quad e_n = \int_0^\infty x p_n^2(w, x)w(x)dx.
\] 
\[(44)\]

Although the coefficients \(\{b_k\}_k, \{e_k\}_k\) are not always known, there exist efficient numerical procedures to calculate them \([7]\) (see also \([1]\)).

**Proof of Proposition 2.1.** Consider the Fourier expansion of \(x p_m(w, x)\) in the system \(\{p_j(w)\}_{j=0}^\infty\):
\[
x p_m(w, x) = d_m p_m(w, x) + d_{m+1} p_{m+1}(w, x), \quad d_j = \int_0^\infty x p_m(w, x)p_j(w)w(x)dx
\]
and for \(x = x_k, k = 1, 2, \ldots, m + 1\)
\[
x p_m(w, x_k) = \frac{\gamma_m(w)}{\gamma_m(w)} p_m(w, x_k).
\] 
\[(45)\]

Similarly, by the Fourier expansion of \(p_{m+1}(w)\) in the system \(\{p_j(w)\}\),
\[
p_{m+1}(w, x) = c_m p_m(w, x) + c_{m+1} p_{m+1}(w, x), \quad c_i = \int_0^\infty p_{m+1}(w, x)p_i(w)w(x)dx
\]
and for \(x = y_k, k = 1, 2, \ldots, m\), we have
\[
p_{m+1}(w, y_k) = \frac{\gamma_{m+1}(w)}{\gamma_{m+1}(w)} p_{m+1}(w, y_k).
\] 
\[(46)\]

By the three-term recurrence relation (44)
\[
p_{m+1}(w, y_k) = \frac{\gamma_{m+1}(w)}{\gamma_{m+1}(w)} p_{m+1}(w, y_k).
\]
and replacing in (46)
\[ p_{m+1}(w, y_k) = -\frac{\gamma_{m-1}(w)\gamma_{m-1}(w)}{\gamma_m(w)} p_{m-1}(\tilde{w}, y_k). \] (47)
By (45) and (47) it follows that \( Q_{2m+1} = p_{m+1}(w)p_m(w) \) has simple zeros. Moreover,
\[ Q_{2m+1}'(x_k) > 0, \quad Q_{2m+1}'(y_k) < 0 \] (48)
by which it follows that the zeros of \( p_{m+1}(w) \) interlace with those of \( p_m(\tilde{w}) \).
Now we prove (3). Using \( x_{k+1} - y_k < x_{k+1} - x_k \), and \( y_k - x_k < y_{k+1} - y_k \), by (4) it follows that
\[ \Delta z_k \leq \frac{\sqrt{a_m}}{m} \Delta z_k, \quad z_k \leq \omega m. \] (49)

To prove the converse inequality in (3), we first prove
\[ x_{k+1} - y_k \geq \frac{\sqrt{a_m}}{m} \sqrt{x_{k+1}}, \quad k = 1, 2, \ldots, j \] (50)
with \( \omega \neq \omega(m) \). By (48) we have
\[ 0 < Q_{2m+1}'(x_k) - Q_{2m+1}'(y_k) = (x_k - y_k)Q_{2m+1}'(\xi_k), \]
where \( \xi_k \in (y_k, x_{k+1}) \). Therefore
\[ \frac{1}{(x_k - y_k)} \leq \frac{Q_{2m+1}'(\xi_k)}{Q_{2m+1}'(x_k)}, \] (51)
By (41) and (38) it follows that
\[ |p_m'(w, \xi_k)| \sqrt{W(\xi_k)} \leq \frac{m}{\sqrt{a_m}} \frac{1}{\sqrt{\xi_k}(a_m - \xi_k)}, \]
and therefore
\[ |Q_{2m+1}'(\xi_k)| \sqrt{W(\xi_k)}W(\xi_k) \leq \frac{m^2}{a_m \xi_k} \frac{1}{\sqrt{\xi_k}(a_m - \xi_k)}. \]

By (37)
\[ \frac{1}{|p_m(w, x_{k+1})| \sqrt{W(x_{k+1})}} \leq \frac{1}{x_{k+1}(a_m - x_{k+1})} \] (52)
and by (40) we get
\[ \frac{1}{Q_{2m+1}'(x_{k+1}) \sqrt{W(x_{k+1})}} \leq \frac{1}{x_{k+1}(a_m - x_{k+1})} \Delta x_{k+1}. \] (53)
Using \( \sqrt{W(x_{k+1})}W(x_{k+1}) \sim \sqrt{W(\xi_k)}W(\xi_k) \), it follows that
\[ \frac{|Q_{2m+1}'(\xi_k)|}{Q_{2m+1}'(x_{k+1})} \leq \frac{m^2}{a_m \xi_k} \Delta x_{k+1}. \] (54)
Since by (4)
\[ \Delta x_{k+1} \sim \frac{\sqrt{a_m}}{m} \sqrt{x_{k+1}}, \quad k \leq j, \]
it follows that
\[ \frac{1}{(x_k - y_k)} \leq \frac{m}{\sqrt{a_m x_{k+1}}}, \] (55)
Since the estimate
\[ \frac{1}{(y_k - x_k)} \leq \frac{m}{\sqrt{a_m y_k}}, \] (56)
follows by similar arguments, the proposition is completely proved.  

**Proof of Proposition 2.2.** Let be \( m > m_0 \) and \( k \) s.t. (7) holds. Let \( g_m \) be the function defined as
\[ g_m(x) = \begin{cases} \frac{x - x_{m,k-1}}{(y_k - x_{m,k-1})}, & \xi_{m,k-1} \leq x \leq \xi_{m,k}, \\ \frac{x - x_{m,k-1}}{(y_k - x_{m,k-1})}, & \xi_{m,k} \leq x \leq \xi_{m,k+1}, \\ 0, & \text{otherwise.} \end{cases} \]
Lemma 5.3. \[ L \]

Let \( Q \) prove Lemma 4.1, p. 36.

Therefore, assuming \( \sum \) and using (59) and (38),

\[
\text{In view of (7) } u_j(\xi_{m,k}) \sim u_j(\xi_{m,k}^2) \sqrt{\xi_{m,k}^2} \sim \sqrt{\xi_{m,k}} \text{ and }
\]

\[
\varrho \leq \left( \frac{\sqrt{a_m}}{m} \right) \| L_{2m+1}^j(x, g_m)u_j \| _\infty
\]

and we can conclude

\[
\| L_{2m+1}^j(x) \| _{C^0} \geq \varrho \left( \frac{m}{\sqrt{a_m}} \right) ^\eta.
\]

The following lemmas will be needed for the proofs of the main results.

**Lemma 5.1.** Let \( \{z_k\}_{k=1}^{2m+1} \) be the zeros of \( Q_{2m+1} \) and denote with \( z_d \) a zero closest to \( x \), \( \Delta z_k = z_{k+1} - z_k \). Assuming \( 0 \leq \rho, \sigma \leq 1 \), for \( x \in (0, a_m) \) and for \( m \) sufficiently large, we have

\[
\sum_{k=1}^{2m+1} \frac{\Delta z_k}{|x - z_k|} \left( \frac{a_m - x}{a_m - z_k} \right)^\rho x_n^\sigma \frac{z_k}{z_k} \leq \varrho \log m,
\]

where \( \varrho \neq C(m, x) \).

We omit the proof of the previous Lemma since it can be easily obtained following the same arguments used in [16] to prove Lemma 4.1, p. 36.

**Lemma 5.2.** Let \( Q_{2m+1} = p_{m+1}(w) \text{p}_m(w) \) and \( z_k, k = 1, 2, \ldots, 2m + 1 \) the zeros of \( Q_{2m+1} \)

\[
\frac{1}{|Q_{2m+1}(z_k)| u_j(z_k)} \leq \varrho \sqrt{\frac{a_m - z_k}{z_k}} \Delta z_k, \quad z_k < z_j, \quad \varrho \neq \varrho(m).
\]

**Proof.** Using (37)

\[
\frac{1}{|p_m(w, x_k)| \sqrt{w(x_k)}} \leq \varrho \sqrt{x_k}, \quad x_k < z_j,
\]

so, by (40), it follows that

\[
\frac{1}{|Q_{2m+1}(x_k)u_j(x_k)|} \leq \frac{1}{\sqrt{x_k}} \Delta x_k, \quad x_k < z_j.
\]

An analogous estimate holds replacing \( x_k \) with \( y_k \). Using then \( \Delta x_k \sim \Delta y_k \sim \Delta z_k \), \( x_k, y_k, z_k < z_j \), the lemma follows.

**Lemma 5.3.** For \( x \in (z_1, z_{2m+1}) \) and denoted with \( z_d \) the zero of \( Q_{2m+1} \) closest to \( x \), we have

\[
\frac{|Q_{2m+1}(x)|}{|Q_{2m+1}(z_d)(x - z_d)|} \leq \varrho, \quad \varrho \neq \varrho(m, x).
\]

**Proof.** Denoted by \( x_k \) a zero of \( p_{m+1}(w) \) closest to \( x \in [x_1, x_{m+1}] \), in [14] it was proved that

\[
\frac{|p_{m+1}(w, x)|}{|p_{m+1}(w, x_k)(x - z_d)| \sqrt{w(x_k)}} \sim 1.
\]

Therefore, assuming \( z_d \) is a zero of \( p_{m+1}(w) \), we have

\[
\frac{|Q_{2m+1}(x)|}{|Q_{2m+1}(z_d)(x - z_d)|} \leq \varrho \frac{|p_{m}(w, x)|}{|p_{m}(w, z_d)| \sqrt{w(z_d)}}
\]

and using (59) and (38).
it follows that

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

and using (11)

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

and using (11)

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

By (39)

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

and recalling (58) we have

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

Combining last inequality with (63), (12) follows.

Proof of Theorem 3.1. First, we prove the sufficient condition. By (42) we have

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

and using (11)

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

By (39)

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

and proving that (12) implies right hand condition in (13), consider the same function $g$ previously defined and let $\bar{x} = \theta a_m$.

We have

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

and by (40) and (37) it follows that

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

Using $a_m - \bar{x} > a_m - x_k$ and $\bar{x} - x_k \leq x_k$ we have

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

Since (12) holds,

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

it follows that $\gamma - z - 1 \geq 0$.

To prove that (12) implies right hand condition in (13), consider the same function $g$ previously defined and let $\bar{x} = \theta a_m$. We have

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

Using (37)

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$

and using (66) again and $a_m - x_k < a_m - \bar{x} < x_k$, $a_m - x_k > (1 - \theta)a_m$ we have

$$\leq C_24 \frac{2^m}{m^2} \leq C_22 \frac{2^m}{m^2}$$
Since (12) holds,
\[ \varrho \log m \geq \|L_{2m+2}(w, w)u_1\|_\infty \geq \varrho (m^{-2})^{-\gamma + \delta} \]

it follows that \( \gamma - \alpha - 1 \leq 1 \).

Let us prove (14). Let \( P \in \mathcal{P}_{2m+1} \). Since \( L_{2m+2}(w, \bar{w}) : f \in C(u_1) \rightarrow \mathcal{P}_{2m+1} \) is a projector,
\[ \|f - L_{2m+2}(w, \bar{w})f\|_\infty \leq \|f - P\|_\infty + \|L_{2m+2}(w, w, f - P)\|_\infty. \]

Using then (15) and (12), the proof is complete. \( \Box \)

**Proof of Theorem 3.3.** By (42) and (22) we have
\[
\|L_{2m+2}(w, w, f)u_1\|_\infty \leq \|f\|_\infty \times \max_{x, y \in [0, m]} \left\{ \sum_{i=1}^{s} \frac{|Q_{2m+1}(x)(a_m - x)B_i(x)|}{(a_m - t_i)B_i(t_i)(x - t_i)} \frac{u_i(x)}{u_i(t_i)} \right\} + \left( \frac{1}{m} \right) \int Q_{2m+1}(z_k)(a_m - z_k)B_k(z_k)(x - z_k) \frac{u_k(x)}{u_k(z_k)} \right| \leq \|f\|_\infty \times \max_{x, y \in [0, m]} \left\{ \frac{C_1(x) + C_2(x)}{m^2} \right\}.
\]

By (37)
\[ |p_m(x, t_i)e^{-\xi^i/2}(a_m/m^2) \right\|_\infty \leq \frac{t_i - x_1}{|x_1 - x_2|} \geq \varrho, \quad i = 1, \ldots, s \]

and consequently, since \( t_i \sim \frac{a_m}{m} \),
\[ Q_{2m+1}(t_i)|u_1(t_i) \geq \frac{\varrho t_i}{\sqrt{a_m - t_i}} \geq \sqrt{a_m - \frac{m^2}{m^2}} \]

Using
\[ B_i(t_i) \leq \sum_{j \neq i} \left| x - t_j \right| / \left| t_i - t_j \right| \leq \varrho \left( \frac{m^2}{a_m} \right)^{s-1} \]

\[ \sqrt{a_m - x} \leq \sqrt{a_m - a_n}, \text{ and (63), we have} \]
\[ C_1(x) \leq \varrho x^{\gamma - \alpha - 2 + s} \frac{m^2/x}{a_m} \]

and taking into account the assumption \( \gamma - \alpha - 2 + s \geq 0 \) and \( x \geq \frac{m^2}{a_m} \), it follows that
\[ C_1(x) \leq \varrho. \] (69)

By (63)–(58) again and using
\[ B_i(x) = \sum_{l=1}^{s} |x - t_l| / (x - t_1) \leq \varrho x^s, \]
\[ B_i(z_k) \geq \left( z_k - t_1 \right)^s \sim z_k^s, \]

we have
\[ C_2(x) \leq \varrho \left\{ \sum_{k=1}^{s} \frac{z_k}{(|x - z_k| \sqrt{a_m - x}} \right\}^{x^\alpha - \gamma - 1 + s} + \frac{\|p_m(x, w)B_i(x)(a_m - x)u_i(x)\|}{B_i(z_k)\|p_m(x, w)B_k(z_k)(a_m - z_k)\|u_i(z_d)} \].

Using Lemma 5.1, under the assumption \( 0 \leq \gamma - \alpha - 1 + s \leq 1 \), by \( B_i(x) \sim B_i(x_d) \) and Lemma 5.3, it follows that
\[ \varrho_2(x) \leq \varrho \log m. \] (70)

Combining (68)–(70), (24) follows. \( \Box \)

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References