Integral representations of displacements in linear elasticity

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Abstract

We present some new formulae for the solution of boundary value problems for a two-dimensional isotropic elastic body. In particular, using the so-called Dbar formalism and the method introduced by Fokas (2008) [4], we obtain integral representations of the Fourier type for the displacements appearing in boundary value problems formulated in an arbitrary convex polygon with n sides.

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1. Introduction

This note aims to introduce some novel formulae for the solution of boundary value problems for the two-dimensional Navier equations, which are the equations describing the displacements in the interior of an isotropic elastic body. Our results are based on the reduction of the Navier equations to a system of Dbar equations. An introductory discussion of the Dbar problem can be found in [1], Chapter 7.

The reduction of the Navier equations to Dbar equations is motivated by the application of a fundamental physical principle, called the Principle of Virtual Work, to the case of an isotropic elastic body. Since there exist many physical systems where the Principle of Virtual Work can be applied, we believe that similar ideas could also turn out to be useful in the case of other physical systems.

In recent years, there has been considerable activity regarding the solution of classical problems of mathematical physics via a reformulation in terms of Riemann–Hilbert problems; see e.g. [2,3]. Here we use the formalism of one of the most important recent developments in this direction, namely of the so-called Fokas method [4]. This method yields the solution of boundary value problems for a large class of linear and integrable nonlinear equations (see e.g. [5]). Among the equations that can be treated using the Fokas method are linear elliptic Partial Differential Equations in two dimensions in such complicated domains as the equilateral and isosceles triangles (see e.g. [6,7]). This method was extended from the case of homogeneous to the case of inhomogeneous linear elliptic PDEs in [8] (see also [9,10]). Also in the same paper, the relation between this method and the Dbar formalism was investigated for the case of linear elliptic PDEs. In particular, it was shown that this method can be formulated in either the physical plane (the complex z-plane) or the complex Fourier plane (the complex k-plane).

The Fokas method was applied to the solution of boundary value problems for the semi-infinite elastic strip in [11]. Since the approach of [11] is based on the Airy stress function, the relevant integral representations involve derivatives of the displacements. On the other hand, here we present expressions for the displacements themselves.

Using the methodology of [8], we obtain certain integral representations for the displacements in a two-dimensional elastic body (see Proposition in Section 2 below). Although this Proposition is formulated in the case of an arbitrary bounded
polygons, a similar result is valid in the case of an unbounded polygon. Using the Fokas method, the above results can yield the solution of boundary value problems for the Navier equations.

2. Linear elasticity in two dimensions

The governing equations of linear plane elasticity are given by:

\[
\begin{align*}
\mu \Delta u + (\lambda + \mu)(u_{xx} + v_{xy}) + \rho X &= 0 \\
\mu \Delta v + (\lambda + \mu)(u_{yy} + v_{yx}) + \rho Y &= 0,
\end{align*}
\]

(2.1a)

(2.1b)

where subscripts denote partial derivatives, \(u, v\) and \(X, Y\) are the displacement and body forces components along the \(x\)- and \(y\)-axes, respectively, \(\mu, \lambda\) are the Lamé coefficients and \(\rho\) is the density of the elastic body occupying a domain \(D \subset \mathbb{R}^2\). Multiplying Eq. (2.1b) by \(i\) and adding the resulting equation to Eq. (2.1a) we find

\[
\mu \Delta (u + iv) + (\lambda + \mu)(u_{xx} + v_{xy} + i(u_{xy} + v_{yy})) + \rho (X + iY) = 0.
\]

(2.2)

If \(z = x + iy\), \(\overline{z} = x - iy\), and we define the function \(\Psi(z, \overline{z})\) by

\[
\Psi(z, \overline{z}) = u(x, y) + iv(x, y),
\]

then writing the \(x, y\) derivatives in terms of the \(z, \overline{z}\) derivatives we find,

\[
\Psi_{zz} + \nu \Psi_{\overline{z}z} = -\frac{\rho(X + iY)}{3\mu + \lambda},
\]

(2.3)

where \(\nu\) is a constant given by

\[
\nu = \frac{\lambda + \mu}{3\mu + \lambda}, \quad \nu \neq 1.
\]

(2.4)

Let \(W_1(z, \overline{z}, k)\) be the following differential 1-form (for a treatment of the theory of differential forms, please see [12])

\[
W_1(z, \overline{z}, k) = e^{ikz}(\Psi_z + \nu \Psi_{\overline{z}} - Q)dz,
\]

(2.5)

where \(Q(z, \overline{z})\) satisfies

\[
Q_x = -\frac{\rho(X + iY)}{3\mu + \lambda}, \quad Q|_{\partial D} = 0.
\]

(2.6)

Then, Eq. (2.3) implies that \(dW_1 = 0\), i.e., this form is closed. Indeed,

\[
dW_1 = (e^{ikz}(\Psi_z + \nu \Psi_{\overline{z}} - Q))_z d\overline{z} \wedge dz
\]

\[
= e^{ikz}(\Psi_{zz} + \nu \Psi_{\overline{z}z} - Q_x)dz \wedge dz
\]

\[
= 0,
\]

(2.7)

where

\[
dz \wedge d\overline{z} = -d\overline{z} \wedge dz = -2i dxdy.
\]

Assume that \(D\) is a simply connected domain. Then, since \(W_1\) is a closed form

\[
\int_{\partial D} e^{ikz}(\Psi_z + \nu \Psi_{\overline{z}})dz = -\int_{D} \frac{\rho(X + iY)}{3\mu + \lambda} d\overline{z} \wedge dz,
\]

(2.8)

where \(\partial D\) denotes the boundary of \(D\). If we define the function \(f\) by

\[
f = \Psi_z + \nu \Psi_{\overline{z}},
\]

(2.9)

then

\[
\Psi_{\overline{z}} = \frac{\nu f - \overline{f}}{\nu^2 - 1}.
\]

(2.10)

Let \(W_2(z, \overline{z}, k)\) be the differential 1-form

\[
W_2(z, \overline{z}, k) = e^{ikz}(\overline{\Psi}(z, \overline{z}) - P)dz, \quad k \in \mathbb{C},
\]

(2.11)

where the function \(P(z, \overline{z})\) is defined by the equations

\[
P_{\overline{z}} = \frac{\nu f - \overline{f}}{\nu^2 - 1}, \quad P|_{\partial D} = 0.
\]

(2.12)
Then
\[ dW_2 = \left( e^{ikz} (\overline{\nu} - p) \right) z_x d\overline{\nu} \wedge dz = e^{ikz} (\overline{\nu} - p) d\overline{\nu} \wedge dz = 0. \]  

(2.13)

Hence, if (2.10) is satisfied, then the 1-form \( W_2(z, \overline{\nu}, k) \) is closed. Therefore, it follows that

\[ \int_{\partial D} e^{ikz} \overline{\nu} dz = \int_{\partial D} \frac{vf - \overline{f}}{\nu^2 - 1} d\overline{\nu} \wedge dz. \]  

(2.14)

Assume that \( D \) is the interior of the convex polygon \( \Omega \) specified by the corners \( z_1, \ldots, z_n, z_{n+1} = z_1 \) (see Fig. 1). Using Proposition 2.1(b) of [8] as well as Eqs. (2.8) and (2.14) we readily obtain the following main result:

**Proposition** (A Novel Representation of Displacements in the Fourier Plane). Let the real-valued functions \( u(x, y) \) and \( v(x, y) \) satisfy Eqs. (2.1) in the interior of the convex polygon \( \Omega \) specified by the corners \( z_1, \ldots, z_n, z_{n+1} = z_1 \); see Fig. 1. Then, the functions \( u \) and \( v \) admit the integral representations

\[ u - iv = P(z, \overline{z}) + \frac{1}{2\pi} \sum_{j=1}^{n} \int_{\nu_j} e^{ikz} \tilde{\nu}_j(k) dk, \quad z \in \Omega, \]  

(2.15)

where \( \nu_j \) are the rays in the complex \( k \)-plane defined by

\[ \nu_j = \{ k \in \mathbb{C} : \arg(k) = -\arg(z_j - z_{j+1}) \}, \quad j = 1, \ldots, n, \]  

(2.16)

and the functions \( \tilde{\nu}_j(k), \tilde{P}_j(k) \) are defined by the following integrals along the boundary of the polygon:

\[ \tilde{\nu}_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz} (u - iv) dz, \]  

(2.17)

\[ \tilde{P}_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz} P(z, \overline{z}) dz, \quad j = 1, \ldots, n, \quad k \in \mathbb{C}. \]  

(2.18)

The function \( P(z, \overline{z}) \) is defined by the equations

\[ P_\overline{z} = \frac{vf - \overline{f}}{\nu^2 - 1}, \quad P |_{\partial D} = 0. \]  

(2.19)

where \( f(z, \overline{z}) \) satisfies the equation

\[ f = Q(z, \overline{z}) + \frac{1}{2\pi} \sum_{j=1}^{n} \int_{\nu_j} e^{ikz} \tilde{f}_j(k) dk - \frac{1}{2\pi} \sum_{j=1}^{n} \int_{\nu_j} e^{ikz} \tilde{Q}_j(k) dk, \quad z \in \Omega. \]  

(2.20)

The functions \( \tilde{f}_j(k), \tilde{Q}_j(k) \) are defined by

\[ \tilde{f}_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz} \left[ (1 + v)(u_x + v_y) + i(1 - v)(v_x - u_y) \right] dz, \]  

(2.21)

\[ \tilde{Q}_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz} Q(z, \overline{z}) dz, \quad j = 1, \ldots, n, \quad k \in \mathbb{C}, \]  

(2.22)
and $Q(z, ar{z})$ satisfies
\[ Q_r = -\frac{\rho(X + iY)}{3\mu + \lambda}, \quad Q_{|\partial D} = 0. \]  
(2.23)

Furthermore, the following Fokas relations are valid for all complex $k$,
\[ \sum_{j=1}^{n} \hat{\psi}_j(k) = \sum_{j=1}^{n} \hat{p}_j(k), \]  
(2.24)
\[ \sum_{j=1}^{n} \hat{f}_j(k) = \sum_{j=1}^{n} \hat{Q}_j(k), \quad k \in \mathbb{C}. \]  
(2.25)

We are currently investigating the use of the above integral representations to solve boundary value problems for the Navier equations in analogy with the plethora of boundary value problems that have already been solved for other elliptic equations, such as the Laplace, Poisson, Helmholtz, modified Helmholtz and biharmonic equations using analogous integral representations to the above ones (see for example [6,13,8]). The domain of interest is an arbitrary convex polygon with $n$ sides (Fig. 1) and for illustration purposes here we first assume that the elastic body undergoes all-round compression or tension. Therefore, all the principal stresses at any point in the interior of the elastic body are equal and the displacements satisfy the condition
\[ u_x + v_y = 0. \]  
(2.26)

We now focus on the following boundary value problem: assume that the $x$-component of the displacement, $u$, is given on the boundary of a simply connected and bounded domain $D$, namely
\[ u(s) = U(s), \quad s \in \partial D, \]  
(2.27)
where $U(s)$ is a known function. Then, the solution of this problem can be found as follows: first, Eq. (2.25) yields $v_x - u_y$ on the boundary. Then, the integral representation (2.20) evaluates $f$ everywhere in the domain $D$ and (2.24) offers $v$ on the boundary. Finally, Eq. (2.15) yields $u - iv$ everywhere in $D$. The above problem would fall into the well-known class of ‘canonical’ problems for linear elasticity if we have not imposed the simplifying constraint (2.26); waiving this restriction renders the problem considerably harder, however we suggest below how this could be resolved. In general, boundary value problems for the Navier equations are of considerable difficulty [14] as it can be seen by considering the high complexity of methods used for the solution even of special cases of them. For example, the canonical boundary value problems for the semi-infinite strip with some special boundary conditions, namely traction-free conditions along the semi-infinite sides, have been solved using combinations of Fourier integrals associated with half-plane and infinite strip solutions as well as the Papkovich–Faddeev eigenfunctions (see e.g. [15–17]). In particular, in [15] eigenfunctions associated with the semi-infinite strip were constructed and each problem was reduced to certain integral equations under the following additional assumptions for the tractions:
\[ \int_0^l p_{1i}(0,y)dy = 0, \quad i = 1, 2, \quad \int_0^l y p_{11}(0,y)dy = 0. \]  
(2.28)

However, for more complicated domains, the above approaches fail, because they do not provide general methods for the construction of the appropriate eigenfunctions. On the other hand, the approach advocated here is based on a general method for constructing such eigenfunctions. For the fundamental problems of elasticity, where either the normal and tangential components $u$ and $v$ of the displacements, or $\lambda(u_x + v_y) + 2\mu u_x$ and $\mu(u_y + v_x)$, are prescribed as boundary conditions (where we assumed that the normal to the boundary is parallel to the $x$-axis) the relations (2.24) and (2.25) do not provide an effective way of determining the unknown boundary values. However, if one has an alternative approach of determining the unknown boundary values $u_x + v_y$ or $v_x - u_y$, then Eqs. (2.15) and (2.20) do provide useful expressions for the solution. Such an alternative approach is offered by the analysis of the Fokas relations (Eqs. 6.8 of [8]) associated with the non-homogeneous biharmonic equation
\[ \psi_{xx} = -16v(X_x + Y_y), \]  
(2.29)
where the real-valued function $\psi$ is defined in terms of $u$ and $v$ by the following set of compatible equations [18],
\[ \psi_{xx} = \lambda(u_x + v_y) + 2\mu v_y, \]  
\[ \psi_{yy} = \lambda(u_x + v_y) + 2\mu u_x, \]  
\[ \psi_{xy} = -\mu(u_y + v_x). \]  
(2.30)

We conclude this section with some further remarks: for a general discussion of uniqueness of solutions to boundary value problems in linear elasticity we refer the reader to [19-22]. Also, the solution of Eqs. (2.1) can have singularities at
the corners of the semi-strip. This issue was addressed in [11] where it was shown that the assumption that there exist no internal singularities is not restrictive. However, there exist certain problems where singularities are important. Such problems can be studied using the results introduced here, which should then be supplemented with the methodology of [11] for treating corner singularities. Alternatively, one can resort to certain appropriate methods such as the method of projection appearing in [23].

3. Conclusions

There exists a vast literature on the use of complex methods for plane elasticity (see for example [24–26]). It should be noted that there exist three crucial differences between the approaches presented in these books and the approach followed here. The first difference is the use of the Dbar formalism as opposed to the analysis of singular integral equations. The author is aware of only [27], where sporadic connections between the Dbar formalism and elliptic problems are included. The second difference of the present approach in comparison with previous works (including [27]) is that the solution relies on the analysis of certain relations which couple Fourier type transforms of known and unknown boundary values (the Fokas relations). The third and maybe most important difference is that the integral representations are formulated in the Fourier plane, (the complex k-plane) as opposed to the physical plane (the complex z-plane). Namely, the representations obtained here involve Fourier kernels with a spectral parameter k as opposed to Cauchy type kernels formulated in the space of the coordinates x, y. The relation between representations for elliptic problems in the physical and Fourier planes was elucidated in [8], where it was also shown that the Fourier plane approach can yield the solution in the case of more complicated problems such as those formulated in the interior of an equilateral triangle.

Also, this approach seems quite efficient for numerical computations [28,18,29,30] because the integrals involve exponential kernels and the relevant contours of integration can be deformed to yield strong decay. In particular, this approach offers a novel way to numerically compute the Dirichlet to Neumann map for elliptic PDEs and is the analogue of the well-known boundary integral method in the Fourier plane. The latter method provides the solution of elliptic equations by exploiting certain linear integral equations involving the unknown boundary values formulated in the physical plane. These equations are the precise analogues of what we here call Fokas relations (and which are formulated in the Fourier plane). For more details about how this approach relates to standard methods we refer the reader to [31].

Our results were motivated by a rather unexpected connection between the Dbar formalism and a fundamental physical principle, called the Principle of Virtual Work. Namely, letting k = 0 in Eq. (2.8), we find

$$\int_{\partial D} \left[ (1 + v) \left( u_x + v_x \right) + i(1 - v) \left( v_x - u_y \right) \right] \, dz = \int_{\partial D} \left( -\frac{\rho}{3\mu + \lambda} \right) \left( X + iY \right) \, d\zeta \wedge dz. \tag{3.1}$$

We call this equation the nonlocal form of the Navier equations. It can be shown [32] that this equation can be obtained starting from the Principle of Virtual Work as applied to the case of an isotropic elastic body. This result provides the basis for the association of a family of differential equations with variational principles in a manner different from Noether theorems and Eulerian versus Lagrangian formulation. Equivalently, Eq. (3.1) can be obtained by integrating the relevant equilibrium equations $\partial \sigma_{ij}/\partial x_j + \rho F_i = 0$, over $\partial D$. It is straightforward to show that Eq. (3.1) is equivalent to a Dbar equation, namely an equation of the form

$$f_{\bar{z}} = \phi, \quad z \in D \subset \mathbb{R}^2, \tag{3.2}$$

with

$$f = (1 + v) \left( u_x + v_x \right) + i(1 - v) \left( v_x - u_y \right) \quad \text{and} \quad \phi = \left( -\frac{\rho}{3\mu + \lambda} \right) \left( X + iY \right). \tag{3.3}$$

In the absence of body forces, $\phi = 0$ and the function $f$ is analytic. It is shown in [8] that starting from Eq. (3.2) it is possible to construct algorithmically integral representations of the unknown function $f$ as well as relations involving the boundary values of $f$ in both the physical and spectral planes. In the particular case of the Navier Eqs. (2.1), the relevant integral representation and the relations involving the boundary values in the spectral plane (which we call the Fokas relations) are given by (2.15) and (2.24)–(2.25), respectively.

This unexpected connection between the Dbar formalism and the Principle of Virtual Work suggests that one could consider the broad class of physical systems where the Principle of Virtual Work can be applied and attempt to construct similar integral representations for systems where a linear constitutive law is valid (and maybe other systems in the same class).

One should then study the specific nonlocal equation for each system which results from the application of the Principle of Virtual Work and investigate whether it can be written in the form of a nonlocal Dbar equation. If this can be done, one can then use the methodology of [8] to construct integral representations and Fokas relations associated with the particular physical system under consideration.
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