Summation of a family of finite secant sums

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Abstract

We use contour integrals and the Cauchy residue theorem in order to derive several summation formulas, in terms of the higher-order Bernoulli polynomials and the ordinary Bernoulli and Euler polynomials, for a remarkably general family of secant sums. Numerous (known or new) special cases are shown to follow readily from the summation formulas presented in this paper.

Keywords: Secant sums; Finite summation formulas; Contour integration; Cauchy residue theorem; Higher-order Bernoulli polynomials; Euler polynomials and numbers

1. Introduction

Recently, by making use of the generating-function method (see, for details [2,9]), Chen [3] evaluated the following sum:

\[ S_{2n}(q) := \sum_{p=0}^{q-1} \sec^n \left( \frac{p\pi}{q} \right). \] (1.1)

On the other hand, Chu [4] has written a brief Maple program to compute symbolically the following finite sum:

\[ A_n(q) := \sum_{p=0}^{q-1} \sec^n \left( \frac{2p\pi}{q} \right) \quad (n, q \in \mathbb{N} := \{1, 2, 3, \ldots\}; \ q \text{ is odd}) \] (1.2)

and other related finite trigonometric sums. In addition, Chu [4] derived the generated function of the sequence \( \{A_n(q)\}_{n=0}^{\infty} \) defined by (1.2).

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In this sequel to the recent investigations by Chen [3] and Chu [4], we consider a substantially more general finite secant sum given by

\[ S_{2n}(q, r) := \sum_{p=0}^{q-1} \cos \left( \frac{2p\pi r}{q} \right) \sec^{2n} \left( \frac{p\pi}{q} \right) \quad (n \in \mathbb{N}; \ q \in \mathbb{N} \setminus \{1\}; \ r = 0, 1, 2, \ldots, q - 1) \]  

(1.3)

and its various special cases including (for example) \( S_n(q) \) and \( A_n(q) \) defined by (1.1) and (1.2), respectively. The object of this paper is to sum \( S_{2n}(q, r) \) in closed form. We use contour integrals and the Cauchy residue theorem in order to derive the summation formula in terms of the higher-order Bernoulli polynomials and the ordinary Bernoulli and Euler polynomials.

2. Preliminaries and statements of the main results

In what follows, we denote by \( B_n^{(m)}(x) \) and \( E_n^{(m)}(x) \), respectively, the Bernoulli polynomial of order \( m \) and degree \( n \) in \( x \) and the Euler polynomial of order \( m \) and degree \( n \) in \( x \), defined by means of the following generating functions (see, for details, [6, p. 53 et seq.; 8, Chapter 1, Section 1.6]):

\[ \left( \frac{t}{e^t - 1} \right)^m e^x = \sum_{n=0}^{\infty} B_n^{(m)}(x) \frac{t^n}{n!} \quad (|t| < 2\pi; \ m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) \]  

(2.1)

and

\[ \left( \frac{2}{e^t + 1} \right)^m e^x = \sum_{n=0}^{\infty} E_n^{(m)}(x) \frac{t^n}{n!} \quad (|t| < \pi; \ m \in \mathbb{N}_0). \]  

(2.2)

We note that

\[ B_n^{(m)} := B_n^{(m)}(0) \quad (m, n \in \mathbb{N}_0) \]  

(2.3)

denotes the \( n \)th Bernoulli number of order \( m \). For \( m = 1 \), we have

\[ B_n(x) := B_n^{(1)}(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x) \quad (n \in \mathbb{N}_0), \]  

(2.4)

where \( B_n(x) \) and \( E_n(x) \) are, respectively, the relatively more familiar (ordinary) Bernoulli and (ordinary) Euler polynomials (see, for instance, [1, p. 804 et seq.]). The (ordinary) Bernoulli numbers \( B_n \) and the (ordinary) Euler numbers \( E_n \) are given by

\[ B_n := B_n(0) \quad \text{and} \quad E_n := 2^n E_n \left( \frac{1}{2} \right) \quad (n \in \mathbb{N}_0), \]  

(2.5)

respectively.

We use the floor function denoted by \([x]\), also called the greatest integer function or the integer value, which gives us the largest integer less than or equal to \( x \).

Our main results are now being presented here as follows.

**Theorem 1.** Let \( B_n^{(m)}(x) \) be the Bernoulli polynomial of order \( m \) and degree \( n \) in \( x \) defined by (2.1) and let \( B_n(x) \) and \( E_n(x) \) be the (ordinary) Bernoulli and the (ordinary) Euler polynomial defined as in (2.4). Then the sums \( S_{2n}(q, r) \) in (1.3) are given by

\[ S_{2n}(q, r) = \mathcal{S}_{2n}(q, r) := \begin{cases} \mathcal{S}^o_{2n}(q, r) & (q \text{ is odd}), \\ \mathcal{S}^e_{2n}(q, r) & (q \text{ is even}), \end{cases} \]  

(2.6)

where

\[ \mathcal{S}^o_{2n}(q, r) = \frac{(-1)^{n+r} 2^{2n-1}}{(2(n-1))!} \sum_{x=0}^{n-1} \left( \frac{2(n-1)}{2x} \right) E_{2x+1} \left( \frac{r}{q} \right) B_n^{(2x)} \left( 2(n-1)-2x \right)(n) \frac{q^{2x+2}}{2x+1} \]
and

$$\mathcal{S}_{2n}^q(q, r) = (-1)^{n+r-1} \frac{2^{2n}}{(2n)!} \sum_{x=0}^{n} \left( \frac{2n}{2x} \right) B_{2n} \left( \frac{r}{q} \right) B_{2n-2x}^{(2n)}(n) q^{2x} \quad (n \in \mathbb{N}; q \in \mathbb{N} \setminus \{1\}; r = 0, 1, 2, \ldots, q - 1).$$

**Corollary 1.** Let $B_m^{(n)}(x)$ be the Bernoulli polynomial of order $m$ and degree $n$ in $x$ defined by (2.1) and let $B_n$ be the Bernoulli number defined as in (2.3). Then the sums $S_{2n}(q)$ in (1.1) are given by

$$S_{2n}(q) = \mathcal{S}_{2n}^q(q) := \begin{cases} \mathcal{S}_{2n}^q(q) & (q \text{ is odd}), \\ \mathcal{S}_{2n}^q(q) & (q \text{ is even}), \end{cases}$$

where

$$\mathcal{S}_{2n}^q(q) = (-1)^{n-1} \frac{2^{2n}}{(2n-1)!} \sum_{x=0}^{n-1} \left( \frac{2(n-1)}{2x} \right) B_{2n+2} B_{2(n-1)-2x}^{(2n)}(n) \frac{(2^{2x+2} - 1) q^{2x+2}}{(2x + 2)(2x + 1)}$$

and

$$\mathcal{S}_{2n}^q(q) = (-1)^{n-1} \frac{2^{2n}}{(2n)!} \sum_{x=0}^{n} \left( \frac{2n}{2x} \right) B_{2n} B_{2n-2x}^{(2n)}(n) q^{2x} \quad (n \in \mathbb{N}; q \in \mathbb{N} \setminus \{1\}).$$

**Corollary 2.** Consider the following cosecant sums:

$$C_{2n}(q, r) := \sum_{p=1}^{q-1} \cos \left( \frac{2rp\pi}{q} \right) \csc^{2n} \left( \frac{p\pi}{q} \right) \quad (n \in \mathbb{N}; q \in \mathbb{N} \setminus \{1\}; r = 0, 1, 2, \ldots, q - 1).$$

If $q$ is an even positive integer, then the sums $C_{2n}(q, r)$ are given, in terms of $B_{m}^{(n)}(x)$ and $B_{n}(x)$ [see (2.1) and (2.4)], by

$$C_{2n}(q, r) = (-1)^{n+r-1} \frac{2^{2n}}{(2n)!} \sum_{x=0}^{n} \left( \frac{2n}{2x} \right) B_{2n} \left( \frac{r}{q} \right) B_{2n-2x}^{(2n)}(n) q^{2x}.$$  

**Corollary 3.** Let $\mathcal{S}_{2n}^q(q)$ be given by (2.7) and let $\lfloor x \rfloor$ be the floor (or the greatest integer) function. If $n$ and $q$ are positive integers, then

$$\sum_{p=0}^{q} \sec^{2n} \left( \frac{2p\pi}{2q + 1} \right) = \frac{1}{2} \lfloor \mathcal{S}_{2n}^q(2q + 1) + 1 \rfloor,$$

$$\sum_{p=0}^{q} \sec^{2n} \left( \frac{p\pi}{2q + 1} \right) = \frac{1}{2} \lfloor \mathcal{S}_{2n}^q(2q + 1) + 1 \rfloor,$$

$$\sum_{p=1}^{q} \sec^{2n} \left( \frac{2p - 1}{2q + 1} \right) = \frac{1}{2} \lfloor \mathcal{S}_{2n}^q(2q + 1) - 1 \rfloor,$$

$$\sum_{p=0}^{q-1} \sec^{2n} \left( \frac{p\pi}{2q} \right) = \frac{1}{2} \lfloor \mathcal{S}_{2n}^q(2q) + 1 \rfloor$$

and

$$\sum_{p=0}^{\lfloor (q-1)/2 \rfloor} \sec^{2n} \left( \frac{p\pi}{q} \right) = \frac{1}{2} \lfloor \mathcal{S}_{2n}^q(q) + 1 \rfloor.$$

**Remark 1.** As matter of fact, the summation formula asserted by Corollary 2 is valid for any positive integer $q \ (q \neq 1)$, as it has already been shown by Cvijović et al. [5]. However, here in our present investigation, this
Assume that 

**Lemma 1.** Assume that \( \lambda \) is a real number and \( m \) is a nonnegative integer.

(i) Let the Bernoulli polynomials \( B_n^{(m)}(x) \) of order \( m \) and degree \( n \) be defined as in (2.1). If \(|z| < \pi\), then

\[
\left( \frac{z}{\sin z} \right)^m \cos(2\lambda z) = \sum_{n=0}^{\infty} (-1)^n B_{2n}^{(m)} \left( \lambda + \frac{1}{2} m \right) \frac{(2z)^{2n}}{(2n)!}
\]

and

\[
\left( \frac{z}{\sin z} \right)^m \sin(2\lambda z) = \sum_{n=0}^{\infty} (-1)^n B_{2n+1}^{(m)} \left( \lambda + \frac{1}{2} m \right) \frac{(2z)^{2n+1}}{(2n+1)!}
\]

(ii) Let the Euler polynomials \( E_n^{(m)}(x) \) of order \( m \) and degree \( n \) be defined as in (2.2). If \(|z| < \frac{\pi}{2}\), then

\[
\left( \frac{1}{\cos z} \right)^m \cos(2\lambda z) = \sum_{n=0}^{\infty} (-1)^n E_{2n}^{(m)} \left( \lambda + \frac{1}{2} m \right) \frac{(2z)^{2n}}{(2n)!}
\]

and

\[
\left( \frac{1}{\cos z} \right)^m \sin(2\lambda z) = \sum_{n=0}^{\infty} (-1)^n E_{2n+1}^{(m)} \left( \lambda + \frac{1}{2} m \right) \frac{(2z)^{2n+1}}{(2n+1)!}
\]

**Lemma 2.** In terms of the polynomials \( B_n^{(m)}(x) \) in (2.1), the following expansion formula holds true:

\[
\left( \frac{z}{\sin z} \right)^m = \sum_{n=0}^{\infty} (-1)^n B_{2n}^{(m)} \frac{1}{2^m} \frac{(2z)^{2n}}{(2n)!} \quad (|z| < \pi; \; m \in \mathbb{N}).
\]

**Proof of Lemma 1.** Set

\[
\theta = 2iz \quad \text{or} \quad z = -\frac{i\theta}{2} \quad (i := \sqrt{-1}).
\]

(i) We first note that each of the functions on the left-hand sides of (3.1) and (3.2) possesses a removable singularity at \( z = 0 \). Thus it is easy to obtain

\[
\left( \frac{z}{\sin z} \right)^m \left[ \cos(2\lambda z) + i \sin(2\lambda z) \right] = \left( \frac{\theta}{e^\theta - 1} \right)^m e^{(\lambda + \frac{1}{2} m)\theta} = \sum_{n=0}^{\infty} B_{2n}^{(m)} \left( \lambda + \frac{1}{2} m \right) \frac{\theta^n}{n!} \quad (|\theta| < 2\pi; \; m \in \mathbb{N}_0),
\]

where we have made use of the definition (2.1). Upon writing \( \theta = 2iz \) in the last member of (3.6), we are led directly to the assertions (3.1) and (3.2) of Lemma 1.

(ii) In this case, we have

\[
\left( \frac{1}{\cos z} \right)^m \left[ \cos(2\lambda z) + i \sin(2\lambda z) \right] = \left( \frac{2}{e^\theta + 1} \right)^m e^{(\lambda + \frac{1}{2} m)\theta} = \sum_{n=0}^{\infty} E_{2n}^{(m)} \left( \lambda + \frac{1}{2} m \right) \frac{\theta^n}{n!} \quad (|\theta| < \pi; \; m \in \mathbb{N}_0).
\]

The required expansions in (3.3) and (3.4) readily follow from (3.7) by a similar argument as in the preceding proof. \( \square \)

**Proof of Lemma 2.** In its special case when \( \lambda = 0 \), the power-series expansion (3.1) immediately yields (3.5). \( \square \)
4. Proofs of the main results

Proof of Theorem 1. Our derivation of the summation formula (2.6) would make use of contour integration and calculus of residues.

Let \( C \) denote the positively-oriented indented square with vertices at

\[
\frac{\pi}{2} \pm \frac{i}{2} \quad \text{and} \quad -\frac{\pi}{2} \pm \frac{i}{2}
\]

and two semicircular indentations, both of radius \( R_q \)

\[
0 < R_q < \frac{\pi}{q},
\]

one to the left of \( \frac{\pi}{2} \) and the other to the right of \( -\frac{\pi}{2} \). Let

\[
\Phi(z) = \frac{q \cos[(2r - q)z]}{\sin(qz)} \sec^{2n} z \quad (n \in \mathbb{N}; \ q \in \mathbb{N} \setminus \{1\}; \ r = 0, 1, \ldots, q - 1)
\]

and consider the contour integral of \( \Phi(z) \) along \( C \).

First of all, upon using the Cauchy residue theorem, we find that

\[
\oint_C \Phi(z) \, dz = 2\pi i \sum_{p=-[(q-1)/2]}^{[(q-1)/2]} \text{Res}(\Phi(z)) = 2\pi i \sum_{p=-[(q-1)/2]}^{[(q-1)/2]} \cos \left( \frac{2rpn\pi}{q} \right) \sec^{2n} \left( \frac{pn\pi}{q} \right) = 2\pi i \mathcal{S}_{2n}(q, r), \quad (4.1)
\]

where

\[
z_p = \frac{p\pi}{q} \quad (p = 0, \pm 1, \pm 2, \ldots, \pm [(q-1)/2]; \ q \in \mathbb{N} \setminus \{1\}).
\]

This can easily be obtained because the only singularities of \( \Phi(z) \) (\( p, q \) and \( r \) are fixed) that lie inside the contour \( C \) are at \( z_p \) and the residues are given by

\[
\text{Res}(\Phi(z)) = \text{Res}_{z=z_p} \left( q \cos(2rz) \cot(qz) \sec^{2n} z + \sin(2rz) \sec^{2n} z \right) = \text{Res}_{z=z_p} \left( q \cos(2rz) \cot(qz) \sec^{2n} z \right)
\]

\[
= \cos \left( \frac{2rpn\pi}{q} \right) \sec^{2n} \left( \frac{pn\pi}{q} \right).
\]

Secondly, we show that, in fact, we have

\[
\oint_C \Phi(z) \, dz = \int_{\Gamma_1} \Phi(z) \, dz + \int_{\Gamma_2} \Phi(z) \, dz = -\int_{\Gamma} \Phi(z) \, dz,
\]

(4.2)

where \( \Gamma, \Gamma_1 \) and \( \Gamma_2 \) stand for the contours given below:

\[
\Gamma: \text{the circle } |z - \frac{\pi}{2}| = R_q \quad (0 < R_q < \frac{\pi}{q}),
\]

\[
\Gamma_1: \text{the semicircle } |z - \frac{\pi}{2}| = R_q \quad (\Re(z) \leq \frac{\pi}{2}; \ 0 < R_q < \frac{\pi}{q}),
\]

\[
\Gamma_2: \text{the semicircle } |z + \frac{\pi}{2}| = R_q \quad (\Re(z) \geq -\frac{\pi}{2}; \ 0 < R_q < \frac{\pi}{q})
\]

and where the contour \( \Gamma \) is traversed in the positive (counter-clockwise) direction while \( \Gamma_1 \) and \( \Gamma_2 \) are negatively-oriented contours.

It can indeed be easily verified that the function \( \Phi(z) \) (for fixed \( q, r \) and \( n \)) is a periodic function given by

\[
\Phi(z + k\pi) = \Phi(z) \quad (k \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots \})
\]

as well as an odd function of \( z \). Now, since \( \Phi(z) \) is an odd function, both the integrals over horizontal sides of the indented square contour \( C \) vanish. Furthermore, in view of the periodicity of \( \Phi(z) \), we find that, except for the integrals over the semicircular indentations, all other integrals over the indented vertical sides of \( C \) cancel. We also have.
\[
\int_{\Gamma_{1}} \Phi(z) \, dz + \int_{\Gamma_{2}} \Phi(z) \, dz = -\int_{\Gamma} \Phi(z) \, dz,
\]

because \( \Phi(z) \) is a periodic function of \( z \). We recall here that the circle \( \Gamma \) is traversed in the positive (counterclockwise) direction while the semicircles \( \Gamma_{1} \) and \( \Gamma_{2} \) are negatively-oriented contours. In this way we have arrived at a very simple result given by (4.2).

Thirdly, we find from (4.1) and (4.2) that

\[
S_{2n}(q, r) = \frac{1}{2\pi i} \oint_{\Gamma} \Phi(z) \, dz = -\frac{1}{2\pi i} \oint_{\Gamma} \Phi(z) \, dz = -\text{Res}_{z=\frac{1}{2}} \left( \Phi(z) \right).
\]

This means that, in order to sum \( S_{2n}(q, r) \), we have to compute the required residue. Upon the substitution \( \zeta = z + \frac{\pi}{2} \), this computation can be separated into the following two cases:

\[
\text{Res}_{z=\frac{1}{2}}(\Phi(\zeta)) = \begin{cases} 
\text{Res}_{z=0}(\Phi^{\text{odd}}(z)) & (q \text{ is odd}), \\
\text{Res}_{z=0}(\Phi^{\text{even}}(z)) & (q \text{ is even}),
\end{cases}
\]

where

\[
\Phi^{\text{odd}}(z) = (-1)^{q} q \cdot \left( \frac{\sin((2r-q)z)}{\cos(qz)} \right) \csc^{2n}z
\]

and

\[
\Phi^{\text{even}}(z) = (-1)^{q} q \cdot \left( \frac{\cos((2r-q)z)}{\sin(qz)} \right) \csc^{2n}z.
\]

Finally, we compute the residues in (4.4). First, we find the coefficients in the Laurent-series expansions of \( \Phi^{\text{odd}}(z) \) and \( \Phi^{\text{even}}(z) \). In the case of \( \Phi^{\text{odd}}(z) \), by applying the expansions (3.4) (with \( m=1 \)) and (3.5), we have

\[
\Phi^{\text{odd}}(z) = z^{-2n}(1) \cdot \frac{1}{\cos(qz)} \sin \left[ 2 \left( \frac{r}{q} - \frac{1}{2} \right) qz \right] (z \csc z)^{2n}
\]

\[
= z^{-2n}(1) \sum_{k=0}^{\infty} (-1)^{k} E_{2k+1}^{(1)} \left( \frac{r}{q} \right) (2qz)^{2k+1} (2x+1)! \sum_{\beta=0}^{\infty} (-1)^{\beta} B_{2\beta}^{(2n)}(n) \frac{(2z)^{2\beta}}{(2\beta)!}
\]

\[
= \sum_{k=0}^{\infty} z^{2k-2n+1}(1) \cdot \frac{q}{(2k+1)!} \left( \frac{q}{2x} \right) E_{2k+1}^{(1)} \left( \frac{r}{q} \right) B_{2k-2n}^{(2n)}(n) \frac{q^{2k+1}}{2x+1}
\]

\[
= \sum_{k=0}^{\infty} z^{2k-2n+1}(1) \cdot \frac{2k+1}{(2k)!} \sum_{l=0}^{k} \left( \frac{2k}{2l} \right) E_{2l+1}^{(1)} \left( \frac{r}{q} \right) B_{2l-2n}^{(2n)}(n) \frac{q^{2k+1}}{2x+1}.
\]

Observe here that we have made use of the Cauchy product of two power series and also of the fact that

\[
E_{n}(x) := E_{n}^{(1)}(x).
\]

Next, by extracting the coefficient of \( z^{-1} \) in the Laurent series (4.5), we find that the residue of \( \Phi^{\text{odd}}(z) \) at \( z = 0 \) is equal to

\[
\frac{(-1)^{n+1} 2^{n-1}}{(2(n-1))!} \sum_{l=0}^{n-1} \left( \frac{2(n-1)}{2l} \right) E_{2l+1}^{(1)} \left( \frac{r}{q} \right) B_{2l(n-1)-2l}^{(2n)}(n) \frac{q^{2l+1}}{2l+1},
\]

which, in light of (4.3) and (4.4), yields the required summation formula (2.6) for \( S_{2n}(q, r) \) when \( q \neq 1 \) is an odd positive integer.

In a similar manner, we can compute the residue for the function \( \Phi^{\text{even}}(z) \) by extracting the coefficient of \( z^{-1} \) in its Laurent-series expansion obtained by making use of the power-series expansions (3.2) (with \( m=1 \)) and (3.5). This completes our proof of Theorem 1.
Proof of Corollary 1. Since 
\[ S_{2n}(q) := S_{2n}(q, 0), \]
the asserted formulas would follow at once from Theorem 1. In the case when \( q \) is an odd integer, we also need the following well-known relation [1, p. 805, Eq. (23.1.20)]:
\[ E_n(0) = -\frac{2}{n+1} (2^{n+1} - 1) B_{n+1}. \]

Proof of Corollary 2. It is an easy exercise to show that 
\[ S_{2n}(q, r) = C_{2n}(q, r) \]
when \( q \) is an even positive integer. Thus the assertion (2.9) follows readily from (2.7). \( \Box \)

Proof of Corollary 3. We deduce the asserted summation formulas by applying some elementary series identities in conjunction with Corollary 1. \( \Box \)

5. Concluding remarks and observations

We have shown that the sum \( S_{2n}(q, r) \) can be evaluated in closed form in terms of the higher-order Bernoulli polynomials and the ordinary Bernoulli and Euler polynomials. The closed-form expression for \( S_{2n}(q, r) \) gives rise to polynomials in \( q \) of degree \( 2n \) with rational coefficients. As illustrative examples, we list a few of these closed-form expressions in each case when \( q \) is an odd positive integer and when \( q \) is an even positive integer.

First, for an odd positive integer \( q \), we have
\[
S_2(q, r) = (-1)^{r-1} \left[ 2E_1 \left( \frac{r}{q} \right) q^2 \right];
\]
\[
S_4(q, r) = (-1)^{r-1} \left[ \frac{4}{3} E_3 \left( \frac{r}{q} \right) q^4 - \frac{4}{3} E_1 \left( \frac{r}{q} \right) q^2 \right];
\]
\[
S_6(q, r) = (-1)^{r-1} \left[ \frac{4}{15} E_5 \left( \frac{r}{q} \right) q^6 - \frac{4}{3} E_3 \left( \frac{r}{q} \right) q^4 + \frac{16}{15} E_1 \left( \frac{r}{q} \right) q^2 \right];
\]
\[
S_8(q, r) = (-1)^{r-1} \left[ \frac{8}{315} E_7 \left( \frac{r}{q} \right) q^8 - \frac{16}{45} E_5 \left( \frac{r}{q} \right) q^6 + \frac{56}{45} E_3 \left( \frac{r}{q} \right) q^4 - \frac{32}{35} E_1 \left( \frac{r}{q} \right) q^2 \right];
\]
\[
S_{10}(q, r) = (-1)^{r-1} \left[ \frac{4}{2835} E_9 \left( \frac{r}{q} \right) q^{10} - \frac{8}{189} E_7 \left( \frac{r}{q} \right) q^8 + \frac{52}{135} E_5 \left( \frac{r}{q} \right) q^6 - \frac{656}{567} E_3 \left( \frac{r}{q} \right) q^4 + \frac{256}{315} E_1 \left( \frac{r}{q} \right) q^2 \right].
\]

Next, for the case when \( q \) is an even positive integer, we have
\[
S_2(q, r) = (-1)^r \left[ 2B_2 \left( \frac{r}{q} \right) q^2 - \frac{1}{3} \right];
\]
\[
S_4(q, r) = (-1)^{r-1} \left[ \frac{2}{3} B_4 \left( \frac{r}{q} \right) q^4 - \frac{4}{3} B_2 \left( \frac{r}{q} \right) q^2 + \frac{11}{45} \right];
\]
\[
S_6(q, r) = (-1)^r \left[ \frac{4}{45} B_6 \left( \frac{r}{q} \right) q^6 - \frac{2}{3} B_4 \left( \frac{r}{q} \right) q^4 + \frac{16}{15} B_2 \left( \frac{r}{q} \right) q^2 - \frac{191}{945} \right];
\]
\[
S_8(q, r) = (-1)^{r-1} \left[ \frac{2}{315} B_8 \left( \frac{r}{q} \right) q^8 - \frac{16}{135} B_6 \left( \frac{r}{q} \right) q^6 + \frac{28}{45} B_4 \left( \frac{r}{q} \right) q^4 - \frac{32}{35} B_2 \left( \frac{r}{q} \right) q^2 + \frac{2497}{14, 175} \right];
\]
\[
S_{10}(q, r) = (-1)^r \left[ \frac{4}{14, 175} B_{10} \left( \frac{r}{q} \right) q^{10} - \frac{2}{189} B_8 \left( \frac{r}{q} \right) q^8 + \frac{52}{405} B_6 \left( \frac{r}{q} \right) q^6 - \frac{328}{567} B_4 \left( \frac{r}{q} \right) q^4 + \frac{256}{315} B_2 \left( \frac{r}{q} \right) q^2 - \frac{14, 797}{93, 555} \right].
\]
Similarly, a few examples of application of Corollary 2 include

\[
S_2(q) = 6B_2q^2, \\
S_4(q) = -(10B_4q^4 - 4B_2q^2), \\
S_6(q) = \frac{28}{5}B_6q^6 - 10B_4q^4 + \frac{16}{15}B_2q^2, \\
S_8(q) = -\left(\frac{34}{21}B_8q^8 - \frac{112}{15}B_6q^6 + \frac{28}{3}B_4q^4 - \frac{96}{35}B_2q^2\right)
\]

and

\[
S_{10}(q) = \frac{1364}{4725}B_{10}q^{10} - \frac{170}{63}B_8q^8 + \frac{364}{45}B_6q^6 - \frac{1640}{189}B_4q^4 + \frac{256}{105}B_2q^2
\]

for the case when \(q\) is an odd positive integer, and

\[
S_2(q) = 2B_2q^2 - \frac{1}{3}, \\
S_4(q) = -\left(\frac{2}{3}B_4q^4 - \frac{4}{3}B_2q^2 + \frac{11}{45}\right), \\
S_6(q) = \frac{4}{45}B_6q^6 - \frac{2}{3}B_4q^4 + \frac{16}{15}B_2q^2 - \frac{191}{945}, \\
S_8(q) = -\left(\frac{2}{315}B_8q^8 - \frac{16}{135}B_6q^6 + \frac{28}{45}B_4q^4 - \frac{32}{35}B_2q^2 + \frac{2497}{14,175}\right)
\]

and

\[
S_{10}(q) = \frac{4}{14,175}B_{10}q^{10} - \frac{2}{189}B_8q^8 + \frac{52}{405}B_6q^6 - \frac{328}{567}B_4q^4 + \frac{256}{315}B_2q^2 - \frac{14,797}{93,555}
\]

for the case when \(q\) is an even positive integer.

We conclude our investigation by remarking further that a first few of the sums \(S_{2n}(q)\) can be found in the literature. Formulas for \(S_2(q)\) and \(S_4(q)\) are listed in such standard tables of series as those by Prudnikov et al. [7], while Chen [3, p. 188, Eqs. (25) and (26) and p. 190, Eqs. (A1) and (A2)] gave closed-form expressions for \(S_{2n}(q)\) \((n = 1, 2, 3, 4, 5)\).

All of these known sums are in complete agreement with the corresponding closed-form expressions given above.

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References


