New Bounds on the Total-Squared-Correlation of Quaternary Signature Sets and Optimal Designs

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Abstract—We derive new bounds on the total squared correlation (TSC) of quaternary (quadrature phase) signature/sequence sets for all lengths \( L \) and set sizes \( K \). Then, for all \( K, L \), we design minimum-TSC optimal sets that meet the new bounds with equality. Direct numerical comparison with the TSC value of the recently obtained optimal binary sets shows under what \( K, L \) realizations gains are materialized by moving from the binary to the quaternary code-division multiplexing alphabet. On the other hand, comparison with the Welch TSC value for real/complex-field sets shows that, arguably, not much is to be gained by raising the alphabet size above four for any \( K, L \).

Index Terms—Code-division multiplexing, quaternary alphabet, sequences, Welch bound.

I. INTRODUCTION

We consider the problem of designing sets of code sequences (signatures) for code-division-multiplexing applications from the quaternary alphabet \( \{\pm 1, \pm j\} \). A fundamental measure of the cross-correlation properties of a signature set (and subsequently overall code-division system performance) is the total squared correlation (TSC). If \( S = \{s_1, s_2, \ldots, s_K\} \), \( s_k \in \mathbb{C}^L \), \( |s_k| = 1 \), \( k = 1, 2, \ldots, K \), is a set of \( K \) normalized (complex, in general) signatures of length (processing gain) \( L \), then the TSC of \( S \) is the sum of the square magnitudes of all inner products between signatures,

\[
TSC(S) = \sum_{m=1}^{K} \sum_{n=1}^{K} |s_m^H s_n|^2 \quad \text{where } "H" \text{ denotes the conjugate transpose operation.}
\]

For real/complex-valued signature sets \( S \in \mathbb{C}^{L \times K} \), the TSC is bounded from below by [1]-[3]

\[
TSC(S) \geq \frac{KM}{L} \tag{1}
\]

where \( M = \max\{K, L\} \). The bound in (1) is called the “Welch bound” and the signature sets that satisfy (1) with equality are called Welch-bound-equality (WBE) sets. While for real/complex-valued signature sets the Welch bound is always achievable [4]-[9], this is not the case in general for finite-alphabet signatures. Recently, new bounds were derived for the TSC of binary (alphabet \( \{\pm 1\} \)) signature sets for all lengths \( L \) and set sizes \( K \) together with optimal set designs for (almost) all \( K \) and \( L \) [10]-[12]. The sum capacity, total asymptotic efficiency, and maximum squared correlation of minimum-TSC optimal sets were found in [13]-[14]. Minimum-TSC and other digital sequence sets are studied and utilized in [15]-[16].

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The gap in TSC between minimum-TSC binary signature sets and Welch-bound-equality real/complex sets can be reduced if we utilize alphabets of larger size. Certainly there is a tradeoff between system performance and transceiver complexity that is associated with our selection of the alphabet size. The quaternary (or quadrature or 4-phase) alphabet appears as a good candidate since most code-division multiplexing (CDM) systems already employ quadrature signaling. Thus, utilizing quaternary signature sets would not incur significant additional cost over binary signature sets.

In this paper we consider a quaternary alphabet and investigate under what \( K, L \) realizations gains can be materialized by moving from the binary to the quaternary code-division multiplexing alphabet. In particular, we first derive new bounds on the TSC of any quaternary signature matrix \( S = [s_1, s_2, \ldots, s_K] \in \frac{1}{\sqrt{L}} \{\pm 1, \pm j\}^{L \times K} \) for all possible \( K \) and \( L \) values.

II. NEW BOUNDS ON THE TSC OF QUATERNARY SIGNATURE SETS

We recall that the Karystinos-Pados bounds on the TSC of a binary signature set can be given compactly by the following expression [10]-[12]

\[
TSC(S_B) \geq \frac{KM}{L} + \begin{cases} 0, & M \equiv 0 (mod 4) \\ \frac{1}{16} \left( \frac{m(m-1)}{2} \right)^2 + \left[ \frac{m}{4} \right]^2 - m, & M \equiv 1 (mod 2), M \equiv 2 (mod 4) \end{cases} \tag{2}
\]

where \( K \) is the number of signatures, \( L \) is the signature length, \( M = \max\{K, L\} \), and \( m = \min\{K, L\} \). The subscript “\( B \)” in \( S_B \) identifies a binary signature set \( S_B \in \frac{1}{\sqrt{L}} \{\pm 1\}^{L \times K} \), and \([x], [x] \) stand for the closest to \( x \) integer that is less than or equal to \( x \) and greater than or equal to \( x \), respectively.

In this paper, we consider quaternary signature sets \( S_Q = [s_1, s_2, \ldots, s_K] \in \frac{1}{\sqrt{L}} \{\pm 1, \pm j\}^{L \times K} \), \( j = \sqrt{-1} \), where the subscript “\( Q \)” in \( S_Q \) stands for quaternary. Since binary signature sets are special cases of quaternary signature sets \( \{\pm 1\} \subset \{\pm 1, \pm j\} \), for any \( K \) and \( L \) any achievable lower bound on the TSC of \( S_Q \) lies between the Welch bound and the Karystinos-Pados bound. Thus, using (2), whenever \( M \equiv 0 (mod 4) \) the bound on TSC of any set \( S_Q \) is

\[
TSC(S_Q) \geq \begin{cases} K, & K \leq L \text{ and } L \equiv 0 (mod 4), \\ K^2/L, & K > L \text{ and } K \equiv 0 (mod 4). \end{cases} \tag{3}
\]

In the rest of this section, we derive new bounds on the TSC of quaternary signature sets for all possible combinations of the

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values of $K$ and $L$ when $M \equiv 1 \pmod{2}$ and $M \equiv 2 \pmod{4}$. Then, via quaternary Hadamard matrix transformations, we design optimal quaternary signature sets that achieve the new bounds.

A. Underloaded system ($K \leq L$)

Theorem 1 below provides new lower bounds on the TSC of any signature set $S_Q$ when $K \leq L$ (underloaded systems) and $\max\{K, L\} = L$ is not a multiple of 4.

**Theorem 1:** Let $S_Q$ be an arbitrary quaternary signature set $S_Q \triangleq \{s_1, s_2, \ldots, s_K\} \subseteq \{\pm 1, \pm j\}^{L \times K}$, $j = \sqrt{-1}$, $K \leq L$. Then,

$$TSC(S_Q) \geq \frac{K + \frac{K(K-1)}{L^2}}{K}, \quad L \equiv 1 \pmod{2}, \quad L \equiv 0 \pmod{2}.$$  \hspace{1cm} (4)

**Proof:** The TSC of $S_Q$ can be expressed as

$$TSC(S_Q) = K + \sum_{m=1}^{K} \sum_{n=m+1}^{K} |s_m^H s_n|^2$$ \hspace{1cm} (5)

where the doublesummation term is the TSC between different signatures in $S_Q$ (if this term has zero value, i.e. all pairs of signatures have zero cross-correlation, the lower bound of TSC($S_Q$) reduces to the Welch bound). To obtain a lower bound on the double-summation term in (5), we consider the set $C$ of all non-ordered pairs of signatures $\{s_m, s_n\}, m \neq n$, with non-zero cross correlation, i.e. $C = \{(s_1, s_2, \ldots, s_K) \triangleq \{\{s_m, s_n\} \text{ such that } m \neq n \text{ and } s_m^H s_n \neq 0, m = 1, 2, \ldots, K, n = 1, 2, \ldots, K\}$. Then,

$$TSC(S_Q) \geq K + 2|C(S_Q)||A|^2$$ \hspace{1cm} (6)

where $A$ is a lower bound on the cross-correlation of any two signatures in $S_Q$ (i.e., $|s_m^H s_n|^2 \geq A$) and $\sum_{i=1}^L s_m^H s_n = 0$, where $L$ denotes the cardinality of the set $C(\cdot)$. Since the quaternary signature alphabet $\{\pm 1, \pm j\}$ is closed under multiplication and conjugation (denoted by $^*$), signature cross-correlations can be expressed as

$$s_m^H s_n \triangleq \sum_{i=1}^L s_m[i] s_n[i] = \left(\frac{1}{L}\right) a + \left(\frac{1}{L}\right) b + \left(\frac{1}{L}\right) c + \left(\frac{1}{L}\right) d$$

for some integers $a$, $b$, $c$, $d$ such that $a + b + c + d = L$. Then, $|s_m^H s_n|^2 \geq \frac{L}{4} \sqrt{(a^2 + b^2 + (c - d)^2)}$. If $L \equiv 1 \pmod{2}$, $a + b + c + d \equiv 1 \pmod{2}$ which implies that $(a - b)^2 + (c - d)^2 \equiv 1 \pmod{2}$ and since $(a - b)^2 + (c - d)^2 \geq 0$, we have $(a - b)^2 + (c - d)^2 \geq 1$. Thus, $|s_m^H s_n|^2 \geq \frac{L}{4}$ for any $s_m, s_n \in S_Q$. We conclude that if $L \equiv 1 \pmod{2}$ then the cross-correlation between any two signatures in $S_Q$ is non-zero ($|A| = \frac{1}{L}$), therefore $|C(S_Q)| = \left(\frac{L}{4}\right) = K(K-1)/2$. On the other hand, if $L \equiv 0 \pmod{2}$ there may be signature pairs in $S_Q$ that exhibit zero cross-correlation.

The new bounds on the TSC of quaternary signature sets for underloaded systems ($K \leq L$) are summarized in Table I. Table I can also be viewed as proof that when the signature length is not even, no orthogonal quaternary signature set exists.

<table>
<thead>
<tr>
<th>Length</th>
<th>Number of Sequences</th>
<th>Lower Bound on TSC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L \equiv 0 \pmod{2}$</td>
<td>Any $K$</td>
<td>$K$</td>
</tr>
<tr>
<td>$L \equiv 1 \pmod{2}$</td>
<td>Any $K$</td>
<td>$K + \frac{K(K-1)}{L}$</td>
</tr>
</tbody>
</table>

**TABLE I**

**UNDERLOADED QUATERNARY SEQUENCE SETS ($K \leq L$)**

**B. Overloaded system ($K > L$)**

Let $d_i = [s_1(l), s_2(l), \ldots, s_K(l)]^T \in \frac{1}{\sqrt{L}} \{\pm 1, \pm j\}^K$ denote the transpose of the $i$th row, $l = 1, 2, \ldots, L$, of the signature matrix $S_Q$. Due to the “row-column equivalence” property, $TSC(S_Q) = \sum_{m=1}^{K} \sum_{n=1}^{K} |s_m^H s_n|^2 = \sum_{l=1}^{L} \sum_{i=1}^{L} |d_i^H d_i|^2$.

Therefore, we can proceed with the calculation of TSC($S_Q$) as follows

$$TSC(S_Q) = \sum_{l=1}^{L} |d_i^H d_i|^2 + \sum_{l=1}^{L} \sum_{r=1, r \neq l}^{L} |d_i^H d_r|^2$$

$$= \frac{K^2}{L} + \sum_{l=1}^{L} \sum_{r=1, r \neq l}^{L} |d_i^H d_r|^2$$

$$= \frac{K^2}{L} + 2|C(\{d_1, d_2, \cdots, d_L\})| |d_i^H d_i|^2$$ \hspace{1cm} (7)

By Theorem 1, we obtain that

$$TSC(S_Q) \geq \begin{cases} \frac{K^2}{L}, & K \equiv 0 \pmod{2} \\ \frac{K^2}{L} + \frac{L}{L-1}, & K \equiv 1 \pmod{2} \end{cases}$$ \hspace{1cm} (8)

The new bounds on the TSC of quaternary signature sets for overloaded ($K > L$) systems are summarized in Table II.

We observe that when $M \triangleq \max\{K, L\} = 0 \pmod{4}$ or $1 \pmod{2}$ (multiple of four or odd) the TSC lower bounds for quaternary signature sets in Tables I and II are identical to the tight binary Karystinos-Pados (KP) bounds [10], while for $M \equiv 2 \pmod{4}$, the lower bounds of TSC($S_Q$) in Table I and Table II are less than the corresponding KP bounds for binary signature sets. We also recall that $K = L \equiv 1 \pmod{4}$ is an open standing problem (the only one) [10],[12] in optimal binary signature set design and KP-bound-equality sets may in fact exist only for values $K = L = 2x(x+1), \quad x = 1, 2, \ldots$ (i.e. $K = L = 5, 13, 25, 41, 61, \ldots$). We conclude that we can benefit in TSC by moving from the binary to the quaternary alphabet if $\max\{K, L\}/2$ is an odd integer or $K = L \neq 2x(x+1) + 1, \quad x = 1, 2, \ldots$. Particularly, when $\max\{K, L\}/2$ is odd, the reduction in TSC is approximately $\frac{4m(m-1)}{4m^2}$, $m = \min\{K, L\}$. Moreover, for any $K$ and $L$, half of the minimum-TSC quaternary sets reach the Welch bound while only a quarter of the minimum-TSC binary sets do. To illustrate the potential reduction in TSC by quaternary designs,
in Fig. 1 we consider systems with four different signature lengths \( L = 34, 46, 54, 66 \) and plot \( \text{TSC}(S) - K \) (i.e. total squared cross-correlation) for both binary and quaternary sets as a function of the number of signatures \( K \). To materialize the potential TSC/multiplexing improvements we need designs that meet the corresponding new quaternary bounds in Tables I and II with equality as described in the following section.

III. DESIGN OF MINIMUM TSC QUATERNARY SIGNATURE SETS

Our designs of optimal quaternary signature sets are based on transformations of quaternary Hadamard matrices as in [10].

**Definition 1:** (Quaternary Hadamard matrix)

Let \( H_Q \) be an \( N \)-order square matrix over the quaternary alphabet, i.e. \( H_Q \in \{ \pm 1, \pm j \}^{N \times N} \), \( j \triangleq \sqrt{-1} \), \( N > 0 \). \( H_Q \) is a quaternary Hadamard matrix if \( H_Q H_Q^H = N I_N \) where \( I_N \) is the \( N \times N \) identity matrix.

Tables I and II indicate that when \( M = \max \{ K, L \} \) is even, the lower bound on the TSC of quaternary signature sets is equal to the real/complex Welch bound and can be achieved by any set \( S_Q \) that has orthogonal rows when \( K \geq L \) or orthogonal columns when \( K \leq L \). If \( M \) is not even, our lower bounds on the TSC of quaternary signature sets in (4), (8) are strictly larger than the Welch bound which implies that there is no quaternary matrix \( S_Q \) that has orthogonal rows when \( K \geq L \) or orthogonal columns when \( K \leq L \). In other words, a necessary condition for a quaternary Hadamard matrix to exist is that its size is even\(^1\); equivalently, if \( M \) is not even, then quaternary Hadamard matrices do not exist.

As is the case for binary Hadamard matrices, there is no universal procedure to generate quaternary Hadamard matrices for all even orders. Binary Hadamard matrices can be considered as a special case of quaternary Hadamard matrices. The generation of binary Hadamard matrices of orders that are multiples of four has been well studied. To generate a quaternary Hadamard matrix with order that is a multiple of four, we may multiply by \( \pm j \) any column or row of a binary Hadamard matrix of the same order. Unfortunately, it is not easy to generate a quaternary Hadamard matrix of even order \( N \) that is not a multiple of four. Below, we propose two alternative methods for this task. First, we suggest a modified version of the analytical procedure of Dit&lacute; [17]-[18] that generates \( (N-2)^2 \) polynomial equations. Our modification incorporates constraints that restrict the phase of the matrix elements to be in the set \( \{ 0, \pi/2, \pi, 3\pi/2 \} \). Then, the solutions of the constrained system of \( (N-2)^2 \) polynomial equations can be used as the entries of a quaternary Hadamard matrix.

We note that solving such a constrained system of \( (N-2)^2 \) polynomial equations is, in general, a computationally complex process and no specific algorithm is available in the literature for this task.

On the other hand, exhaustive search to return all quaternary Hadamard matrices of order \( N \) has prohibitively high (equal to \( 4^N \)) complexity even for moderate values of \( N \). The second method that we suggest herein has significantly less computational complexity (equal to \( 4^N \)) than exhaustive search. Our method is based on the following lemma [19].

**Lemma 1:** If \( A_1, A_2 \) are two circulant matrices such that \( A_1, A_2 \in \{ \pm 1, \pm j \}^{N \times N} \), \( N \in \mathbb{N} \), and \( A_1, A_2 \) satisfy

\[
A_1 A_1^H + A_2 A_2^H = N I_N/2
\]

where \( I_N/2 \) is an \( N/2 \times N/2 \) identity matrix, then the construction

\[
H_Q = \begin{bmatrix} A_1 & A_2 \\ A_2^H & -A_1^H \end{bmatrix}
\]

is an \( N \)-order quaternary Hadamard matrix.

As an example, if \( A_1 = \begin{bmatrix} j & 1 \\ 1 & j \end{bmatrix} \) and \( A_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \), then

\[
H_Q = \begin{bmatrix} j & 1 & -1 & -j \\ 1 & j & -1 & -j \\ -1 & -j & j & 1 \\ -j & -1 & -1 & j \end{bmatrix}
\]

is a quaternary Hadamard matrix. There are only \( 4^2/2 \) distinct \( 2 \times 2 \) circulant matrices over the quaternary alphabet and each of them can be identified by its first row only. If two \( 2 \times 2 \) circulant matrices \( A_1 \) and \( A_2 \) that satisfy (9) exist, they can be found by examining all \( 4^N \) possible pairs of circulant matrices. Then, a quaternary Hadamard matrix with order \( N \) can be generated by \( A_1 \) and \( A_2 \) as given by Lemma 1. Examples of the first rows of \( A_1 \) and \( A_2 \)-type matrices for different values of \( N \) are given in Table III. Additional quaternary Hadamard matrices generated by this method can be found in [20].

In the rest of this section we present a sufficient condition under which the new TSC lower bounds of Tables I and II become tight. Then, we outline a design procedure of quaternary signature sets that achieve the bounds.

**Proposition 1:** Set \( N \triangleq 2 \max \{ K, L \}/2 \) and \( P \triangleq 2 \max \{ K, L \}/2 \). If there exists a quaternary Hadamard ma-

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\(^1\)A necessary condition for a binary Hadamard matrix to exist is that its size is a multiple of four, except for the trivial cases of size one or two.
TABLE III
EXAMPLES OF $A_1$, $A_2$ CIRCULANT MATRICES

<table>
<thead>
<tr>
<th>$N/L$</th>
<th>First row of $A_1$</th>
<th>First row of $A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3/3</td>
<td>$[j 1 1]$</td>
<td>$[-1 1 1]$</td>
</tr>
<tr>
<td>5/5</td>
<td>$[1 j -1 -1 j]$</td>
<td>$[1 -1 j j -1]$</td>
</tr>
<tr>
<td>7/7</td>
<td>$[1 1 1 -1 -1 1]$</td>
<td>$[1 j -j j -j 1]$</td>
</tr>
<tr>
<td>9/9</td>
<td>$[1 -1 1 1 -1 -1 1]$</td>
<td>$[1 j 1 1 1 1 -1]$</td>
</tr>
<tr>
<td>11/11</td>
<td>$[1 -1 -1 j 1 j 1 j -1]$</td>
<td>$[j 1 -j j -1 j -j 1]$</td>
</tr>
<tr>
<td>13/13</td>
<td>$[1 1 -1 j j -1 -1 -1 j j 1]$</td>
<td>$[j 1 -1 j j -1 j j -1]$</td>
</tr>
<tr>
<td>15/15</td>
<td>$[1 -1 j 1 -j 1 -j 1 -1 1]$</td>
<td>$[j 1 1 j 1 -1 -1 -1 j j 1]$</td>
</tr>
<tr>
<td>17/17</td>
<td>$[1 -1 -1 j -j 1 1 j j 1 1 j -j 1]$</td>
<td>$[j 1 -1 j -j -1 j -j 1 1 j 1]$</td>
</tr>
</tbody>
</table>

![Diagram](image)

Fig. 2. Optimal quaternary signature set design procedure.

trix of size $N$, then for any $K$ and $L$ there exists a quaternary signature set $S_Q = [s_1, s_2, \ldots, s_K] \in \frac{1}{\sqrt{P}} \{\pm 1, \pm j\}^{L \times K}$ that achieves the TSC lower bound in Table I or II. If there exists a quaternary Hadamard matrix of size $P$, then there exists a quaternary signature set $S_Q = [s_1, s_2, \ldots, s_K] \in \frac{1}{\sqrt{P}} \{\pm 1, \pm j\}^{L \times K}$ with $K \neq L$ that achieves the TSC lower bound in Table I or II.

For underloaded systems, $K \leq L$, let $N = 2\lceil K/2 \rceil$ and generate an $N$-order quaternary Hadamard matrix $H_Q$. Either $L = N$ or $L = N - 1$. If $L = N$, then a quaternary set $S_Q$ can be formed by selecting and normalizing by $\frac{1}{\sqrt{L}}$ any $K$ columns of $H_Q$; if $L = N - 1$, then we first truncate $H_Q$ by one row and then form $S_Q$ by selecting and normalizing by $\frac{1}{\sqrt{L}}$ any $K$ columns from the truncated matrix. For overloaded systems, $K \geq L$, let $N = 2\lceil K/2 \rceil$ and generate an $N$-order quaternary Hadamard matrix $H_Q$. Then, $K = N$ or $K = N - 1$. If $K = N$, we may choose any $L$ rows of $H_Q$ and normalize them by $\frac{1}{\sqrt{L}}$; this is our $S_Q$. If $K = N - 1$, we may proceed by truncating $H_Q$ by one column and then form $S_Q$ by choosing and normalizing by $\frac{1}{\sqrt{L}}$ any $L$ rows of the truncated matrix.

By Proposition 1, a minimum-TSC quaternary signature set can also be designed based on a $P = 2\lceil \max\{K, L\}/2 \rceil$ order quaternary Hadamard matrix if it exists. Since $P = N$ when $\max\{K, L\} \equiv 0 \pmod{2}$, we focus on the case $\max\{K, L\} \equiv
In this paper, we derived new bounds on the TSC of quaternary signature sets for both underloaded and overloaded code-division multiplexing systems (summarized in Tables I and II, respectively). We showed that the new bounds on the TSC of quaternary signature sets are lower than the corresponding binary signature set bounds (same number of signals \( K \) and signature length \( L \)) for all \( \max\{K,L\} \equiv 2(\mod 4) \) or \( K = L \equiv 1(\mod 4) \neq 2x(x+1)+1 \) cases. We then designed minimum-TSC optimal quaternary sets that meet the new bounds for all \( K, L \). Our design procedure depends on the existence of a quaternary Hadamard matrix of size \( 2\lceil\frac{\max\{K,L\}}{2}\rceil \). The sum-capacity, maximum squared correlation, and total asymptotic efficiency of minimum TSC quaternary sets are evaluated in closed-form and contrasted against minimum-TSC optimal binary and real/complex sets in [21].

References