Note

Characterizing minimally $n$-extendable bipartite graphs

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Abstract

In this paper, it is proved that let $G$ be a bipartite graph with bipartition $(X, Y)$ and with a perfect matching $M$, let $G$ be an $n$-extendable graph, then $G$ is minimally $n$-extendable if and only if, for any two vertices $x \in X$ and $y \in Y$ such that $xy \in E(G)$, there are exactly $n$ internally disjoint $(x, y)$ $M$-alternating paths $P_1, P_2, \ldots, P_n$ such that $P_i$ $(1 \leq i \leq n)$ starts and ends with edges in $E(G) \setminus M$.

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In this paper, we consider finite, undirected, connected and simple graphs unless otherwise stated. In particular, we consider bipartite graphs. For the terminology and notation not defined in this paper, the reader is referred to [3].

Let $G$ be a graph with a perfect matching $M$, and let $n \leq (\nu(G) - 2)/2$ be a positive integer. Then the graph $G$ is said to be $n$-extendable if any matching of size $n$ in $G$ is contained in a perfect matching of $G$. The graph $G$ is said to be minimally $n$-extendable if $G$ is $n$-extendable but $G - e$ is not $n$-extendable for any edge $e \in E(G)$. Let $x$ and $y$ be two vertices of $G$. An $(x, y)$ path $P$ is a path from $x$ to $y$ in $G$. The path $P$ is said to be an $M$-alternating path if the edges on $P$ from $x$ to $y$ appear in $M$ and $E(G) \setminus M$ alternately. A cycle $C$ is said to be an $M$-alternating cycle if the edges on $C$ appear in $M$ and $E(G) \setminus M$ alternately along $C$. Let $u, v \in V(P)$, we use $P(u, v)$ to denote the segment of $P$ from $u$ to $v$, including $u$ and $v$. Let $R$ be another path in $G$ such that $E(R) \cap E(P) = \emptyset$. We denote $R \cup P$ by $R + P$. Let $S$ and $T$ be two sets. We use $S \triangle T$ to denote the symmetric difference of $S$ and $T$.

The concept of $n$-extendable graphs was introduced by Plummer [7] in 1980. Since then, an extensive research on this topic has been done (see [8,9]).

Motivated by a remark in [6], Aldred et al. [1] found a Menger type characterization of the $n$-extendable bipartite graphs, which shows the relation between $k$-connected graphs and $n$-extendable bipartite graphs. We state this result in the following.

Theorem 1. Let $G$ be a bipartite graph with bipartition $(X, Y)$ which has a perfect matching. Then $G$ is $n$-extendable if and only if, for any perfect matching $M$ and for each pair of vertices $x \in X$ and $y \in Y$, there are $n$ internally disjoint $M$-alternating paths connecting $x$ and $y$, furthermore, these $n$ paths start and end with edges in $E(G) \setminus M$. 
However, the sufficiency of Theorem 1 requires us to check every perfect matching. Later, Lou et al. [5] obtained the following theorem which claims that we have only to check just one perfect matching which is arbitrarily chosen since it forces the existence of \( n \) internally disjoint alternating paths for all the other perfect matchings.

**Theorem 2.** Let \( G \) be a bipartite graph with bipartition \((X, Y)\) and with a perfect matching, and let \( x \in X \) and \( y \in Y \). Let \( M_0 \) and \( M \) be two perfect matchings of \( G \). If \( G \) has \( n \) internally disjoint \((x, y)\) \( M_0 \)-alternating paths starting and ending with edges in \( E(G) \setminus M_0 \), then \( G \) has \( n \) internally disjoint \((x, y)\) \( M \)-alternating paths starting and ending with edges in \( E(G) \setminus M \).

By Theorems 1 and 2, we get the following corollary.

**Corollary 3.** Let \( G \) be a bipartite graph with bipartition \((X, Y)\) and with a perfect matching \( M \). Then \( G \) is \( n \)-extendable if and only if, for any two vertices \( x \in X \) and \( y \in Y \), there are \( n \) internally disjoint \((x, y)\) \( M \)-alternating paths starting and ending with edges in \( E(G) \setminus M \).

Lou [4] also obtained some structural results on minimally \( n \)-extendable bipartite graphs which are similar to the results of minimally \( k \)-connected graphs (see [2]). Motivated by the similarity, in this paper, we obtain the following theorems to characterize the minimally \( n \)-extendable bipartite graphs. Before we show the main results of this paper, we give a known result which will be used in the proof of our main theorems.

**Theorem 4.** Let \( x \) and \( y \) be two vertices of a digraph \( D \), such that \( x \) is not joined to \( y \). Then the maximum number of internally disjoint directed \((x, y)\) paths in \( D \) is equal to the minimum number of vertices whose deletion destroys all directed \((x, y)\) paths in \( D \).

**Proof.** See [3], Theorem 11.6. \( \square \)

In the following, we prove the main theorems of this paper.

**Theorem 5.** Let \( G \) be a bipartite graph with bipartition \((X, Y)\) and with a perfect matching \( M \). Let \( u \in X \) and \( v \in Y \) be two vertices of \( G \) and \( e = xy \in E(G) \setminus M \). If there are \( n (n \geq 1) \) internally disjoint \((u, v)\) \( M \)-alternating paths \( Q_1, Q_2, \ldots, Q_n \) such that \( e = xy \in E(Q_n) \) and \( Q_i \) \((1 \leq i \leq n)\) starts and ends with edges in \( E(G) \setminus M \), if, in \( G - e \), there are \( n \) internally disjoint \((x, y)\) \( M \)-alternating paths \( P_1, P_2, \ldots, P_n \) such that \( P_i \) \((1 \leq i \leq n)\) starts and ends with edges in \( E(G) \setminus M \), then there are \( n \) internally disjoint \((u, v)\) \( M \)-alternating paths \( Q'_1, Q'_2, \ldots, Q'_n \) in \( G' = Q_1 \cup Q_2 \cup \cdots \cup Q_{n-1} \cup (Q_n - e) \cup P_1 \cup P_2 \cup \cdots \cup P_n \) such that \( Q'_i \) \((1 \leq i \leq n)\) starts and ends with edges in \( E(G) \setminus M \).

**Proof.** Let \( Q_i = u x_i, 0 \ y_i, 1 x_i, 1 y_i, 2 x_i, 2 y_i, \ldots, y_i, m_i, x_i, m_i v = y_i, m_i + 1 \), where \( y_i, j x_i, j \in M \), \( 1 \leq i \leq n \), \( 1 \leq j \leq m_i \) and \( m_i \geq 0 \). When \( m_i = 0 \), \( Q_i = uv \) is an edge in \( E(G) \setminus M \). For \( 1 \leq i \leq n \), we give \( Q_i \) an orientation from \( u \) to \( v \). For \( 1 \leq i \leq n \), we give \( P_i \) an orientation from \( x \) to \( y \). Notice that \( E(P_1) \setminus E(Q_1 \cup Q_2 \cup \cdots \cup Q_{n-1} \cup (Q_n - e)) \) forms several segments each of which goes from an \( x_{i,j} \) to a \( y_{r,s} \) \((1 \leq i \leq r \leq n, 0 \leq j \leq m_i \) and \( 1 \leq s \leq m_r + 1 \) and \( 1 \leq m_i + 1 \) and \( 1 \leq m_r + 1 \) and \( 1 \leq i \leq n \). So \( G' \) is a directed graph. By the same reason as above, we notice that any directed path from \( u \) to \( v \) in \( G' \) is an \( M \)-alternating path from \( u \) to \( v \) starting and ending with edges in \( E(G) \setminus M \).

**Case 1:** Suppose \( uv \) is not one of \( Q_1, Q_2, \ldots, Q_{n-1} \).

Notice that if \( uv \neq Q_0 \), then \( uv = xy \).

To the contrary, suppose that there are less than \( n \) internally disjoint \((u, v)\) \( M \)-alternating paths starting and ending with edges in \( E(G) \setminus M \) in \( G' \), which means that there are less than \( n \) internally disjoint directed paths from \( u \) to \( v \) in \( G' \). By Theorem 4, \( G' \) has a cutset \( S \subseteq V(G') \setminus \{u, v\} \) with \( |S| \leq n - 1 \) such that there is no \((u, v)\) directed path in \( G' - S \). Obviously, \( |S| \) cannot be less than \( n - 1 \). Otherwise, one of \( Q_1, Q_2, \ldots, Q_{n-1} \) does not intersect \( S \), and hence is a directed \((u, v)\) path in \( G' - S \), a contradiction. Now suppose \( |S| = n - 1 \). Then \( |S \cap V(Q_i)| = 1 \) \((i = 1, 2, \ldots, n - 1)\), otherwise one of \( Q_i \) \((i = 1, 2, \ldots, n - 1)\) is a directed \((u, v)\) path in \( G' - S \), a contradiction. But there are \( n \) internally
Theorem 6. Let $G$ be a bipartite graph with bipartition $(X, Y)$ and with a perfect matching $M$. And let $G$ be an $n$-extendable graph. Then $G$ is minimally $n$-extendable if and only if, for any two vertices $x \in X$ and $y \in Y$ such that $xy \in E(G)$, there are exactly $n$ internally disjoint $(x, y)$ $M$-alternating paths $P_1, P_2, \ldots, P_n$ such that $P_i$ ($1 \leq i \leq n$) starts and ends with edges in $E(G)\setminus M$.

Proof. We prove necessity first. Suppose that $G$ is a minimally $n$-extendable bipartite graph and $e = xy \in E(G)$ such that $x \in X$ and $y \in Y$. To the contrary, suppose there are not exactly $n$ internally disjoint $(x, y)$ $M$-alternating paths as required in the theorem. By Corollary 3, there are more than $n$ internally disjoint $(x, y)$ $M$-alternating paths starting and ending with edges in $E(G)\setminus M$.

Case 1: $xy \notin M$.

Since $xy$ is an $(x, y)$ $M$-alternating path starting and ending with edges in $E(G)\setminus M$, by the above argument, in $G - xy$, there are at least $n$ internally disjoint $(x, y)$ $M$-alternating paths $P_1, P_2, \ldots, P_n$ such that $P_i$ ($1 \leq i \leq n$) starts and ends with edges in $E(G)\setminus M$.

Since $G$ is minimally $n$-extendable, $G - xy$ is not $n$-extendable. By Corollary 3, there are two vertices $u \in X$ and $v \in Y$ such that there are $n$ internally disjoint $(u, v)$ $M$-alternating paths $Q_1, Q_2, \ldots, Q_n$ starting and ending with edges in $E(G)\setminus M$ in $G$, but there are less than $n$ internally disjoint $(u, v)$ $M$-alternating paths starting and ending with edges in $E(G)\setminus M$ in $G - xy$. So $e = xy$ lies on one of $Q_1, Q_2, \ldots, Q_n$. Without loss of generality, assume that $e \in E(Q_n)$. Let $G' = Q_1 \cup Q_2 \cup \ldots \cup Q_{n-1} \cup (Q_n - e) \cup P_1 \cup P_2 \cup \ldots \cup P_n$. By Lemma 5, there are $n$ internally disjoint $(u, v)$ $M$-alternating paths $Q'_1, Q'_2, \ldots, Q'_n$ starting and ending with edges in $E(G)\setminus M$ in $G'$ and hence in $G - xy$, which contradicts the assumption that $G$ is minimally $n$-extendable.

Case 2: $xy \in M$.

By the argument before Case 1, there are at least $n + 1$ internally disjoint $(x, y)$ $M$-alternating paths $P_1, P_2, \ldots, P_n, P_{n+1}$ starting and ending with edges in $E(G)\setminus M$. Then $C = xy + P_{n+1}$ is an $M$-alternating cycle. Let $M' = M \triangle E(C)$. Then $M'$ is another perfect matching of $G$. By Theorem 2, $G$ satisfies the hypotheses we had before with respect to $M'$. Since $xy \notin M'$, by the same argument as in Case 1 and replacing $M$ by $M'$, we have the conclusion that there are exactly $n$ internally disjoint $(x, y)M'$-alternating paths $Q'_1, Q'_2, \ldots, Q'_n$ starting and ending with edges in $E(G)\setminus M'$. By Theorem 2, there are exactly $n$ internally disjoint $(x, y) M'$-alternating paths starting and ending with edges in $E(G)\setminus M$. Suppose not. There are more than $n$ internally disjoint $(x, y) M'$-alternating paths starting and ending with edges in $E(G)\setminus M$. Then, by Theorem 2 again, there are more than $n$ internally disjoint $(x, y) M'$-alternating paths starting and ending with edges in $E(G)\setminus M'$, contradicting the conclusion we got above. The necessity is then proved.

Now we prove sufficiency. Suppose that, for any $e = xy \in E(G)$ such that $x \in X$ and $y \in Y$, there are exactly $n$ internally disjoint $(x, y) M$-alternating paths $P_1, P_2, \ldots, P_n$ starting and ending with edges in $E(G)\setminus M$. We shall prove that $G$ is minimally $n$-extendable.
Case 1: \( e = xy \notin M \).
Then, in \( G - xy \), there are only \( n - 1 \) internally disjoint \((x, y)\) -alternating paths starting and ending with edges in \( E(G) \setminus M \) since \( xy \) is an \((x, y)\) -alternating path. By Theorem 1, \( G - xy \) is not \( n \)-extendable.

Case 2: \( e = xy \in M \).
By the hypothesis, \( C = P_n + e \) is an \( M \)-alternating cycle. Let \( M' = M \cup E(C) \). By Theorem 2 and the same reason as the end of proof of necessity, for \( e = xy \in E(G) \), there are exactly \( n \) internally disjoint \((x, y)\) -alternating paths \( P_1', P_2', \ldots, P_n' \) starting and ending with edges in \( E(G) \setminus M' \). But now \( e = xy \notin M' \). By the same argument as in Case 1, we have a conclusion that, in \( G - xy \), there are only \( n - 1 \) internally disjoint \((x, y)\) -alternating paths starting and ending with edges in \( E(G) \setminus M' \). By Theorem 1, \( G - xy \) is not \( n \)-extendable.

In both cases, for any edge \( e \in E(G) \), \( G - e \) is not \( n \)-extendable. So \( G \) is minimally \( n \)-extendable. The sufficiency is then proved.

The proof of this theorem is complete. \( \square \)

References