Aggregate Bandwidth Estimation in Video Distribution Systems

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Abstract—In this paper we propose various original algorithms for evaluating, in a video distribution system, a statistical estimation of aggregate bandwidth needed by a given number of smoothed video streams. Each video stream is characterized by the marginal distribution, approximated by a histogram derived from real video traces. Independence among video streams is the common assumption for the developed algorithms, whose results are compared with simulation and with other results, based on the same assumptions, already presented in the literature, discussing the pros and cons of the considered solutions.

Index terms-- Video Distribution, Bandwidth Evaluation, Analytical and Simulation Model.

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I. INTRODUCTION

A large part of traffic in future broadband networks will be presumably generated by multimedia applications. In such systems, capital importance will play the issue of Quality of Service (QoS), which imposes strict constraints on admission control, bandwidth allocation, capacity planning, etc. In this paper, we have considered a video distribution system, which forms the core of applications like Video on Demand, Distance Learning, Internet video broadcast, etc.[1,2]. In particular, we suppose to distribute stored video, codified by standard MPEG algorithms, which, as well known, produce a variable bit rate (VBR) traffic. As an example, in Fig. 1 is reported the bit rate of the first 15000 frames of a MPEG-1 coded trace of the film “Star Wars”.

As reported in [3], work-ahead smoothing algorithms, exploiting client buffer, can be employed for optimizing network resources utilization. As an example, in Figs. 1b-1c is reported the bit rate of the same film “Star Wars” smoothed with a client buffer, respectively, of 256K and 1.0M Bytes.
The technique utilized to reach this purpose consists in transmitting as long as possible pieces of the same film with a constant bit rate, that varies from piece to piece. The variations of the bit rate are done accordingly to a scheduling algorithm that assures the minimum variability of the bit rate between all the CBR pieces. In this way it can be demonstrated (see [2]) that the smoothing is the best possible. As can be inferred, smoothing technique reduces consistently the rate variability and so improves significantly the bandwidth resources allocation on the network exploiting also statistical multiplexing gains in the aggregate bandwidth. For this reason we suppose that all the films will be present on the network in a smoothed manner, as calculated by the smoothing algorithm. The consequence of this smoothing algorithm is that the rate marginal distribution of smoothed sequences is quite radical different when compared with original unsmoothed sequence. As an example, in figures 2a-2c are reported the rate histograms relative to the figures 1a-1c. As highlighted in [2,3] a consistent gain in network resource utilization can be obtained by exploiting smoothing, which will likely be implemented in real systems. A significant problem in such systems result the aggregate bandwidth estimation needed for planning the transmission infrastructure and for admission control. Such a problem, in the literature, has been attacked by exploiting Chernoff bound, in the hypothesis of independence among video streams and bufferless systems [3]. This last hypothesis of bufferless network switch, can be validly justified by considering that the real-time distribution of smoothed video traffic imposes stringent delay requirements and as a consequence very small buffer can be employed in the intermediate network nodes. As a matter of facts, in this paper we model the network switch as a bufferless multiplexer.

Various original algorithms for evaluating the aggregate bandwidth needed by N independent streams are proposed comparing the obtained results with simulation and with other results, based on the same assumptions, already presented in the literature, discussing the pros and cons of the considered solutions. Each stream is characterized by its marginal distribution approximated by a histogram derived from real video traces. In particular, the analytical models and the relative results for homogeneous streams are developed and presented in section II, while in section III the case of heterogeneous systems is considered.
II. BANDWIDTH ESTIMATION FOR HOMOGENEOUS STREAMS

In this section we consider a system that can deliver only statistically homogeneous streams, typically originating from the same source, while in the next section a more general system, considering heterogeneous streams, will be evaluated, exploiting the analytical results obtained in this section.

A. Analytical model

We consider $N$ homogeneous streams, either smoothed or unsmoothed, whose rate can be statistically characterized by the histogram method, that consists in defining, for each of them, a discrete r.v. that can assume the $K$ values $r_1 < r_2 < \ldots < r_K$ with probability $p(r_i) = p_i$ ($i=1, 2, \ldots, K$). The system state can be
defined by the following vector \((\alpha_1, \alpha_2, \ldots, \alpha_N)\) where the generic \(\alpha_i (\alpha_i = 1, 2, \ldots, K)\) identifies the stream rate \(i\). In the simplifying hypothesis that the \(N\) video streams are independent, the state probability is given by

\[
p(\alpha_1, \alpha_2, \ldots, \alpha_N) = \prod_{i=1}^{N} p_{\alpha_i}
\]  

(1)

The aggregate bandwidth, needed in this specific state, results \(\Lambda = r_{\alpha_1} + r_{\alpha_2} + \ldots + r_{\alpha_N}\). By considering that several streams can exploit simultaneously the same rate, the system state can be expressed, more simply, by the vector \((n_1, n_2, \ldots, n_N)\), where the generic \(n_i\) indicates the number of streams that exploit rate \(r_i\). Obviously, the total number of streams must be \(N\), i.e., \(n_1 + n_2 + \ldots + n_N = N\). The needed bandwidth for this state is given by \(\Lambda = n_1 r_1 + n_2 r_2 + \ldots + n_N r_N\).

This last result can be justified by simple combinatorial considerations, observing that the \(n_1\) streams that exploit rate \(r_1\), can occupy \(n_1\) of the \(N\) streams and thus the number of combinations is \(\binom{N}{n_1}\). Similarly the \(n_2\) streams that exploit rate \(r_2\), can occupy \(n_2\) of the \(N-n_1\) free streams and thus the number of combinations is \(\binom{N-n_1}{n_2}\), etc.

The needed bandwidth can be evaluated only statistically. In particular, let \(A_s\) be the available bandwidth, we have

\[
p(A \leq A_s) = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} \ldots \sum_{n_K=0}^{N} p(n_1, \ldots, n_K) = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} \ldots \sum_{n_K=0}^{N} \prod_{i=1}^{K} \frac{p_i^{n_i}}{n_i!}
\]  

(3)

with the constraints

\[
\begin{cases}
n_1 + n_2 + \ldots + n_K = N \\
n_1 r_1 + n_2 r_2 + \ldots + n_K r_K \leq A_s
\end{cases}
\]

By imposing a uniform quantization in the histogram, i.e., \(r_2 = 2 r_1\), \(r_3 = 3 r_1\), \ldots, \(r_K = K r_1\), the second constraint becomes \(n_1 + 2 n_2 + \ldots + K n_N \leq A_s/r_1\). The previous expression (3), including the constraints, can be synthetically written as:

\[
p(A \leq A_s) = \sum_{n_1 + 2 n_2 + \ldots + K n_K = N}^{n_1 + n_2 + \ldots + n_K = N} \prod_{i=1}^{K} \frac{p_i^{n_i}}{n_i!}
\]  

(4)

It is quite intuitive to realize that the previous expression can be rewritten evaluating the number of terms required by the disequation in the constraints. We have

\[
p(A \leq A_s) = \sum_{j=0}^{\left\lfloor \frac{A_s}{r_1} \right\rfloor} \sum_{n_1 + 2 n_2 + \ldots + K n_K = j}^{n_1 + n_2 + \ldots + n_K = N} \prod_{i=1}^{K} \frac{p_i^{n_i}}{n_i!}
\]  

(5)

where \(h = \left\lfloor \frac{A_s}{r_1} \right\rfloor\).
The same expression can be obtained by a more formal (and much more complex) approach [5]. It is evident that when \( \Lambda_s \) is equal to the maximum possible rate for \( N \) streams, i.e., \( \Lambda = NKr \), the previous probability must be one.

A brute force evaluation of the previous expression is practically impossible, due to the exponential number of terms in the summation. With the target to find a simple iterative method let us rewrite the previous expression (5) indicating the inner summation by the symbol \( a \). We have:

\[
p(\Lambda \leq \Lambda_s) = \left( \sum_{j=N}^{K} a_{j-N} \right)
\]

(6)

Various alternative iterative algorithms are proposed in the sequel for evaluating these terms \( a_j \). The simple iteration of the first proposed method is reported below (see appendix A).

\[
p_1 a_j = \sum_{i=1}^{K-1} \left( \frac{(N+1)i}{j} - 1 \right) p_{i;i} a_{j-i} \quad \text{where} \quad \begin{cases} j=1,\ldots,(K-1)N \\ a_{j-i}=0; \quad j-i<0 \end{cases}
\]

(7)

with initial condition \( a_0 = p_1^N \).

The terms \( a \) have a precise physical meaning. In fact, \( a_0 \) represents the probability that all \( N \) streams are using their smallest bit rate, \( r \), and thus the global bit rate results \( Nr \). Likewise, the term \( a_1 \) represents the probability that \( N-1 \) streams exploit rate \( r \), while one stream transmits using rate \( r_2 = 2r \), thus the needed bandwidth result \( (N+1)r \). In general we can say that the term \( a_j \) represents the probability that the \( N \) streams exploit globally a bandwidth given by \( (N+j)r \).

As mentioned before, the sum of all \( a \) must be one, i.e., \( \sum_{j=0}^{(K-1)N} a_j = 1 \), indeed, we have verified this condition in the numerical examples. The probability \( P(\Lambda \leq \Lambda_s) \) in (6) can be evaluated by summing all values of \( a \) for \( j \) varying from 0 to \( h-N \). Alternatively, we may evaluate \( 1- p(\Lambda \leq \Lambda_s) \), by summing the values of \( a \) for \( j \) varying from \( h-N+1 \) to its maximum value \( (K-1)N \).

The previous iterative expressions, very simple and efficient, present some problems of error propagation when the number of streams exceeds a few hundreds or even less in some limit case. To overcome this problem and thus generalize the solution a new iterative expression has been derived exploiting the potentiality of \( Z \) transform.

Let \( F(z) \) be the \( Z \) transform of the coefficients \( a_j \); it results (see Appendix B):

\[
F(z) = \sum_{j=0}^{\infty} a_j z^j = \left( \sum_{i=1}^{K} p_i z^{i-1} \right)^N
\]

(8)

Actually, we are interested in defining an alternative procedure to find the coefficients \( a \). Let us rewrite (8) with a minor modification in the symbolism. Let us denote with

\[
F_N(z) = \left( \sum_{i=1}^{K} p_i z^{i-1} \right)^N = P(z)^N = \sum_{j=0}^{(K-1)N} a_j^{(N)} z^j
\]

(9)

Likewise, let us denote by

\[
F_{N-1}(z) = P(z)^{N-1} = \sum_{j=0}^{(K-1)(N-1)} a_j^{(N-1)} z^j
\]

(10)

Obviously, it results
\[ F_N(z) = F_{N-1}(z)P(z) = F_{N-1}(z)[p_1 + p_2z + \ldots + p_Kz^{K-1}] \]

And more explicitly we have

\[
\sum_{j=0}^{(K-1)N} a_j^{(N)}z^j = \sum_{j=0}^{(K-1)(N-1)} a_j^{(N-1)}z^j[p_1 + p_2z + \ldots + p_Kz^{K-1}] = \\
= p_1 \sum_{j=0}^{(K-1)(N-1)} a_j^{(N-1)}z^j + p_2 \sum_{j=0}^{(K-1)(N-1)} a_j^{(N-1)}z^{j+1} + \ldots + p_K \sum_{j=0}^{(K-1)(N-1)} a_j^{(N-1)}z^{j+K-1}
\]

By imposing the equality between the coefficients for the same power of \( z \) in left and right hand side, we have:

\[
a_j^{(N)} = p_1a_j^{(N-1)} + p_2a_{j-1}^{(N-1)} + \ldots + p_Ka_{j-K+1}^{(N-1)}
\]

where the initial condition result \( F_1(z) = P(z) \) and thus we have \( a_0^{(1)} = p_1; a_1^{(1)} = p_2; \ldots; a_{K-1}^{(1)} = p_K \).

From equation (11), we have

\[
a_j^{(2)} = \sum_{m=0}^{K-1} a_m^{(1)}a_{j-m}^{(1)} \quad j=0, 1, \ldots, 2(K-1)
\]

which represents the convolution of the sequence \( a \) with itself; in matrix form \( a^{(2)} = a^{(1)} \ast a^{(1)} \). In general we have:

\[
a_j^{(n)} = \sum_{m=0}^{K-1} a_m^{(n-1)}a_{j-m}^{(n-1)} \quad k=0,1, \ldots, n(K-1)
\]

In matrix form we have

\[
a^{(N)} = a^{(N-1)} \ast a^{(1)}
\]

This last result should not surprise because it is well known that the Z transform of a convolution is a product of the Z transform of the convoluted sequences, as reported in (8).

Still the result reported in (14) can be justified by considering that \( a^{(N)} \) represents the p.m.f. of a sum of \( N \) independent random variables and thus it is given by the convolution among the p.m.f. of the summed random variables [4].

Another technique to find all the terms \( a_j \) acts as follows. We have to calculate:

\[
a_j = \sum_{n+n_2+\ldots+n_{KN}=N} \frac{N!}{n!} \prod_{r=1}^{K} \frac{p_r^{n_r}}{n_r!}, \quad N \leq j \leq KN
\]

Now we define the following function:

\[
g_{\nu, \lambda}(n) = \sum_{n+n_2+\ldots+n_{KN}=N} \frac{n!}{n!} \prod_{r=1}^{\lambda} \frac{p_r^{n_r}}{n_r!}
\]

It is obvious that \( a_j = g_{\nu, \lambda}(N) \).
We can also note that the generic function $g_{s, j}(n)$ will contain all the contributes relative to the function $g_{s-1, j}(n)$ with added some terms necessarily containing all the conditions on $n$, with $n$ assuming all the values between 0 and N and respecting all the boundaries under the symbol of the summation notation in (16).

In particular, for $n_0 = 0$ we have:

$$g_{s, j}(n)|_{n_0=0} = \sum_{n_1=1}^{s} n_1! \prod_{r=1}^{n_1} \frac{p_{s, r}}{n_r!} = \sum_{n_1=1}^{s} n_1! \prod_{r=1}^{n_1} \frac{p_{s, r}}{n_r!} = g_{s-1, j}(n)$$

Now let us consider the contributes to $g_{s, j}(n)$ relative to the case $n_0 = 1$. Since $n_0 = 1$, we will obtain:

$$g_{s, j}(n)|_{n_0=1} = \sum_{n_1=1}^{s} n_1! \prod_{r=1}^{n_1} \frac{p_{s, r}}{n_r!} = \sum_{n_1=1}^{s} \frac{(n-1)!}{(1-n)!} n_1! \prod_{r=1}^{n_1} \frac{p_{s, r}}{n_r!} = \left(\frac{n_1!}{(1-n)!} \cdot p_{s, 1}\right) g_{s-1, j-1}(n-1)$$

Following the same procedure we find that, in general:

$$g_{s, j}(n)|_{n_0=q} = \left(\frac{n!}{(n-q)!} \cdot \frac{p_{s, q}}{q!}\right) g_{s-1, j-q}(n-q) \quad 0 \leq q \leq \frac{j-n}{s-1}$$

The parameter $q$ can arrive only to $\frac{j-n}{s-1}$ since, for the generic $g_{s, j}(n)$, it must be $j \geq n$; similarly, for the function $g_{s-1, j-q}(n-q)$ we have that $j-q \geq n-q$.

If now we call: $I = \frac{j-n}{s-1}$, we have:

$$g_{s, j}(n) = \sum_{p=0}^{I} g_{s, j}(n)|_{n_0=p} = \left(\frac{n!}{(n-q)!} \cdot \frac{p_{s, q}}{q!}\right) g_{s-1, j-q}(n-q) \quad (17)$$

The equation (17) has to be used for $s = 1, 2, ..., K$ and for $n = 1, 2, ..., N$. The initial conditions to be applied are:

$g_{s, 0}(0) = 1, \quad s = 1, 2, ..., K$

$g_{1, 1}(n) = p_{1}^{n}, \quad n = 1, 2, ..., N$

Iterating in this way, we finally obtain all the functions $g_{s, j}(N)$ for each $j$, corresponding to all the terms $a_{ij}$ we want to find.

### B. Numerical Results

In this section the algorithms developed in the previous section will be applied to specific cases in order to evaluate the aggregate bandwidth needed by a given number of streams. In particular, 15000 frames of the film Star Wars have been considered which have been smoothed with a user buffer of 1 MB. The evaluated probability histogram considers 5 bins ($K=5$) giving the following probabilities: $p_1 = 0.13193$; $p_2 = 0.16376$; $p_3 = 0.0048$; $p_4 = 0.0567$; while the rate of $r$, results 147552 bit/s.

The coefficients $a_{ij}$ have been derived by the iterative algorithms exposed in the previous section for 20 streams of the film Star Wars and the results are reported in Fig. 3, which represents the p.m.f. of the global bandwidth needed by the 20 streams. In fact, as discussed before, a generic value of $a_{ij}$ gives the probability that the bandwidth needed by the considered 20 streams is equal to $(N+j)r$, i.e., $a_{ij} = p_{s, 1}^{A=(N+j)r}$. It is not surprising that the shape of the figure is similar to a Gaussian. This derives by an application of the central limit theorem to the sum of 20 random variables supposed independent.
The previous data are exploited to evaluate the aggregate bandwidth comparing the results of the proposed approximations based on iterative methods and on the application of central limit theorem with those obtained by Chernoff bound and simulation for various values of main system parameters, namely, loss probability and number of streams. The reader is referred to reference [3] for the details about the Chernoff bound.

As can be observed from the previous figures, the result obtained by the analytical methods developed in this paper are very close to the results obtained by the Chernoff method. The results of all methods become enough more close to the simulation results increasing the number of bins in the histogram, although after a given value there is a saturation with small improvements.
III. BANDWIDTH ESTIMATION FOR ETEROGENEOUS STREAMS

A. Analytical model

In this section the aggregate bandwidth in case of streams that have different statistical characteristics (as usual with variegated sources), will be evaluated. We suppose that the video server has \( M \) different types of films and each one is characterized by the histogram method, that consists in defining, for a generic film \( i \), a discrete r.v. that can assume the \( K_i \) values \( r_{i1} < r_{i1} < \ldots < r_{ik} \) with probability \( p(r_{ij}) = p_j \) \((j=1, \ldots, K)\). We suppose that a user can choose a film \( i \) with probability \( \beta_i \) (obviously, \( \beta_1 + \beta_2 + \ldots + \beta_M = 1 \)).

The system state can be defined by the following vector \((\alpha_1, \alpha_2, \ldots, \alpha_N)\) where the generic \( \alpha_i \), is a couple of indices \((i,j)\) that identifies the type of stream and the relative rate. In the simplifying hypothesis that the \( N \) video streams are independent, the state probability is given by

\[
p(\alpha_1, \alpha_2, \ldots, \alpha_N) = \prod_{i=1}^{N} p_{\alpha_i}\]

The aggregate bandwidth needed in this specific state, results \( \Lambda = r_{a1} + r_{a2} + \ldots + r_{as} \). By considering that several streams can exploit simultaneously the same rate, the system state can be expressed, more simply, by \((n_{11}, n_{12}, \ldots, n_{ik}, n_{21}, \ldots, n_{MK})\) where the generic \( n_j \) indicates the number of streams that exploit the rate \( r_{ij} \). Obviously, the total number of streams must be \( N \).

The needed bandwidth for this state is given by

\[
A = n_{11}r_{11} + n_{12}r_{12} + \ldots + n_{ik}r_{ik} + \ldots + n_{MK}r_{MK}.
\]

Following the same concepts developed for the single type of streams, the state probability results

\[
p(n_{11}, n_{12}, \ldots, n_{MK}) = \binom{N}{n_{11}} \binom{N-n_{11}}{n_{12}} \ldots \binom{N-n_{11} \ldots - n_{MK}}{n_{MK}} \prod_{m=1}^{M} \beta_m^{n_m} \prod_{r=1}^{r_m} p_{mr}^{n_{mr}} = N! \prod_{m=1}^{M} \prod_{r=1}^{r_m} \left( \frac{\beta_m p_{mr}}{n_{mr}} \right)^{n_{mr}}\]

The needed aggregate bandwidth can be evaluated only statistically and in particular we have

\[
P(A \leq A^*) = \sum_{n_{11}=0}^{N} \sum_{n_{12}=0}^{N} \ldots \sum_{n_{MK}=0}^{N} N! \prod_{m=1}^{M} \prod_{r=1}^{r_m} \left( \frac{\beta_m p_{mr}}{n_{mr}} \right)^{n_{mr}}\]

with the constraints

\[
\begin{cases}
n_{11}r_{11} + n_{12}r_{12} + \ldots + n_{ik}r_{ik} + \ldots + n_{MK}r_{MK} = N \\
n_{11}r_{11} + n_{12}r_{12} + \ldots + n_{ik}r_{ik} + \ldots + n_{MK}r_{MK} \leq A^*
\end{cases}
\]

By imposing the uniform quantization among all films, i.e.,

\[
\begin{cases}
r_{11} = r_{21} = \ldots = r_{M1} = r_1 \\
r_{12} = r_{22} = \ldots = r_{M2} = r_2 = 2r_1 \\
\ldots \\
r_{1K} = r_{2K} = \ldots = r_{MK} = r_K = Kr_1
\end{cases}
\]

and exploiting an approach similar to the homogeneous case, it is possible to find the following result [5].
\[ P(\Lambda \leq \Lambda_s) = \sum_{j=N}^{K} a_{j-N} \]  

(21)

\[ a_{j-N} = \sum_{n_1 \ldots n_j = N}^{N} \frac{N! \prod_m^{K} (\beta_m p_m)^{n_m}}{n_m!} \]  

(22)

The previous expression, like the homogeneous case, cannot be directly exploited due to its complexity. Also in this case an iterative approach can be pursued. In particular, we can find by (22):

\[ a_0 = \left( \sum_{m=1}^{M} (\beta_m p_{m1}) \right)^N \]

\[ a_1 = N \left( \sum_{m=1}^{M} \beta_m p_{m2} \right) \left( \sum_{m=1}^{M} \beta_m p_{m1} \right)^{N-1} \]

Indicating by

\[ p_1 = \sum_{m=1}^{M} (\beta_m p_{m1}); \quad p_2 = \sum_{m=1}^{M} \beta_m p_{m2} \]

we have

\[ a_0 = p_1^N; \quad a_1 = N p_2 p_1^{N-1} \]

which correspond to the first two terms of the iterative method obtained in the case of homogeneous streams.

In general, let

\[ p_j = \sum_{m=1}^{M} \beta_m p_{mj} \quad \forall \ 1 \leq m \leq K \]  

(23)

With the previous position, the expression (22) becomes equivalent to the corresponding equation (6) of the homogeneous case and thus a similar approach can be followed to find an iterative solution.

In general, the following iterative algorithm can be found

\[ p_1 a_j = \sum_{i=1}^{K-1} \frac{(N+1)i!}{j!} p_{mi} a_{j-i} \quad \text{where} \quad \begin{cases} j=1, \ldots, (K-1)N \\ a_{j-i} = 0; \quad j-i < 0 \end{cases} \]  

(24)

with \( a_0 = p_1^N \).

The coefficients \( a_{j-N} \) (likewise the homogeneous case) have a specific meaning. In fact, \( a_0 \) represents the probability that all \( N \) streams are using their smallest bit rate, \( r_s \), and thus the global bandwidth results \( N r_s \). Likewise, the term \( a_1 \) represents the probability that \( N-1 \) streams are exploiting the rate \( r_s \), while one stream is using the rate \( r_{s+1} = 2 r_s \), thus the needed bandwidth result \( (N+1)r_s \). In general we can say that the term \( a_i \) represents the probability that the \( N \) streams exploit a bandwidth given by \((N+k) r_s\).

Likewise the homogeneous case, to overcome some problems of error propagation a new iterative expression has been derived exploiting the potentiality of \( Z \) transform. The derivation is the same of the homogeneous case and, obviously, gives the same results:
\[ a_j^{(N)} = p_j a_j^{(N-1)} + p_2 a_{j-1}^{(N-1)} + \ldots + p_K a_{j-K+1}^{(N-1)} \]

(25)

where the initial conditions result
\[ a_0^{(1)} = p_1; a_1^{(1)} = p_2; \ldots; a_{K-1}^{(1)} = p_K. \]

Equation (25) represents the convolution of the sequence \( a^{(N-1)} \) with the sequence \( a \); in matrix form
\[ a^{(N)} = a^{(N-1)} * a^{(1)} \]

(26)

We can also extend the results obtained using the (20) to the heterogeneous case, but we have to use the (23) to obtain the probabilities \( p_i \) with \( i \) from 1 to \( K \). We will find the same values of the terms \( a_j \) obtained using the (25).

\[ B. \quad \text{Numerical Results} \]

In this section the algorithms developed in the previous section will be applied to specific cases in order to evaluate the aggregate bandwidth needed by a given number of streams. In particular, 15000 frames of the films Star Wars, Asterix and Mr Bean have been considered which have been smoothed with a user buffer of 1 MB. The evaluated probability histogram considers 5 bars (\( K = 5 \)) giving the following probabilities:

<table>
<thead>
<tr>
<th>Star Wars</th>
<th>Asterix</th>
<th>Mr. Bean</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 = 0.93433333 )</td>
<td>( p_1 = 0.31448763 )</td>
<td>( p_1 = 0.69785276 )</td>
</tr>
<tr>
<td>( p_2 = 0.06566667 )</td>
<td>( p_2 = 0.68544570 )</td>
<td>( p_2 = 0.29807949 )</td>
</tr>
<tr>
<td>( p_3 = 0 )</td>
<td>( p_3 = 0.00006667 )</td>
<td>( p_3 = 0.00400107 )</td>
</tr>
<tr>
<td>( p_4 = 0 )</td>
<td>( p_4 = 0 )</td>
<td>( p_4 = 0 )</td>
</tr>
<tr>
<td>( p_5 = 0 )</td>
<td>( p_5 = 0 )</td>
<td>( p_5 = 0.00006668 )</td>
</tr>
</tbody>
</table>

The value of \( r_1 \), common to all considered films results: \( r_1 = 588336 \) bit/s.

The coefficients \( a_j \) have been derived by the algorithms exposed in the previous section for 30 streams of the films, 10 streams for each film (Star War, Asterix, Mr Bean) and the results are reported in Fig. 6, which represents the p.m.f. of the global bandwidth needed by the 30 streams. In fact, as discussed before, a generic value of \( a_j \) gives the probability that the bandwidth needed by the considered 30 streams is equal to \( (N+k)r_1 \), i.e., \( a_j = p_j \lambda = (N+k)r_1 \). The shape of the figure is similar to a Gaussian, likewise the homogeneous case. This derives by an application of the central limit theorem to the sum of 30 random variables supposed independent.

![Figure 6 – p.m.f. of the global bandwidth needed by 30 video streams uniformly distributed among the three films smoothed with a buffer of 1 MB.](image-url)
The previous data are exploited to evaluate the aggregate bandwidth comparing the results of the proposed method with those obtained by Chernoff bound and simulation for various values of main system parameters, namely, loss probability and number of streams.

![Figure 7](image1.png)

**Figure 7** - Aggregate bandwidth for the three films with a loss probability of $10^{-6}$ (with different evaluation methods).

![Figure 8](image2.png)

**Figure 8** - Aggregate bandwidth for the three films with a loss probability of $10^{-6}$ (with the iterative method and different number of bins).

As can be observed from previous figures, the result obtained by the analytical methods developed in this paper, similarly to the homogeneous case, are quite close to the results obtained by the Chernoff method and both become enough more close to the simulation results increasing the number of bins in the histogram, although after a given value there is a saturation without any further improvements.

### IV. CONCLUSIONS

The original algorithms presented in this paper for bandwidth estimation in VBR video distribution systems, open, in the opinion of authors, a new approach to the problem that can be fruitfully explored and that makes use of a probabilities distribution of the single film as obtained with the histogram method. We can also note that this method is valid independently from the shape of the probability distribution and can be used even if the distribution is not referable to the ones usually known in the literature. Nevertheless, the authors are still working to the refinements of these methods. In particular, the first hypothesis that is not imposed is the independence among streams and the authors are confident that a suitable solution can be obtained with variants of the algorithms presented in this paper.

### REFERENCES


The goal of this section is to find an iterative method to evaluate the likelihood that a system composed by a number $N$ of flows assumes a determined bandwidth. We know that:

$$p(\Lambda \leq \Lambda_{sup}) = \sum_{j=N}^{\text{max} \Lambda} \left( \frac{\sum_{i=1}^{K} \prod_{n=1}^{N} \frac{p_{ni}}{n!}}{\prod_{n=1}^{Kn} \frac{1}{n!}} \right) \quad \text{(A.1)}$$

supposing the histogram method and the following subdivision of the rates:

$$r_i = r_k / K; \quad r_2 = 2r_k / K = 2r_i; \quad r_k = Kr_k / K = K r_i$$

We can make the following assumption:

$$a_{j-N} = \sum_{n_1+n_2+...+n_k=N}^{N!} \prod_{i=1}^{K} \frac{p_{ni}}{n_i!} \quad \forall 0 \leq j-N \leq (K-1)N$$

Let us consider the first addend of the external sum; it is obtained for $n_i=N$ and $n_i=n_i=...=n_k=0$. In fact, from this assumption follows that $j=N$. The value of this term will be:

$$a_0 = N!(p_1^N / N!) = p_1^N$$

and clearly represents the probability that all the flows are present in the network with their minimum rate $r_i$ (given the independence of all the flows present in the network).

Similarly, the second addend of the external sum of (A.1) will be obtained assuming $n_i=N-1$, $n_i=1$ and $n_i=...=n_k=0$. With this position, in fact, we will obtain $j=N+I$. In this case, we have:

$$a_1 = N!(p_1^{N-1} p_2) / (N-1)! = N p_1^{N-1} p_2$$

and therefore:

$$p_1 a_1 = N p_1^N p_2 = N p_2 a_0 .$$

Now let us find the term $a_2$. The conditions to be respected are the following:

$$\begin{cases} n_1+n_2+...+Kn_k = N+2 \\ n_1+n_2+...+n_k = N \end{cases}$$

There are two cases that respect the conditions mentioned above. The first is $n_i=N-2$, $n_i=2$ and $n_i=...=n_k=0$ and the second is $n_i=N-1$, $n_i=1$ and $n_i=...=n_k=0$. For this reason, the term $a_2$ will be the sum of two addends:

$$a_2 = \frac{N! p_1^{N-2} p_2^2}{(N-2)!} = \frac{N(N-1)}{2} p_1^{N-2} p_2^2$$

Therefore:

$$p_1 a_2 = [(N-1)/2] p_2 a_1 + N p_2 a_0 .$$

Now if we observe the structure of the terms $p_1 a_1$ and $p_1 a_2$, we can suppose also the structure of $p_1 a_3$. Let us imagine that the term $p_1 a_3$ is structured in this way:

$$p_1 a_3 = \frac{N-2}{3} p_2 a_2 + C_1(N) p_3 a_2 + C_2(N) p_4 a_2$$
where $C_1(N)$ and $C_2(N)$ are generic coefficients depending only on the number $N$ of flows. To find the value of these coefficients, let us replace the expressions of $a_2$, $a_1$ and $a_0$ into the expression of $p_1a_3$. We find that:

$$a_3 = \frac{N(N-1)(N-2)}{6}p_1^{N-3}p_2^3 + \frac{N(N-2)}{3}p_1^{N-1}p_2p_3 + NC_1(N)p_1^{N-2}p_2p_3 + C_2(N)p_1^{N-1}p_4.$$  \hspace{1cm} (A.2)

This expression suggests us all the combinations of $n$ indexes that verify the constraints necessary to find $a_3$, which are:

\[
\begin{align*}
n_1 + 2n_2 + \ldots + Kn_k &= N + 3 \\
n_1 + n_2 + \ldots + n_k &= N
\end{align*}
\]

In fact, the first addend of $a_3$ suggests us that $n_1=N-3$, $n_2=3$ and $n_i=\ldots=n_k=0$; the second and the third addend suggest that $n_1=N-2$, $n_2=1$ and $n_i=\ldots=n_k=0$; the forth one suggests that $n_1=N-1$, $n_2=1$ and $n_i=\ldots=n_k=0$. We can easily observe that all these combinations of $n_i$ verify the two constraints described above.

Then we can calculate the exact expression of $a_3$ through the $n_i$ combinations found above and utilizing (A.1); the result is given by:

$$a_3 = \frac{N(N-1)(N-2)}{6}p_1^{N-3}p_2^3 + N(N-1)p_1^{N-3}p_2p_3 + Np_1^{N-3}p_4.$$  \hspace{1cm} (A.3)

Comparing this equation with (B.2), it is possible to find the values for $C_1(N)$ and $C_2(N)$:

$$C_1(N) = \frac{2N-1}{3}; \hspace{1cm} C_2(N) = N$$

Now we finally can assert that:

$$p_1a_3 = [(N-2)/3]p_2a_2 + [(2N-1)/3]p_3a_1 + Np_4a_0$$

The same procedure illustrated here to find $p_1a_3$ can obviously be adopted to find all the other terms $p_1a_4$, $p_1a_5$, etc. The steps are:

1) Express the term $p_1a_j$ as:

$$p_1a_j = \frac{N-j+1}{j}p_2a_{j-1} + \sum_{i=1}^{K-2} C_i(N)p_{i+2}a_{j-i-1} = \sum_{i=1}^{K-1} C_i(N)p_{i+1}a_{j-i};$$  \hspace{1cm} (A.3)

2) replace all the terms $a_{j-1}$, previously found, in the expression of $p_1a_j$;

3) find all the combinations of $n_i$ indexes that verify the conditions:

\[
\begin{align*}
n_1 + 2n_2 + \ldots + Kn_k &= N + j \\
n_1 + n_2 + \ldots + n_k &= N
\end{align*}
\]

and find the exact expression of $a_j$;

4) find the coefficients $C_i(N)$ comparing the expression found at step 1 with the one found at step 1.

By way of illustration, we report the first five terms found applying this method:
In general, since the probabilities are in number of $K$ (that is the number of the bins of the histogram), for $j$ from $1$ to $K-1$, the terms $p_i a_j$ will present addends containing the terms $a_i$ till the term $a_j$, while for $j>K-1$, all the terms $p_i a_j$ will contain exactly $K-1$ addends, each of them containing the probabilities from $p_i$ to $p_s$.

It is possible to find directly all the coefficients, $C_i^j(N)$, of the generic term $p_i a_j$ with a simple expression. For all $j$, we can express the first coefficient $C_i^j(N)$ in the following way:

$$C_i^j(N) = \binom{N-j+1}{j} = \frac{\left[ \begin{array}{c} j-1 \\ 0 \end{array} \right]_N - \left[ \begin{array}{c} j-1 \\ 1 \end{array} \right]_N}{\binom{j}{1}} \quad (A.4)$$

Now considering all the coefficients $C_i^j(N)$, for $j>2$, we can easily see that:

$$C_2^2(N) = \binom{1}{1} \cdot N;$$

$$C_3^2(N) = \frac{2}{3} \cdot \left\lbrack \begin{array}{c} 2 \\ 1 \end{array} \right\rbrack_N \cdot \frac{1}{2} = \frac{1}{3} \cdot \left\lbrack \begin{array}{c} 2 \\ 1 \end{array} \right\rbrack_N \cdot \frac{1}{2};$$

$$C_4^2(N) = \frac{3}{6} \cdot \left\lbrack \begin{array}{c} 3 \\ 1 \end{array} \right\rbrack_N \cdot \frac{1}{2} = \frac{1}{4} \cdot \left\lbrack \begin{array}{c} 3 \\ 1 \end{array} \right\rbrack_N \cdot \frac{1}{2};$$

and so on. That is:

$$C_i^j(N) = \frac{\left\lbrack \begin{array}{c} j-1 \\ 1 \end{array} \right\rbrack_N - \left\lbrack \begin{array}{c} j-1 \\ 2 \end{array} \right\rbrack_N}{\binom{j}{2}}. \quad (A.5)$$

Adopting the same procedure for all the other $C_i^j(N)$ for $i$ from $3$ to $j-1$, the following important relation is obtained:

$$C_i^j(N) = \frac{\left\lbrack \begin{array}{c} j-1 \\ i-1 \end{array} \right\rbrack_N - \left\lbrack \begin{array}{c} j-1 \\ i \end{array} \right\rbrack_N}{\binom{j}{i}}. \quad (A.6)$$

and with a bit of algebra we have:

$$C_i^j(N) = \frac{(N+1)i}{j}. \quad (A.7)$$
This last expression presents the advantages of a great simplicity and also it is independent from the coefficients $C_{i}^{j-i}(N)$ calculated at the previous step. Using (A.7) in (A.3), we find the expression (7)

$$p_{1}a_{j} = \sum_{i=1}^{K-1} \left[ \frac{(N+1)i}{j} - 1 \right] p_{j-i} a_{j-i}$$  \hspace{1cm} (A.8)$$

with \[ j=1,\ldots,(K-1)N \]
\[ a_{j-i}=0; \quad j-i<0. \]

**APPENDIX B**

The purpose of this section is to find Z transform of of the terms $a$ (considered in detail in Appendix A). The expression (A.8) can be rewritten in the following way

$$jp_{1} a_{j} = (N+1-j)p_{2} a_{j-i} + \ldots + [(K-1)N+(K-1)-j]p_{K} a_{j-K+1}$$

Multiplying left and right hand side by $z^{j}$ and summing for all values of $j$ from 0 to infinity, we obtain:

$$p_{1} \sum_{j=0}^{\infty} ja_{j} z^{j} = p_{2} \sum_{j=1}^{\infty} (N+1-j)a_{j-1} z^{j} + \ldots + p_{K} \sum_{j=K-1}^{\infty} [(K-1)(N+1)-j]a_{j-K+1} z^{j}$$ 

(B.1)

Let $F(z)$ be the Z transform of the coefficients $a$.

$$F(z) = \sum_{k=0}^{\infty} a_{k} z^{k}$$  \hspace{1cm} (B.2)$$

We can express all the addends of (B.1) introducing the Z-transform. In particular, for the first member of (B.1) we say that:

$$p_{1} \sum_{j=0}^{\infty} ja_{j} z^{j} = p_{2} z \sum_{j=1}^{\infty} ja_{j} z^{j-1} = p_{1} z \frac{d}{dz} F(z)$$

Let us continue developing the first addend of the second member of (B.1). We have:

$$p_{2} \sum_{j=1}^{\infty} (N-j+1)a_{j-1} z^{j} = p_{2} \sum_{j=1}^{\infty} (N+1)a_{j-1} z^{j} - p_{2} \sum_{j=1}^{\infty} ja_{j-1} z^{j} =$$

$$= p_{2} (N+1) z F(z) - p_{2} z^{2} \frac{d}{dz} \left[ z F(z) \right]$$

Now let us develop the second addend of the second member of (B.1):

$$p_{2} \sum_{j=2}^{\infty} (2N-j+2) a_{j-1} z^{j} = 2(N+1) p_{2} \sum_{j=2}^{\infty} a_{j-2} z^{j} - p_{2} \sum_{j=2}^{\infty} ja_{j-2} z^{j} =$$

$$= 2(N+1) p_{2}z^{2} F(z) - p_{2} z^{2} \frac{d}{dz} \left[ z^{2} F(z) \right]$$

In the same way we can easily generalize the results for the generic $i$-th addend of the second member of (B.1). We can easily find that:
\[ \sum_{j=i}^{\infty} (iN - j + i) a_{j-i} z^j = ip_{i+1}(N+1)z^{i} F(z) - p_{i+1} z \frac{d}{dz} z^i [F(z)] \]  \hspace{1cm} (B.3)\]

with \(1 \leq i \leq K-1\).

Replacing all the terms (B.3) in (B.2) and grouping together the similar terms, and isolating all the members containing the derivative of \(F(z)\) to the first member, we have:

\[ zF'(z)[p_1 + p_2 z + \ldots + p_K z^{K-1}] = NzF(z)[p_1 + 2p_2 z + \ldots + (K-1)p_K z^{K-2}] \]  \hspace{1cm} (B.4)\]

Let \(P(z)\) be:
\[ P(z) = p_1 + p_2 z + \ldots + p_K z^{K-1} = \sum_{i=1}^{K} p_i z^{i-1} \]

(B.4) becomes:
\[ F'(z)P(z) = NF(z)P'(z) \]

that is
\[ \frac{F'(z)}{F(z)} = \frac{P'(z)}{P(z)} \]  \hspace{1cm} (B.5)\]

Integrating both members of the last equation we obtain:
\[ F(z) = P(z)^N = \left( \sum_{i=1}^{K} p_i z^{i-1} \right)^N. \]  \hspace{1cm} (B.6)