On a Geometry of Ivanov and Shpectorov for the O’Nan Sporadic Simple Group

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We show that, for the O’Nan sporadic simple group, there is no $R_{wpri}$ and $(IP)_2$ geometry of rank 6 with a maximal parabolic subgroup isomorphic to $M_{11}$ and that there is no $R_{wpri}$ and $(IP)_3$ geometry of rank 5 with a maximal parabolic subgroup isomorphic to $J_1$. This last result permits us to show that the Ivanov–Shpectorov geometry is not $R_{wpri}$. The results obtained in this paper rely partially on computer algebra.

1. INTRODUCTION

In 1973 [17], O’Nan provided strong evidence for the existence of a new sporadic group now called $O'N$. Later in the seventies, Sims constructed this group with help of a computer (see [13] for a survey of the story of $O'N$) but his work seems to be unpublished. In 1980, Andrilli published in his Ph.D. Thesis [1], supervised by Sims, an existence and uniqueness proof of $O'N$. Around 1985, the maximal subgroups of $O'N$ were determined independently by Yoshiara in his Master’s Thesis [20], Wilson [19] and Ivanov, Tsaranov and Shpectorov [16]. The two latter references rely partially on computer algebra.

Some definitions of $O'N$, its maximal subgroups and its character table are available in the Atlas of Finite Groups [10] as well as a presentation.

In [5], the first author of this paper gives a flag-transitive geometry for $O'N$ whose diagram is the following.

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1 -- 4 -- 10-7-9 -- 6 -- 2
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148
In 1986, Ivanov and Shpectorov constructed a geometry admitting a flag-transitive action of the O'Nan simple group [15]. This geometry has the following diagram.

In this paper, we prove the following two theorems.

**Theorem 1.1.** Let $G$ be the O'Nan simple sporadic group. Then $G$ has no $\text{Rwpri}$ and $(IP)_2$ geometry of rank greater or equal to 5 with some maximal parabolic subgroup isomorphic to the Janko group $J_1$.

**Theorem 1.2.** Let $G$ be the O'Nan simple sporadic group. Then $G$ has no $\text{Rwpri}$ and $(IP)_2$ geometry of rank greater or equal to 6 with some maximal parabolic subgroup isomorphic to the Mathieu group $M_{11}$.

Because $O'N.1$ has one of its maximal parabolic subgroups isomorphic to $J_1$, we deduce from Theorem 1.1 the following corollary.

**Corollary 1.** The Ivanov-Shpectorov geometry for the O'Nan sporadic simple group is not $\text{Rwpri}$.

The proofs of the two theorems rely on the fact that the only maximal subgroups of $O'N$ that have subgroups isomorphic to $PSL_2(11)$ (which we denote by $L_2(11)$) are $J_1$ and $M_{11}$.

The motivation of this work is to try to give an upper bound on the maximal rank that an $\text{Rwpri}$ geometry could have in $O'N$. Similar work has already been accomplished in [8], where it is shown that $M_{12}$ has no geometry with $\text{Rwpri}$ and $(IP)_2$ of rank greater or equal to 6. This result was quite surprising because of the fact that $M_{11}$ has rank 5 geometries with $\text{Rwpri}$ and $(IP)_2$ [12].

The paper is organized as follows. In Section 2, we recall definitions and we fix some notation for incidence geometry. In Section 3, we explain how to implement the sporadic group $O'N$ on a computer. In Section 4, we state
some lemmas that are used throughout the proof of Theorem 1.1. In Section 5, we prove Theorem 1.1. Finally, in Section 6, we prove Theorem 1.2.

2. DEFINITIONS AND NOTATION

The basic concepts about geometries constructed from a group and some of its subgroups are due to Tits [18] (see also [6], chapter 3).

Let \( G \) be a group together with a finite family of subgroups \((G_i)_{i \in I}\). We define the pre-geometry \( \Gamma = \Gamma(G, (G_i)_{i \in I}) \) as follows. The set \( X \) of elements of \( \Gamma \) consists of all cosets \( gG_i, g \in G, i \in I \). We define an incidence relation \( * \) on \( X \) by:

\[
g_1 G_i * g_2 G_j \quad \text{iff} \quad g_1 G_i \cap g_2 G_j \text{ is non-empty in } G.
\]

The type function \( t \) on \( \Gamma \) is defined by \( t(gG_i) = i \). The type of a subset \( Y \) of \( X \) is the set \( t(Y) \); its rank is the cardinality of \( t(Y) \) and we call \( |I| \) the rank of \( \Gamma \). A flag is a set of pairwise incident elements of \( X \) and a chamber of \( \Gamma \) is a flag of type \( I \). An element of type \( i \) is also called an \( i \)-element. The group \( G \) acts on \( \Gamma \) as an automorphism group by left translation, preserving the type of each element.

As in [11], we call \( \Gamma \) a geometry provided that every flag of \( \Gamma \) is contained in some chamber and we call \( \Gamma \) flag-transitive (FT) provided that \( G \) acts transitively on all chambers of \( \Gamma \), hence also on all flags of any type \( J \), where \( J \) is a subset of \( I \). Assuming that \( \Gamma \) is a flag-transitive geometry and that \( F \) is a flag of \( \Gamma \), the residue of \( F \) is the pre-geometry

\[
\Gamma_F = \Gamma \left( \bigcap_{j \in t(F)} G_j, \left( \bigcap_{j \in t(F)} G_j \right) \right)_{i \in \Gamma \setminus t(F)}
\]

and we readily see that \( \Gamma_F \) is a flag-transitive geometry.

We call \( \Gamma_{(i)} \) the \( G_i \)-residue of \( \Gamma \).

Let \( J \) be a subset of \( I \). The \( J \)-truncation of \( \Gamma \) is the geometry consisting of the elements of type \( j \in J \), together with the restricted type-function and induced incidence relation. In group-geometry terms, the \( J \)-truncation of \( \Gamma(G, (G_i)_{i \in I}) \) is the geometry \( \Gamma(G, (G_j)_{j \in J}) \).

We call \( \Gamma \) firm (F) (resp. thick, thin) provided that every flag of rank \( |I| - 1 \) is contained in at least two (resp. three, exactly two) chambers. We call \( \Gamma \) residually connected (RC) provided that the incidence graph of each residue of rank \( \geq 2 \) is a connected graph. We call \( \Gamma \) primitive (Pri) provided that \( G \) acts primitively on the set of \( i \)-elements of \( \Gamma \), for each \( i \in I \).
As in [7], we call \( \Gamma \) residually primitive (\( \text{Rpr} \)) if each residue \( \Gamma_f \) of a flag \( F \) is primitive for the group induced on \( \Gamma_f \) by the stabilizer \( G_F \) of \( F \).

We call \( \Gamma \) weakly primitive (\( \text{Wpr} \)) provided there exists some \( i \in I \) such that \( G \) acts primitively on the set of \( i \)-elements of \( \Gamma \) and we call \( \Gamma \) residually weakly primitive (\( \text{Rwp} \)) provided that each residue \( \Gamma_f \) of a flag \( F \) is weakly primitive for the group induced on \( \Gamma_f \) by the stabilizer \( G_F \) of \( F \).

If \( F \) is a geometry of rank 2 with \( I = \{0, 1\} \) such that each of its 0-elements is incident with each of its 1-elements, then we call \( F \) a generalized digon.

Following [3] and [4], the diagram of a firm, residually connected, flag-transitive geometry \( \Gamma \) is a graph together with additional structure, whose vertices are the elements of \( I \), which is further described as follows. To each vertex \( i \in I \), we attach the order \( s_i \), which is \( |I_f| - 1 \), where \( F \) is any flag of type \( \Gamma \setminus \{i\} \), the number \( n_i \) of varieties of type \( i \), which is the index of \( G_i \) in \( G \), and the subgroup \( G_i \). Elements \( i, j \) of \( I \) are not joined by an edge of the diagram provided that a residue \( \Gamma_f \) of type \( \{i, j\} \) is a generalized digon. Otherwise, \( i \) and \( j \) are joined by an edge endowed with three positive integers \( d_{ij}, g_{ij}, d_{ji} \) where \( g_{ij} \) (the gonality) is equal to half the girth of the incidence graph of a residue \( \Gamma_f \) of type \( \{i, j\} \) and \( d_{ij} \) (resp. \( d_{ji} \)), the \( i \)-diameter (resp. \( j \)-diameter) is the greatest distance from some fixed \( i \)-element (resp. \( j \)-element) to any other element in the incidence graph of \( \Gamma_f \).

On a picture of the diagram, this structure will often be depicted as follows.

\[
\begin{array}{ccc}
\circ & d_{ij} & g_{ij} & d_{ji} \\
| & s_i & s_j | \\
| & n_i & n_j | \\
G_i & G_j & \end{array}
\]

If \( g_{ij} = d_{ij} = d_{ji} = n \), then \( \Gamma_f \) is called a generalized \( n \)-gon and on a picture, we do not write \( d_{ij} \) and \( d_{ji} \).

If \( (d_{ij}, g_{ij}, d_{ji}) = (5, 5, 6) \) we write \( P \) on the corresponding edge instead of the 3 parameters. This is because such a rank 2 residue is a Petersen graph. We sometimes call a rank 2 residue with parameters \( (d_{ij}, g_{ij}, d_{ji}) \) a \( d_{ij} - g_{ij} - d_{ji} \)-residue.

We say that \( \Gamma \) satisfies the intersection property (\( \text{IP} \)), if every rank 2 residue of \( \Gamma \) is either a partial linear space or a generalized digon.

The (strongly) boolean lattice of a geometry \( \Gamma(G, (G_j), j \in J) \) that is firm and residually connected, is the set of \( 2^n \) subgroups \( \bigcap (G_j), j \in J \), where \( J \) is a subset of \( I \) and \( |J| = n \).
The subgroups appearing in the boolean lattice are called the parabolic subgroups of $I$, and the subgroups $G_i$ are called the maximal parabolic subgroups.

As to notation for groups, we follow the conventions of the Atlas [10] up to slight variations. The symbol ":" stands for split extensions, the "hat" symbol "^" stands for non split extensions and the symbol $\times$ stands for direct products. Sometimes it is not known whether an extension is split or not. In that case, we write ":" or nothing to denote that it is an extension.

3. IMPLEMENTING $O'N$ ON A COMPUTER

It is easy to implement this group on the computer algebra package Magma [2] thanks to the following presentation given in the Atlas [10].

$$F\langle a, b, c, d, e, f, g \rangle := \text{FreeGroup}(7);$$

$$G\langle s, t, u, v, w, x, y \rangle := \text{quo}(F|$$

$$a^2, b^2, c^2, d^2, e^2, f^2, g^2, (a*b)^3, (a*c)^2, (a*d)^2, (a*e)^2, (a*f)^2, (b*c)^2, (b*d)^2, (b*e)^2, (b*f)^2, (c*d)^2, (c*e)^2, (c*f)^2, (d*e)^2, (d*f)^2, (e*f)^2, (a*f)=(c*d)^4, (c*g*c*g*d), (c,d*g*d*g), s*f*g*g, (b*g*b*c*g*c)^2*b, (e*g*e*d*g*d)^2*e, (b*c*d*g)^3);$$

$$H := \text{sub}(G|s, t, u, v, w, x);$$

$$V := \text{CosetSpace}(G, H);$$

$$\text{ON} := \text{CosetImage}(V);$$

By typing these instructions in Magma, we obtain $O'N$ as a permutation group acting on a set of 122760 points. Using the Atlas description of the proper maximal subgroups we can obtain them as permutation subgroups and study their structure further. For example, it is easy to compute the subgroup lattices of the following list of conjugacy classes of proper maximal subgroups that are of interest for this work: $J_1$, $4 \cdot L_3(4):2$, $(3^3:4\times A_6):2$, $3^2:2^4+D_{10}$, $L_2(31)$, $4^1:L_3(2)$, $M_{11}$ (two classes), and $A_7$ (two classes).

We give now a way to construct some of them because we use their subgroup lattices throughout our proofs. Of course, the subgroup lattices of $J_1$, $M_{11}$ and $A_7$ are well known and can be obtained almost instantly in Magma. To construct the subgroup $4 \cdot L_3(4):2$, we type the following line
in Magma after having implemented $ON$ with the preceding lines of Magma-code.

$$g := \text{sub}(ON | \text{ON.1, ON.2, ON.3, ON.4, ON.7});$$

To construct the subgroup $(3^2 : 4 \times A_6) \cdot 2$, we type the following line in Magma as in the previous case.

$$h := \text{sub}(\text{ON | ON.1, ON.2, ON.3, ON.5, ON.6, ON.7});$$

$$g := \text{Normalizer}(\text{ON}, h);$$

And finally, to construct the subgroup $3^4 : 2^{1+4}D_{10}$ we type the following line in Magma.

$$syl := \text{SylowSubgroup}(\text{ON, 3});$$

$$g := \text{Normalizer}(\text{ON, syl});$$

Then we can compute the subgroup lattice of $g$ by using the “SubgroupLattice(g)” function. Because these subgroups are given as permutation groups of degree 122760, it is useful to reduce their degree before computing the subgroup lattice. This is done by looking at an orbit on which it acts faithfully. This speeds up the computation of the subgroup lattice.

4. PRELIMINARY LEMMAS AND USEFUL GEOMETRIES

The first lemma is used several times in the proof of Theorem 1.1 to conclude that a certain boolean lattice does not satisfy RWPRI.

**Lemma 4.1.** Let $G < ON$ be a group such that 240 divides its order and it has subgroups isomorphic to $2 \times A_5$ and $GL_2(3)$. Then $G$ is a subgroup of $4 \cdot L_3(4) : 2$ of order 161280, 80640 or 3840. Moreover, there is only one class of groups of order 161280 (resp. 80640, 3840) in $4 \cdot L_3(4) : 2$.

**Proof.** It view of the fact that 240 divides its order, the group $G$ is contained in one of $4 \cdot L_3(4) : 2$, $(3^2 : 4 \times A_6) : 2$, $3^4 : 2^{1+4}D_{10}$, $L_3(31)$, or $M_{11}$. Because $G$ must contain a subgroup isomorphic to $2 \times A_5$, it cannot be a subgroup of the three latter. Also, since $G$ must contain a subgroup isomorphic to $GL_2(3)$, it cannot be a subgroup of $(3^2 : 4 \times A_6) : 2$. Looking at the subgroup lattice of $4 \cdot L_3(4) : 2$ (easily computed with Magma), we see that the only subgroups satisfying the three hypotheses are groups of order 161280, 80640, or 3840. We also readily see that for each of them, there is only one conjugacy class of such subgroups in $4 \cdot L_3(4) : 2$.

We give in Table 1 the F, RC, FT, (IP)$_2$ and RWPRI rank 4 geometries of $J_1$. These are taken from [14].
 Lemma 4.2. The $F$, $RC$, $FT$, $(IP)_2$ and $RWPI$ rank 4 geometries of $J_1$ have at least one maximal parabolic subgroup isomorphic to $L_2(11)$.

Proof. Obvious thanks to Table I.

Lemma 4.3. Any $F$, $RC$, $FT$, $(IP)_2$ and $RWPI$ rank 4 geometries of $M_{11}$ that can be used to construct a geometry with a maximal parabolic subgroup isomorphic to $J_1$ is one of the five following ones.

<table>
<thead>
<tr>
<th>M_{11}.1</th>
<th>M_{11}.2</th>
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<table>
<thead>
<tr>
<th>$J_{1,1}$</th>
<th>$J_{1,2}$</th>
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<p>| | | | | |</p>
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<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>P</td>
<td>266</td>
<td>1</td>
</tr>
<tr>
<td>1463</td>
<td>2926</td>
<td>1463</td>
<td>1</td>
<td>266</td>
</tr>
<tr>
<td>$L_2(11)$</td>
<td>$2 \times A_5$</td>
<td>$S_3 \times D_{10}$</td>
<td>$2 \times A_5$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$J_{1,2}$</th>
<th>$P^*$</th>
</tr>
</thead>
</table>

| | | | | |
|---|---|---|---|
| 1 | 266 | L_2(11) |

| | | | | |
|---|---|---|---|
| 1 | 22 | A_6 |

| | | | | |
|---|---|---|---|
| 1 | 66 | 220 | 165 |
| $L_2(11)$ | $S_3$ | $S_3 \times S_3$ | $GL_2(3)$ |

| | | | | |
|---|---|---|---|
| 2 | 12 | L_2(11) |

| | | | | |
|---|---|---|---|
| 2 | 66 | 220 | |
| $S_3$ | $S_3 \times S_3$ | |

| | | | | |
|---|---|---|---|
| 1 | 1 \times 1 | 1 | 2 |

| | | | | |
|---|---|---|---|
| 1 | 266 | L_2(11) |

Proof. Geometries $J_{1.1}$ and $J_{1.2}$, given in Table I, have the same $L_2(11)$-residue. It has the following diagram.

We know that the rank 4 geometries of $M_{11}$ we are looking for must have a rank 3 residue corresponding to this one. Thus, a fast review of all rank 4 geometries given in [12] shows that the only five geometries that can fit together with $J_{1.1}$ or $J_{1.2}$ are those given in our list.
Lemma 4.4. If \( G \leq O'N \) is a subgroup containing a subgroup isomorphic to \( 2 \times A_5 \) and a subgroup isomorphic to \( S_5 \), then \( G \) is a subgroup of \( 4 \cdot L_3(4) : 2 \).

Proof. We look at the list of maximal subgroups of \( O'N \) given in [10].

The classes of maximal subgroups whose order is divisible by 120, and which may contain subgroups isomorphic to \( A_5 \) are \( 4 \cdot L_3(4) : 2 \), \((3^2 : 4 \times A_6) : 2 \), \( J_1 \), \( L_2(31) \), \( M_{11} \) and \( A_7 \). The four latter are well known simple groups. Looking at the Atlas of finite groups, we readily see that \( A_7 \), \( L_2(31) \) and \( M_{11} \) do not contain subgroups isomorphic to \( 2 \times A_5 \), while \( J_1 \) does not contain subgroups isomorphic to \( S_5 \). Using Magma, we see that \((3^2 : 4 \times A_6) : 2 \) contains only one conjugacy class of subgroups of order 120 (these are \( 2 \times A_5 \) subgroups), and that \( 4 \cdot L_3(4) : 2 \) contains some conjugacy classes of subgroups isomorphic to \( 2 \times A_5 \) and some isomorphic to \( S_5 \).

5. PROOF OF THEOREM 1.1

It is obvious that there is no RWPR and \((IP)_2 \) geometry of rank greater than 5 with \( J_1 \) as one of the maximal parabolic subgroups because \( J_1 \) does not have RWPR and \((IP)_2 \) geometries of rank greater than 4.

Lemma 4.2 tells us that the rank 4 RWPR and \((IP)_2 \) geometries of \( J_1 \) always have at least one \( G_i \) isomorphic to \( L_3(11) \).

Then, by looking at the list of maximal subgroups of \( O'N \), we readily see that only subgroups isomorphic to \( J_1 \) or \( M_{11} \) contain subgroups isomorphic to \( L_3(11) \). Because \( L_3(11) \) is maximal in \( J_1 \) and in \( M_{11} \), it must be self-normalized in \( O'N \). Thus every subgroup \( L_3(11) \) is contained in one \( J_1 \) and in two non-conjugate \( M_{11} \). So, in order to extend a rank 4 geometry of \( J_1 \) to a rank 5 geometry of \( O'N \), we need to take at least one maximal parabolic subgroup isomorphic to \( M_{11} \). Lemma 4.3 gives us the only rank 4 geometries of \( M_{11} \) that can be used as \( M_{11} \)-residues.

What we do next is try to combine a residue \( J_1, i \) with a residue \( M_{11}, j \) with \( i = 1 \) or 2 and \( j = 1, 2, 3, 4 \) or 5. In the following discussion, \( A, B, C, D \) and \( E \) denote the five maximal parabolic subgroups, \( AB \) denotes \( A \cap B \), etc.

\( J_1, 1 \) and \( M_{11}, 1 \)

Thanks to the diagrams given in Table 1. and Lemma 4.3, the diagram is almost determined. Only one edge is still unknown. It looks as follows.
We already know that $AB \cong L_2(11)$, $AC \cong 2 \times A_5$, $AD \cong S_3 \times D_{10}$, $AE \cong 2 \times A_5$, $BC \cong S_5$, $BD \cong S_3 \times S_3$ and $BE \cong GL_2(3)$. Thanks to $AC$ and $BC$, Lemma 4.4 implies that $C$ must be a subgroup of $4 \cdot L_3(4) : 2$. Looking at the diagram, we see that the residue of $CDE$ must be a thin rank 2 geometry, that is a generalized $p$-gon for some positive integer $p$. Since we want $CDE$ to act residually weakly primitively on this residue, $p$ must be a prime number. Now, looking at the residue of $C$, we get a non-connected diagram. This yields that $|C| = |AC| \cdot p$. So $C$ contains a $2 \times A_5$ maximally. Since $C$ must be a subgroup of $4 \cdot L_3(4) : 2$ of order $120p$, the number $p$ is either 2, 3 or 7. Suppose $p$ is equal to 3 or 7. Because $O_2(2 \times A_5) = 2$ is contained in $O_2(4 \cdot L_3(4) : 2) = 4$, we know that $A_5 < L_3(4) : 2$. Supposing $p \neq 2$ implies that $L_3(4)$ has a subgroup of order $60p$ containing a subgroup $A_5$, which is a contradiction. Thus we may assume $p = 2$. Now, since $C$ contains subgroups isomorphic to $S_5$ and $2 \times A_5$, and $C$ is of order 240, it must be isomorphic to $2 \times S_5$. The diagram is then fully known. It is the $O'N.1$ diagram mentioned in the introduction. We look at the geometries of $S_5 \times 2$ that could be used as $C$-residue (see [9]). This gives us $CD \cong 2 \times D_{12}$ and $CE \cong 2 \times D_8$. Looking at the boolean lattice of $J_1.1$ given in [14], we know that $ADE \cong D_{20}$. A quick look at the diagram gives us $BDE \cong D_{12}$ and $CDE \cong 2^3$. Thus $DE$ has an order divisible by 120. Now we see that $E$ satisfies the hypotheses of Lemma 4.1. Thus, thanks to this lemma, we can conclude that $E$ is a subgroup of $4 \cdot L_3(4) : 2$, of order 161280, 80640 or 3840. Assuming $E$ is of order 161280 or 80640 implies certainly that the boolean lattice does not satisfy the Rwpri condition. So we may assume $E$ is a group of order 3840 contained in a $4 \cdot L_3(4) : 2$. There is a unique class of such subgroups in $4 \cdot L_3(4) : 2$. We take a group of this class. Magma tells us that it has maximal subgroups of order 768, 640, 384 or 240. Thus $DE$ must be a group of order 240 in order to have Rwpri. Looking at maximal subgroups of the classes of subgroups of order 240, we see that none of them is of order 20, 12 or 8. So $DE$ cannot act residually weakly primitively on its residue. Thus it is not possible to construct an Rwpri geometry of rank 5 with those residues.
Remark that if we take $DE \cong 2 \times A_5$ we obtain a boolean lattice corresponding to the Ivanov-Shpectorov geometry. The latter argument shows that the RWPRI condition is not satisfied in this case.

\textbf{J$_1$.1 and M$_{11}$.2}

If we assume $A \cong J_1$ and $B \cong M_{11}$ with the residues as wanted, then thanks to the boolean lattice of J$_1$.1 given in [14] and the maximal parabolics of M$_{11}$.2 given in Lemma 4.3, we may assume $AB \cong L_2(11)$, $AC \cong 2 \times A_5$, $AD \cong S_3 \times D_{10}$, $AE \cong 2 \times A_5$, $BC \cong A_6$, $BD \cong S_3 \times S_3$ and $BE \cong S_5$. Also, we know that $ABE \cong D_{12}$. From this, we see that $E$ must contain a $S_5$ and a $2 \times A_5$. Thus thanks to Lemma 4.4, the subgroup $E$ must be a subgroup of $4 \cdot L_3(4) : 2$. Because of $ABE$, the $S_5$ and the $2 \times A_5$ must intersect in a $D_{12}$. Since there is no $2 \times A_5$ in $L/O_2(L)$, $Z(2 \times A_5) = O_2(D_{12})$ is contained in $O_2(L)$. Clearly, $|S_5 \cap O_2(L)| = 1$. Thus a $S_5 < L$ and a $2 \times A_5 < L$ cannot intersect in a $D_{12}$. Hence $E$ cannot be a subgroup of $4 \cdot L_3(4) : 2$, a contradiction. This means that it is not possible to construct an RWPRI geometry of rank 5 with those residues.

\textbf{J$_1$.1 and M$_{11}$.3}

Assuming $A \cong J_1$ and $B \cong M_{11}$ gives $AB \cong L_2(11)$, $AC \cong 2 \times A_5$, $AD \cong S_3 \times D_{10}$, $AE \cong 2 \times A_5$, $BC \cong L_2(11)$, $BD \cong S_3 \times S_3$ and $BE \cong GL_2(3)$. The subgroup $C$ must contain $BC \cong L_2(11)$. Thus it must be isomorphic to $J_1$ or $M_{11}$. If $C \cong M_{11}$, then its residue must be of type $M_{11}$.1. We have seen already that it is not possible to combine a residue of type $J_1$.1 with a residue of type $M_{11}$.1. Thus we may assume $C \cong J_1$ and the $C$-residue is $J_1$.1. Then we know that $CD \cong S_3 \times D_{10}$ and $CE \cong 2 \times A_5$. Thanks to Lemma 4.1, we know that $E$ is a group of order 3840, the cases where $E$ is of order 161280 or 80640 giving clearly no RWPRI configuration. Looking at the boolean lattice of J$_1$.1 given in [14], we know that $ADE \cong D_{20}$. A quick look at the diagram gives us $BDE \cong D_{12}$ and $CDE \cong D_{12}$. Thus $DE$ has an order divisible by 60. As in the case $J_1$.1 and $M_{11}$.1, we see that $E$ satisfies the hypotheses of Lemma 4.1 and we may assume $E$ is a group of order 3840 contained in a $4 \cdot L_3(4) : 2$. There is only one class of such subgroups in $4 \cdot L_3(4) : 2$. We take a group of this class. MAGMA tells us that it has maximal subgroups of order 768, 640, 384 or 240. Thus $DE$ must be a group of order 240 in order to satisfy RWPRI. Looking at maximal subgroups of the classes of subgroups of order 240, we see that none of them is of order 20 or 12. Thus $DE$ cannot act residually weakly primitively on its residue. Thus it is not possible to construct an RWPRI geometry of rank 5 with those residues.
**J\(_{1.1}\) and M\(_{11.4}\)**

Assuming \( A \cong J_1 \) and \( B \cong M_{11} \) gives \( AB \cong L_2(11) \), \( AC \cong 2 \times A_5 \), \( AD \cong S_3 \times D_{10} \), \( AE \cong 2 \times A_5 \), \( BC \cong L_2(11) \), \( BD \cong S_5 \) and \( BE \cong S_3 \times S_3 \). The subgroup \( C \) contains \( BC \cong L_2(11) \). It must thus be isomorphic to \( J_1 \) or \( M_{11} \). Since the \( BCD \) residue is a \( 5-3-5 \), the subgroup \( C \) cannot be isomorphic to \( J_1 \). But \( C \) contains \( AC \cong 2 \times A_5 \), thus it cannot be isomorphic to \( M_{11} \) either.

**J\(_{1.1}\) and M\(_{11.5}\)**

In this case again, the residue of \( C \) is such that \( C \) must be isomorphic to \( M_{11} \), but \( C \) contains a \( 2 \times A_5 \), a contradiction.

**J\(_{1.2}\) and M\(_{11.1}\)**

Assuming \( A \cong J_1 \) and \( B \cong M_{11} \) gives \( AB \cong L_2(11) \), \( AC \cong L_2(11) \), \( AD \cong S_3 \times D_{10} \), \( AE \cong 2 \times A_5 \), \( BC \cong S_5 \), \( BD \cong S_3 \times S_3 \) and \( BE \cong GL_2(3) \). Since \( AC \cong L_2(11) \), we know that \( C \cong M_{11} \). The \( C \)-residue must then be \( M_{11} \). It gives \( CD \cong S_3 \times S_3 \) and \( CE \cong GL_2(3) \) and the diagram is fully determined. It looks as follows.

The boolean lattice of \( J_1.1 \) (see [14]) tells us that \( ABE \cong D_{12} \). A quick look at the diagram gives us \( BDE \cong D_{12} \) and \( CDE \cong D_{12} \). Thus \( DE \) has an order divisible by 60. The same argument as in the case \( J_1.1 \) and \( M_{11.3} \) shows that it is not possible to construct an Rwp mat geometry of rank 5 with those residues.

**J\(_{1.2}\) and M\(_{11.2}\)**

Assuming \( A \cong J_1 \) and \( B \cong M_{11} \) gives \( AB \cong L_2(11) \), \( AC \cong L_2(11) \), \( AD \cong S_3 \times D_{10} \), \( AE \cong 2 \times A_5 \), \( BC \cong A_6 \), \( BD \cong S_3 \times S_3 \) and \( BE \cong S_5 \). The boolean lattice of \( J_1.2 \) given in [14] tells us that \( ABE \cong D_{12} \). As in the case \( J_1.1 \) and \( M_{11.2} \), the subgroup \( E \) must thus contain a \( 2 \times A_5 \) and a \( S_3 \) intersecting in a \( D_{12} \) which is not possible in \( ON' \).
$J_1$, $2$ and $M_{11}$

Assuming $A \cong J_1$ and $B \cong M_{11}$ gives $AB \cong L_2(11)$, $AC \cong L_2(11)$, $AD \cong S_5 \times D_{10}$, $AE \cong 2 \times A_5$, $BC \cong L_2(11)$, $BD \cong S_5 \times S_3$ and $BE \cong GL_2(3)$. The subgroup $C$ must contain $AC \cong L_2(11)$ so it must be isomorphic to $M_{11}$ but it has to be taken in the other conjugacy class of $M_{11}$ than the one including $B$ because of $BC$. We denote $C \cong M_{11}'$ to keep in mind that it is not conjugate to $B$ in $O_N$. The $C$-residue is $M_{11} \cdot 3$. Thus $CD \cong S_5 \times S_3$ and $CE \cong GL_2(3)$. The diagram is fully determined. It looks as follows.

![Diagram](image)

The boolean lattice of $J_1 \cdot 1$ (see [14]) tells us that $ADE \cong D_{20}$. A quick look at the diagram gives us $BDE \cong D_{12}$ and $CDE \cong D_{12}$. Thus $DE$ has an order divisible by 60. The same argument as in the case $J_1 \cdot 1$ and $M_{11} \cdot 3$ shows that it is not possible to construct an $\text{RWP}_3$ geometry of rank 5 with those residues.

$J_1$, $2$ and $M_{11}$

Assuming $A \cong J_1$ and $B \cong M_{11}$ gives $AB \cong L_2(11)$, $AC \cong L_2(11)$, $AD \cong S_5 \times D_{10}$, $AE \cong 2 \times A_5$, $BC \cong L_2(11)$, $BD \cong S_5$ and $BE \cong S_3 \times S_3$. Since $C$ contains $AC \cong L_2(11)$, it must be isomorphic to $M_{11}$. The only possible $C$-residue is then $M_{11} \cdot 4$. We get $CD \cong S_5$, $CE \cong S_5 \times S_3$ and the diagram is fully known. It looks as follows.

![Diagram](image)
Looking at this diagram, we see that $|D| = 600$. But 600 does not divide the order of $O'N$, thus we have a contradiction.

**J_{1,2} and M_{11.5}**

Assuming $A \cong J_1$ and $B \cong M_{11}$ gives $AB \cong L_2(11)$, $AC \cong L_2(11)$, $AD \cong S_5 \times D_{10}$, $AE \cong 2 \times A_5$, $BC \cong L_2(11)$, $BD \cong S_5$ and $BE \cong GL_2(3)$. Since $C$ contains $AC \cong L_2(11)$, it must be isomorphic to $M_{11}$. The only possible $C$-residue is then $M_{11.5}$. We get $CD \cong S_5$ and $CE \cong GL_2(3)$. The diagram is fully known. It looks as follows.

Looking at the diagram, we see that $|D| = 600$. But 600 does not divide the order of $O'N$, thus we have a contradiction.

This finishes the proof of Theorem 1.1.

6. PROOF OF THEOREM 1.2

There are only four rank 5 geometries that are F, RC, FT, (IP)$_2$ and RWPR for $M_{11}$. Their diagrams are given in Table 2.

All these geometries have at least two maximal parabolic subgroups isomorphic to $L_2(11)$. Thus, if we want to construct a rank 6 geometry for $O'N$, with one maximal parabolic subgroup isomorphic to $M_{11}$, at least three of them must be isomorphic to $M_{11}$. Here, no $J_1$ may be used because $J_1$ does not have rank 5 RWPRI geometries. Because $L_2(11)$ is self-normalized in $O'N$, and because $O'N$ has only two conjugacy classes of subgroups $M_{11}$, we know that at least four of the six maximal parabolics must be subgroups $M_{11}$.

Suppose $G_0 \cong M_{11}$ and its residue is of type $M_{11.6}$ (see Table II.). Looking at the rank 5 geometries of $M_{11}$ (see Table II.), we may assume $G_{01} \cong G_{02} \cong L_2(11)$ which implies that $G_1 \cong G_2 \cong M_{11}$. If we consider that $G_{01}$ is the $L_2(11)$ appearing at the left of the diagram of $M_{11.6}$, the $G_{01}$-residue implies that the $G_1$-residue cannot be of the form $M_{11.8}$ or $M_{11.9}$. 
TABLE II
The Rank 5 RWpRI Geometries of $M_{11}$

$M_{11.6}$

$M_{11.7}$

$M_{11.8}$

$M_{11.9}$
Suppose it is of the form $M_{11}, 6$. Then there is no rank 5 geometry of $M_{11}$ that can be used as $G_2$-residue. And the same holds if we assume the $G_1$-residue to be of the form $M_{11}, 7$. The same kind of discussion permits us to show very easily that it is not possible to extend a rank 5 geometry of $M_{11}$ to a rank 6 geometry of $O'N$. 

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