A new Petrie-like construction for abstract polytopes

Michael I. Hartley a,1, Dimitri Leemans b

a University of Nottingham (Malaysia Campus), Jalan Broga, Semenyih, 43500 Selangor, Malaysia
b Département de Mathématiques, Université Libre de Bruxelles, C.P. 216, Géométrie, Boulevard du Triomphe, B-1050 Bruxelles, Belgium

Received 12 June 2007
Available online 15 January 2008
Communicated by Francis Buekenhout

Abstract

This article introduces a new construction for polytopes, that may be seen as a generalisation of the Petrie dual to higher ranks. Some theoretical results are derived regarding when the construction can be expected to work, and the construction is applied to some special cases. In particular, the generalised Petrie duals of the hypercubes are enumerated.

© 2008 Elsevier Inc. All rights reserved.

Keywords: Abstract regular polytopes; Petrial

1. Introduction

The history of the study of regular polyhedra and regular polytopes began an important turning point when Coxeter popularised, in Section 2 of [1], the concept of the “Petrie polygon” of a polyhedron. Loosely, a Petrie polygon of a polyhedron \( P \) is a polygon whose vertices and edges are selected from those of \( P \) in such a way that any pair of successive edges, but no three consecutive edges, lie on the same face of \( P \). Clearly, a Petrie polygon of a convex polyhedron is not planar. By way of example, the Petrie polygons of the cube are skew hexagons.

The turning point started in earnest when the concept of the “Petrie dual” (or “Petrial”) of a polyhedron was introduced. This is a structure with the same vertices and edges as a given polyhedron, but whose faces are the Petrie polygons. So the Petrie dual of the cube is a polyhedron with 8 vertices, 12 edges, and 4 (skew) hexagonal faces, meeting three per vertex. This poly-

E-mail addresses: mikeh@dugeo.com (M.I. Hartley), dleemans@ulb.ac.be (D. Leemans).

Present address: DownUnder Geosolutions, 80 Churchill Avenue, Subiaco 6008, Australia.

0097-3165/S – see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcta.2007.11.008
The discovery of these “Petrie polyhedra” led eventually to the development of the concept of an abstract regular polypode. Thinking about Petrie polyhedra requires detachment from the concept that the faces of a polyhedron must be planar. We refer to Section 1A of [6] for more details on historical development. Let us just mention the work of Branko Grünbaum [2] who generalised regular skew polyhedra by allowing skew polygons as faces as well as vertex-figures. This leads to detachment from the idea that the faces need any geometric meaning at all. Indeed, an abstract polypode is defined merely as a partially ordered set, with certain properties imposed that are intended to reflect (loosely) the properties one expects the face lattice of a polyhedron to have. For example, the maximal totally ordered subsets (the flags) all have the same size. Also, for any flag $F$, if $F, G \in \Phi$ are such that there is exactly one $H \in \Phi$ with $F < H < G$, then the polytope has exactly one flag $\Phi'$ for which $\Phi \backslash \Phi' = \{H\}$. There are also conditions regarding connectivity. Abstract polytopes, and more specifically regular abstract polytopes, have received a great deal of attention over the 20 years since they were introduced, and are now well-established in the literature. The standard reference for the topic is [6], to which the reader is referred for more details. Embedding a polytope in a ‘space’ is not necessary for the study of abstract polytopes, and in fact forms a distinct branch of the theory (see Chapter 5 of [6]).

According to Jacques Tits [7], an abstract polypode is regular if the poset’s automorphism group acts transitively on the flags. The “fundamental theorem” of abstract regular polytopes links regular abstract polytopes to a class of groups with special generating sets of involutions, the string C-groups.

A string C-group is a group $W = \langle s_0, \ldots, s_{n-1} \rangle$, where the $s_i$ are all involutions, where $s_i$ and $s_j$ commute if $|i - j| > 1$, and which satisfies the so-called intersection property. This property is that $W_I \cap W_J = W_{I \cap J}$, where $W_I$ is defined for any $I \subseteq \{0, \ldots, n-1\}$ via $W_I = \langle s_i : i \in I \rangle$. String Coxeter groups are examples of string C-groups. Following the terminology of Coxeter group theory, subgroups of the form $W_I$ are called parabolic subgroups of $W$. Let $H_i$ be the parabolic subgroup $\langle s_j : j \neq i \rangle$. Then a polytope may be formed as a coset geometry by taking the collection of all cosets $\{wH_j : w \in W, j \in \{0, \ldots, n-1\}\}$, defining a partial order via $uH_i \leq vH_j$ if and only if $i \leq j$ and $uH_i \cap vH_j \neq \emptyset$, and adjoining a maximal and a minimal element to the poset. This polytope is regular, and its automorphism group will be exactly $W$.

In fact, for any regular abstract polytope $\mathcal{P}$, its automorphism group $W$ is a string C-group. The generators $s_0, \ldots, s_{n-1}$ of $W$ arise in a natural way from the structure of $\mathcal{P}$, and the polytope constructed as a coset geometry as per the preceding paragraph is isomorphic to $\mathcal{P}$. In this way there is a one-to-one correspondence between regular abstract polytopes and string C-groups (with specified generating sets). The reader is again referred to [6] for details.

For a subgroup $N$ of the automorphism group $W$ of a polytope $\mathcal{P}$, one may attempt to construct a quotient $\mathcal{P}/N$ in the obvious way. Another important result in the theory of abstract polytopes is that every regular polytope $\mathcal{Q}$ may be written as the quotient of a universal polytope $\mathcal{P}$ having the same vertex figures and facets as $\mathcal{Q}$. Knowing the universal polytope with a particular facet and vertex figure then gives, in principle, all polytopes with that facet and vertex figure. The universal polytopes are therefore of particular interest. For further information, the reader is referred to Sections 2D and 4A of [6].

Revisiting the Petrie dual, a regular polyhedron corresponds to a C-group $W = \langle s_0, s_1, s_2 \rangle$ on three generators. The group of the Petrie dual will be given by $W = \langle s_0s_2, s_1, s_2 \rangle$ (see 7B2.
of [6]). The latter is not always a C-group. Although it is the same abstract group, it has different
generators, and does not always satisfy the intersection property—it is possible for \( \langle s_0s_2, s_1 \rangle \cap 
\langle s_1, s_2 \rangle \) not to equal \( \langle s_1 \rangle \), even if \( \langle s_0, s_1 \rangle \cap \langle s_1, s_2 \rangle \) equals \( \langle s_1 \rangle \).

The Petrie dual is an example of a generalised mixing operation (Section 7B of [6]), where
a new polytope is constructed by manipulating the generating set of the automorphism group of
an old one. This article presents a mixing operation that operates on polytopes of arbitrary rank
greater than 1, and of which the Petrie dual is a special case. It may therefore be regarded as a
Petrie-like operation for higher rank polytopes. Examples of the operation have already appeared
in the literature, but the operation itself has not been specifically studied. For example, in [4] it
helped to tie together the two polytopes that occur amongst the rank 4 thin incidence geometries
of the first Janko group \( J_1 \). Also, [5] (in Section 2) presents a generalised Petrie operation which
is a special case of the operation presented here.

2. Preliminary results

Let \( W = \langle s_0, \ldots, s_{n-1} \rangle \), let \( H \) be a parabolic subgroup of \( W \), and let \( \omega \) be a central involution
of \( H \). That is, \( \omega^2 = 1 \) and \( \omega \) is an element of the centre \( Z(H) = \{ k \in H: kh = hk \text{ for all } h \in H \} \)
of \( H \). For much of what follows we let \( W = H \). Let \( I \) be a subset of \( \{0, \ldots, n - 1 \} \), and for any
subset \( J \) of \( \{0, \ldots, n - 1 \} \), let \( S_J = \{ s_j: j \in J \} \). Note that for what follows, the case \( I = \emptyset \) is
generally not interesting.

Consider \( M_I = \langle s_i \omega^{\eta_i} \rangle \) where \( \eta_i = 1 \) or 0 respectively if \( i \) is, or is not, in \( I \). Likewise, let
\( M'_I = \langle s_i \tau^{\eta_i} \rangle \subseteq W \times \langle \tau \rangle \), where \( \tau \neq W \), \( \tau^2 = 1 \), and \( \tau \) commutes with all of \( W \). Note that
\( \tau \in M'_I \) if and only if \( M'_I = W \times \langle \tau \rangle \). This article will explore the circumstances under which
\( M_I \) is a string C-group. The bulk of this section and the next concern the case
\( \omega \in Z(W) \). This section develops some theory, while Section 4 applies the theory to a particular class of polytopes.
Section 5 considers briefly the case \( \omega \in Z(H) \) where \( H \) is a proper parabolic subgroup, again
developing theory and giving examples of its application.

In what follows, if \( s \) denotes + then \( -s \) denotes – and vice versa. Define \( K^+_I \) and \( K^-_I \) as
follows. If \( I \cap J = \emptyset \) then \( K^+_I \cap J = W_J \), \( K^-_I \cap J = \emptyset \). Otherwise, let \( I \in K^+_I \), and if \( x \in K^+_I \) and \( j \in J \), let \( x s_j \in K^-_{I \cap J} \) if \( j \in I \), or \( x s_j \in K^+_I \cap J \) otherwise.

Thus, \( K^+_I \) is the set of elements of \( W_J \) which may be expressed as words over \( S_J \) which have
an even number of elements from \( S_I \). Likewise, \( K^-_I \) is the set of elements of \( W_J \) which may be
expressed as words over \( S_J \) with an odd number of elements from \( S_I \). If \( J = \{0, \ldots, n - 1 \} \), we
write \( K^+_I \) as \( K^+_I \).

Clearly, \( K^+_I \) is a subgroup of \( W_J \). If \( I \) and \( J \) are disjoint, it equals \( W_J \). If \( I \) and \( J \) are not
disjoint, either \( K^+_I \cap J = W_J \), or \( K^+_I \cup K^-_I = W_J \) and \( K^+_I \cap K^-_I = \emptyset \). The following
lemmas will prove useful.

2.1. Lemma. The group \( M'_I = W \times \langle \tau \rangle \) if and only if \( K^+_I = W \).

Proof. If \( I \in K^-_I \), then there is a word \( v \) in the subgroup \( \langle s_j: j \in J \rangle \) with an odd number of elements
of \( \{ s_i : i \in I \} \), such that \( v = 1 \) in \( W \). Mapping this word to \( M'_I \) yields \( v \tau^{\eta} \) for some
odd \( \eta \), that is, it yields \( \tau \), so \( \tau \in \langle s_j \tau^{\eta} \rangle = M'_I \). It follows that \( M'_I = W \times \langle \tau \rangle \). The converse is trivial. \( \Box \)

A similar result could be shown regarding the parabolic subgroups of \( M'_I \).
2.2. **Lemma.** The parabolic subgroup $\langle s_j \tau^0 \rangle: j \in J$ of $M'_I$ equals $W_I \times \langle \tau \rangle$ if and only if $K^+_{I|J} = W_J$.

If $\omega \in Z(W)$, there is an obvious morphism $\zeta$ mapping $M'_I$ to $M_I$, via $\zeta : s_j \mapsto s_j$ for $j \notin I$, and $\zeta : s_j \tau \mapsto s_j \omega$ for $i \in I$. That is, $\zeta : q \tau^0 \mapsto q \omega^0$ for $q \in W$.

2.3. **Lemma.** The morphism $\zeta$ satisfies $|\ker \zeta| = 1$ or 2, and $|\ker \zeta| = 1$ if and only if $\omega \in K^+_I \neq W$.

**Proof.** Let $x \in M'_I$ be such that $x \zeta = 1$. Either $x = q \in W$, or $x = q \tau \in W$. If $x = q$, then $x \zeta = q \zeta = q$, which can only be 1 if $x = 1$. If $x = q \tau$, then $x \zeta = q \omega = 1$ only if $q = \omega$, that is, $x = \omega \tau$. Thus, $\ker \zeta = \{1\}$ or $\{1, \omega \tau\}$.

Now $\omega \tau \notin M'_I$ if and only if there is no way to express $\omega$ as a word with an odd number of elements of $S_I$. This is so if and only if $K^-_I \neq W$ and $\omega \in K^+_I$. □

Note by way of example that any orientable polytope will have $K^-_I \neq W$, if $I = \{0, \ldots, n-1\}$. When the orientable polytope has a central involution of even length, then $|\ker \zeta| = 1$. Examples of such polytopes include the $n$-gons for even $n$, and the $d$-cubes for even $d$. In fact, if an $n$-gon has $n = 4m$ for some $m$, then $\omega \in K^+_I \neq W$ for any non-empty subset $I$ of $\{0, 1\}$.

2.4. **Lemma.** If $W = G \times C_2$, then there exists a set $I$ and a central involution $\omega$ such that $W = K^+_I \times \langle \omega \rangle$.

**Proof.** If there exists a normal subgroup $G$ of $W$ such that $W = G \times C_2$, we may write $W = G \times \langle \alpha \rangle$. This $\alpha$ will be an involution in the centre $Z(W)$ of $W$, and we may set $\omega = \alpha$. Let $I = \{i : s_i \notin G\}$. Then $G$ satisfies the defining properties of $K^+_I$, so we may write $W = K^+_I \times \langle \omega \rangle$. □

Hence, although $K^+_I$ may be defined for any polytope, the concept arises naturally in polytopes whose groups are of the above form. The lemma has a converse.

2.5. **Lemma.** If $\omega$ is a central involution of $W$, and $\omega \notin K^+_I$, then $W = K^+_I \times \langle \omega \rangle$.

**Proof.** By Lemma 2.3, the group $K^+_I$ has index one or two in $W$. It cannot have index one here, because $\omega \notin K^+_I$. Since it has index two, it is normal in $W$. The group $\langle \omega \rangle$ is also normal, since $\omega$ is a central involution. Reminding ourselves again that $\omega \notin K^+_I$ yields the desired result. □

3. **Main results**

We are interested to know when $M'_I$ and $M_I$ will be string C-groups, that is, groups of polytopes. Certainly $M'_I$ has a string diagram, and $M_I$ sometimes will, in particular if $\omega \in Z(W)$, since if the order of $s_i s_j$ is two, then the orders of $s_i \tau^0 s_j \tau^0$ and $s_i \omega^0 s_j \omega^0$ are still two, at least (for the latter case) if $\omega$ commutes with $s_i$ or $s_j$. It remains to discover when the groups satisfy the intersection property.

3.1. **Theorem.** Let $J$ and $K$ be subsets of $\{0, \ldots, n-1\}$. Then the intersection of $M'_{I|J} = \langle s_j \tau^0 \rangle: j \in J$ and $M'_{I|K} = \langle s_j \tau^0 \rangle: j \in K$ is different from $M'_{I|(J \cap K)} = \langle s_j \tau^0 \rangle: j \in J \cap K$ if and only if $\tau \in M'_I$ and $\tau \in M'_I|K$, but $\tau \notin M'_I|J \cap K$. This is how to prove the theorem.
Proof. If $x \in M'_{I|J} \cap M'_{I|K}$, then $x = s_{j_1} \tau^\varepsilon_1 \ldots s_{j_m} \tau^\varepsilon_m$, where the $j_i$ are in $J$, and $\varepsilon_i$ is 1 or 0 respectively as $j_i$ is, or is not, in $I$. This may be simplified to $x = s_{j_1} \ldots s_{j_m} \tau^\varepsilon$. Likewise $x = s_{k_1} \ldots s_{k_m} \tau^{\varepsilon'}$, where each $k_i$ is in $K$. For these expressions to be equal, it must be that $\varepsilon = \varepsilon'$, and $s_{j_1} \ldots s_{j_m}$ and $s_{k_1} \ldots s_{k_m}$ are equal in $W$, and may therefore be written as $s_{j_1} \ldots s_{j_m} \in W_{J \cap K}$.

Let $r_1 = s_{j_1} \tau^{\varepsilon_1}$. Then $r_1 \ldots r_m \tau^{\varepsilon_m}$ equals either $x$ or $x\tau$. We are assuming that $x \notin M'_{I|J \cap K}$, hence $x \tau \in M'_{I|J \cap K}$. From this, we may conclude that $\tau \in M'_{I|J}$, since $x \in M'_{I|J}$ and $x\tau \in M'_{I|J \cap K} \leq M'_{I|J}$, and likewise, $\tau \in M'_{I|K}$. We may also conclude that $\tau \notin M'_{I|J \cap K}$, since $x\tau \in M'_{I|J \cap K}$ and $x \notin M'_{I|J \cap K}$.

The converse is trivial. \(\square\)

Thus $M'_I$ fails the intersection property if and only if there exists some $J$ and $K$ such that $1 \in K_{I|J}^-$ and $1 \in K_{I|K}^-$, but $1 \notin K_{I|J \cap K}^-$. Note however, that $1 \notin K_{I|i}^-$ for any $i$, and that if $1 \in K_{I|J}^-$ and $J \subseteq K$, then $1 \in K_{I|K}^-$. From the above observations, and from the previous theorem, the following may be deduced.

3.2. Theorem. $M'_I$ satisfies the intersection property if and only if $1 \notin K_{I}^-$, or there exists some $J$ such that $1 \in K_{I|J}^-$ if and only if $J \subseteq K$.

Proof. Omitted. \(\square\)

Note that the $J$ of the above theorem will satisfy $J \cap I \neq \emptyset$.

If $W$ has a central involution, there is a relationship between when $M'_I$ satisfies the intersection property, and when $M_I$ does.

3.3. Theorem. Let $\omega$ be a central involution of $W$, and let $\omega$ be such that $\omega \notin H_0H_{n-1}$. Then $M_I$ passes or fails the intersection property exactly when $M'_I$ does.

Proof. Suppose first that $M'_I$ passes the intersection property, and $M_I$ fails. Then there exist $J$ and $K$ and $x$ such that $x \in \langle s_{j_0} \omega^0 \rangle$: $j \in J$ and $x \in \langle s_{j_0} \omega^0 \rangle$: $j \in K$, but $x \notin \langle s_{j_0} \omega^0 \rangle$: $j \in J \cap K$. Note that neither $J$ nor $K$ can be $\{0, 1, \ldots, n - 1\}$. Now either $x$ or $x\omega \tau$ is an element of $\langle s_{j} \tau^\varepsilon \rangle$: $j \in J$, and likewise either $x$ or $x\omega \tau$ is in $\langle s_{j} \tau^\varepsilon \rangle$: $j \in K$. If in fact $x$ (or $x\omega \tau$) is in both of these, then since $M'_I$ satisfies the intersection property, the same is in $\langle s_{j} \tau^\varepsilon \rangle$: $j \in J \cap K$. Hence either $x$ or $x\omega \tau$ is an element of both $\langle s_{j} \rangle$: $j \in J$ and $\langle s_{j} \rangle$: $j \in K$, and thus also of $\langle s_{j} \rangle$: $j \in J \cap K$.

Suppose first this common element is $x$. Then $x \in \langle s_{j} \rangle$: $j \in J \cap K$, but $x \notin \langle s_{j} \omega^0 \rangle$: $j \in J \cap K$. If it follows therefore that $x\omega \tau$ is in the latter, which is a subgroup of $\langle s_{j} \omega^0 \rangle$: $j \in J$, Thus, both $x$ and $x\omega \tau$ are elements of $\langle s_{j} \omega^0 \rangle$: $j \in J$, hence also of $\langle s_{j} \rangle$: $j \in J$, which is a subset of $H_0H_{n-1}$ unless $J = \{0, \ldots, n - 1\}$. This would contradict the premise of the theorem. A similar argument applies if the common element is $x \omega \tau$.

Suppose now that $x \in \langle s_{j} \omega^0 \rangle$: $j \in J$ and $x\omega \tau \in \langle s_{j} \omega^0 \rangle$: $j \in K$, where without loss of generality $x \in W$. Now $x \in \langle s_{S} \rangle$ and $x\omega \tau \in \langle s_{S} \rangle$: $S \subseteq K$, since $\omega = (x^{-1})(x\omega \tau)$. Then $x^{-1}$ may be written $h_{J+}h_{J-}$, and $x\omega \tau$ written $h_{K+}h_{K-}$, where $h_{X+} \in H_0$ and $h_{X-} \in H_{n-1}$. It follows that $\omega$ equals $h_{K+}h_{K+}+h_{J-}h_{J-}$, which is conjugate to the element $h_{K+}h_{J+}h_{J-}h_{K-}$ of $H_0H_{n-1}$. Since $\omega \in Z(W)$, it is equal to all its conjugates, so this again contradicts the premise of the theorem.

Now, assume that $M_I$ passes the intersection property, but $M'_I$ fails. If $J$ and $K$ are such that $\tau \in \langle s_{j} \tau^\varepsilon \rangle$: $j \in J \cap K$, then $\tau \notin \langle s_{j} \tau^\varepsilon \rangle$: $j \in J \cap K$, then in particular $\tau \in$
For the cube, or

Table 1
The Schläfli types of various polytopes derived from the cube

<table>
<thead>
<tr>
<th>I</th>
<th>Schläfli symbol</th>
<th>Further information</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0]</td>
<td>$[4, 3^n-2]$</td>
<td>$1 \not\in K^I_{I-}$</td>
</tr>
<tr>
<td>[0, ..., n - 1]</td>
<td>$[4, 3^n-2]$</td>
<td>$1 \not\in K^I_{I-}$</td>
</tr>
<tr>
<td>[1, ..., n - 1]</td>
<td>$[4, 3^n-2]$</td>
<td>$1 \not\in K^I_{I-}$</td>
</tr>
<tr>
<td>[0, ..., i]</td>
<td>$[4, 3^i-1, 6, 3^{n-2-i}]$</td>
<td>$1 \in K^I_{I-K}$ iff $[i, i + 1] \subseteq K$</td>
</tr>
<tr>
<td>[1, ..., i]</td>
<td>$[4, 3^i-2, 6, 3^{n-2-i}]$</td>
<td>$1 \in K^I_{I-K}$ iff $[i, i + 1] \subseteq K$</td>
</tr>
<tr>
<td>[0, i, ..., n - 1]</td>
<td>$[4, 3^i-2, 6, 3^{n-i}]$</td>
<td>$1 \in K^I_{I-K}$ iff $[i - 1, i] \subseteq K$</td>
</tr>
<tr>
<td>[i, ..., n - 1]</td>
<td>$[4, 3^i-2, 6, 3^{n-1-i}]$</td>
<td>$1 \in K^I_{I-K}$ iff $[i - 1, i] \subseteq K$</td>
</tr>
</tbody>
</table>

$(s_j \tau^n): j \in J$, so $\omega \in (s_j \omega^n): j \in J$. Since again neither $J$ nor $K$ can be $\{0, 1, \ldots, n - 1\}$, this clearly contradicts the premise of the theorem. □

The following result about the sections of the polytopes $P(M_I)$ and $P(M'_I)$ is useful.

3.4. Theorem. Suppose $\omega$ is a central involution of $W$, and $\omega \not\in H_{0, n-1}$. Suppose further that $M'_I$ (and therefore $M_I$) is a $C$-group. Then the proper sections of the polytope $P(M_I)$ are isomorphic to the corresponding proper sections of $P(M'_I)$.

Proof. It is sufficient to prove the result for the facets and vertex figures. In fact, by duality, it is sufficient to prove the result for the facets. Let $\zeta$ be the map from $M'_I$ to $M_I$ via $q \mapsto q \omega$. By Lemma 2.3, we have ker $\zeta = \{1, \omega \tau \}$. The restriction $\psi$ of $\zeta$ to the parabolic subgroup $K_{n-1}$ of $M'_I$ will be a bijection from $M'_I$ to $M_I$. It is certainly onto, and if $\omega \tau \in \text{ker } \psi$, then $\omega \in H_{n-1}$. □

4. Generalised Petrials of hypercubes

In this section, the new Petrie-like construction is applied to the $n$-cubes. The group of the cube is $W = [4, 3, \ldots, 3] = (s_0, \ldots, s_n)$. We can represent $W$ as a permutation group on the set $\{-n, -1, 1, \ldots, n\}$ via $s_0 = (-1, 1)$, and $s_i = (i, i + 1)(-i, -(i + 1))$. The cube has a central involution given by $(1, -1)(2, -2) \cdots (n, -n) = (s_0 \ldots s_{n-1})^n$.

It is useful to characterise the $I$ for which $M_I$ and $M'_I$ are $C$-groups.

4.1. Theorem. $M_I$ and $M'_I$ are $C$-groups if and only if $I = X \cup Y \neq \emptyset$, where $X = \emptyset$ or $\{1, \ldots, i\}$ or $\{i, \ldots, n - 1\}$, and $Y = \emptyset$ or $\{0\}$.

Proof. For the cube, $\omega \not\in H_{0, n-1}$, so, by Theorem 3.3, $M_I$ and $M'_I$ are either both or neither $C$-groups. However, by Theorem 3.2, the group $M'_I$ is a $C$-group if and only $1 \not\in K^{-}_{I-}$, or there exists $J$ such that $1 \in K^{-}_{J}=K^{-}_{I-K}$ if and only if $J \subseteq K$.

If $i_1, i_2 \in I$ are such that $i_1 - 1 \not\in I$, $i_2 + 1 \not\in I$, $i_2 \neq i_1 - 1$, $i_1 > 1$ and $i_2 < n - 1$, then $1 \in K^{-}_{I-[i_1-1,i_1]^{-1}}$ since $s_{i_1-1}s_{i_1}$ has order 3, and likewise $1 \in K^{-}_{I-[i_2,i_2+1]}$. This contradicts Theorem 3.2 if such $i_1$ and $i_2$ exist. It follows that if $i \in I$, then either $1, 2, \ldots, i \in I$, or $i, i + 1, \ldots, n - 1 \in I$. Therefore $I$ is as stated. The converse result is straightforward. □

The different cases, and the Schläfli symbols of the resulting polytopes, are shown in Table 1. It is worthwhile to identify various sections of these polytopes, especially in the light of Theorem 4.4. First recall Theorem 3.4, which shows that it is sufficient to analyse the sec-
tions of $M'_1$. The sections of type $\{3, 6\}$ have a group $\langle \rho_0, \rho_1, \rho_2 \rangle$, say, generated by $\rho_0 = (i - 1, i)(-i - 1, i)$, $\rho_1 = (i, i + 1)(-i, -(i + 1))$ and $\rho_2 = (i + 1, i + 2)(-i + 1, -(i + 2))(n + 1, n + 2)$, or by $\rho_0 = (i - 1, i)(-i - 1, i)(n + 1, n + 2)$, $\rho_1 = (i, i + 1)(-i, -(i + 1))(n + 1, n + 2)$ and $\rho_2 = (i + 1, i + 2)(-i + 1, -(i + 2))$). These are isomorphic, respectively, to $(1, 2), (2, 3), (3, 4)(5, 6)$ and $(1, 2)(5, 6), (2, 3)(5, 6), (3, 4)$, each of which has order 48. There is only one regular polytope of type $\{3, 6\}$ and a group of order 48, that is $\{3, 6\}_{(2, 0)}$.

In fact, a rank $k$ section of type $\{3^i, 6, 3^j\}$ (with $i + j + 2 = k, i, j \geq 0$) will have a group isomorphic to $S_{k+1} \times C_2$, of order $2(k + 1)!$. The vertex figure will have a group of order $2k!$ or $k!$, depending on whether or not it contains a 6 in the Schläfli symbol, and therefore the rank $k$ sections have $\frac{2}{3}(k + 1)$ vertices, where $p$ is the first entry in the Schläfli symbol of the section (either 3 or 6).

Consider sections of type $\{3, 6, 3\}$ and $\{3, 6, 6\}$. The groups of these sections have order 240. However, they must be respectively quotients of the universal polytopes $\gamma T_{(2, 0)}^{4, 6}$ and $\gamma T_{(2, 0)}^{6, 6}$ of Sections 11B and 11E of [6]. An inspection of Tables 11B1 and 11E1 of [6] reveals that the groups of these universal polytopes have the same size as the sections under consideration. Therefore these sections are universal. Likewise, a proper section of type $\{4, 3, 6\}$ must be the universal polytope $4T_{(2, 0)}^{4, 6}$. This is generalised in Theorem 4.3.

As noted earlier, a section of the form $\{3^i, 6, 3^j\}$, $k = i + j + 2$, has $(k + 1)$ vertices when $i > 0$. The facets of this section will have $k$ vertices (irrespective of whether or not $j = 0$). It may be shown (see Lemma 4.2 below) that the section is therefore weakly neighbourly, that is, every pair of vertices share a common facet. In fact, since the simplex is also weakly neighbourly, we could say that every section whose Schläfli symbol starts with a 3 is weakly neighbourly.

4.2. Lemma. Let $\mathcal{P}$ be a regular polytope with $n + 1$ vertices, and suppose its facets have $n$ vertices. Then $\mathcal{P}$ is weakly neighbourly.

Proof. Let $x$ be a vertex of $\mathcal{P}$, let $F_x$ be the set of all facets of $\mathcal{P}$ containing $x$, and let $V_x$ be the set of vertices of these facets. Note that $|V_x| \geq n$ and (by regularity) does not depend on $x$. If $|V_x| \neq n$ we are done, so assume $|V_x| = n$. Then all facets containing $x$ have the same vertex set $V_x$. Let $y$ be the vertex of $\mathcal{P}$ that is not in $V_x$. There must exist some $v \in V_x \cap V_y$. Then $F_y$ contains a facet containing $x$, and another containing $y$. Therefore, $V_x \cup \{y\} \subseteq V_y$, so $|V_y| \neq |V_x|$, which is a contradiction. □

Chapter 8 of [6] gives a construction $2^{[\mathcal{K}, \mathcal{D}]}$ of a polytope from a regular polytope $\mathcal{K}$ (which becomes the vertex figure of the new polytope) and a Coxeter diagram $\mathcal{D}$ on which the group $\Gamma(\mathcal{K})$ of $\mathcal{K}$ acts. The group of the new polytope is $G(\mathcal{D}) \rtimes \Gamma(\mathcal{K})$, where $G(\mathcal{D})$ is the Coxeter group defined by the diagram. The trivial diagram is the diagram with the same number of vertices as $\mathcal{K}$, where no pair of vertices is joined by an edge. In this case, $G(\mathcal{D}) = C^v_2$ where $v$ is this number of vertices. If $\mathcal{D}$ is the trivial diagram, then the polytope $2^{\mathcal{K}, \mathcal{D}}$ is written $2^\mathcal{K}$. Note by way of example that if $\mathcal{K}$ is the $(n - 1)$-simplex, then $2^\mathcal{K}$ is the $n$-cube.

4.3. Theorem. Let $Q$ be a face of $\mathcal{P}(M'_1)$, and let $\mathcal{K}$ be the vertex figure of $Q$. Suppose the vertex figure of $\mathcal{K}$ starts with 3. Then $Q$ is the universal $2^\mathcal{K}$.

Proof. Corollary 8E6 of [6] shows that since $\mathcal{K}$ is finite and weakly neighbourly, and if $\mathcal{F}$ is a facet of $\mathcal{K}$, then the universal polytope $\{2^\mathcal{F}, \mathcal{K}\}$ is just $2^\mathcal{K}$, hence $2^\mathcal{K}$ is indeed universal. The
group of $2^K$ will have order $2^v|\Gamma(K)|$, where $v$ is the number of vertices of $K$, and $\Gamma(K)$ is the automorphism group of $K$. Here, if $K$ is a simplex, $2^K$ is a cube, and we are done. If, on the other hand, $K$ is a rank $k$ polytope of type $\{3^i, 6, 3^j\}$, then it has $k+1$ vertices and group of order $2(k+1)!$. The group of the universal $2^K$ therefore has order $2^{k+2}(k+1)!$.

Considering the group of the rank $k+1$ polytope $Q$, we may note that the order of its automorphism group is twice that of the $k+1$ cube, that is, $2^{k+2}(k+1)!$. Since $Q$ has the same vertex figures and (by a simple induction) facets as $2^K$, it follows that $Q$ is in fact $2^K$ as required.

Note that the above applies also to the rank $n$ improper face, $Q = \mathcal{P}(M'_I)$. It is useful to note exactly when $\mathcal{P}(M_I) \cong \mathcal{P}(M'_I)$. This will be when $\omega \tau \notin M'_I$. However, if these polytopes have a 6 in their Schläfli symbol, then $\tau \in M'_I$, so $\omega \tau \in M'_I$. Therefore, in this case, the polytope $\mathcal{P}(M_I)$ is not the universal $\mathcal{P}(M'_I)$, but a quotient $\mathcal{P}(M'_I)/(\omega \tau)$. If the polytopes have no 6 in their Schläfli symbol, they must be either cubes or hemicubes. However, the map taking an element $q \tau^N \in M'_I$ to $q \in W$ is onto, so $|M'_I| \geq |W|$, and thus $|M'_I| = |W|$ and $\mathcal{P}(M'_I)$ is a cube. Then $\mathcal{P}(M_I)$ will be a cube also, unless $\omega \tau \in M'_I$, that is, $\omega \in K^-_I$. For the $n$-cube, we have $\omega = (s_0 \cdots s_{n-1})^n$.

The three cases under consideration here are $I = \{0, \ldots, n-1\}$, $I = \{1, \ldots, n-1\}$ and $I = \{0\}$.

For the first case, since the cube is orientable, $\omega \in K^-_I$ if and only if $n$ is odd. Now, let $I = \{0\}$. If $n$ is odd, $\omega \in K^-_0$. For even $n$, note that $\omega \in K^+_0$. If also $\omega \in K^-_0$, then there exist two words for $\omega$ that may be transformed from one to another by only applying a relation of $\{4, 3, \ldots, 3\}$, one with an odd number of $s_0$, the other with an even number. However, none of the relations of $\{4, 3, \ldots, 3\}$ can change the parity (evenness or oddness) of the number of $s_0$ in the expression for $\omega$.

If $I = \{1, \ldots, n-1\}$ a similar argument applies, except that $\omega \in K^+_I$ for all $I$.

The above argument leads to the following result.

**4.4. Theorem.** The $n$-polytope $\mathcal{P}(M'_I)$ is universal. However, $\mathcal{P}(M_I)$ is universal if and only if $I = \{0, \ldots, n-1\}$, or $n$ is even and either $I = \{0\}$ or $\{0, \ldots, n-1\}$. Otherwise $\mathcal{P}(M_I) \cong \mathcal{P}(M'_I)/(\omega \tau)$.

**Proof.** As noted earlier, the proof of Theorem 4.3 extends to $\mathcal{P}(M'_I)$, to show that it is universal. The facets and vertex figures of $\mathcal{P}(M'_I)$ are the same as those of $\mathcal{P}(M'_I)$, hence the latter covers the former. The covering map will be an isomorphism unless $\omega \tau \in M'_I$. If the polytope contains a 6 in the Schläfli symbol, then $\tau$, and hence also $\omega \tau$ are elements of $M'_I$. For $\mathcal{P}(M_I)$ to be universal, then, we must have $I = \{0\}$, $\{0, \ldots, n-1\}$ or $\{1, \ldots, n-1\}$. For the cube, $\omega = (s_0 \cdots s_{n-1})^n$, from which it immediately follows that if $0 \notin I$ and $n$ is odd, $\omega \in K^-_I$, so $\omega \tau \in M'_I$. To show that $M_I$ is universal in the remaining cases, it is sufficient to note that no relation of $W$ contains an odd number of elements of $I$, so $\omega$ cannot be rewritten with an odd number of symbols from $I$. □

5. Proper parabolic subgroups

In the previous sections, we have mostly concentrated on the case where $\omega \in Z(W)$. However, the construction may also, at times, be applied when $\omega$ centralises a proper parabolic subgroup $W_J$ of $W$. In fact, the Petrie operation for polyhedra is a special case of this. The Petrial of a polytope with group $\langle s_0, s_1, s_2 \rangle$ is the polytope with group $\langle s_0^\omega, s_1, s_2 \rangle$, where $\omega = s_2 \in Z(\langle s_2 \rangle)$. 
Likewise, the generalised Petrial of Eq. (2.2) of [5] is the special case \( J = n - 1 \) and \( I = n - 3 \). First we move to some preliminary results, some stated here without proof.

5.1. Lemma. If \( \omega \) centralises \( W_J \), then \( \omega \) also centralises \( W_J \), where \( J' \subseteq J \) and if \( i \pm 1 \notin J' \) then \( i \in J \).

This \( W_J \) may be seen as a maximal parabolic subgroup of \( W \) centralised by \( \omega \). The next few lemmas show, in terms of \( J \), for what \( I \) the construction presented in this article may be attempted. Since (as noted earlier) the case \( I = \emptyset \) is not interesting, we shall assume \( I \neq \emptyset \) here.

5.2. Lemma. Suppose \( \omega \) is an involution which centralises \( W_J \), but not \( W_{J'} \) for any \( J' \) with \( J \subset J' \). Then \( M_J \) will be a group generated by involutions if and only if \( I \subseteq J \) and \( \omega \neq s_i \) for any \( i \in I \).

Proof. Suppose \( M_J \) is a group generated by involutions. Let \( i \in I \). If \( s_i \omega \) is an involution, then \( s_i \neq \omega \) and \( s_i \omega = \omega s_i \), so \( i \in J \). Thus \( I \subseteq J \). Conversely, if \( I \subseteq J \) and \( s_i \neq \omega \) for any \( i \in I \), then \( s_i \omega \) is trivially an involution. \( \square \)

5.3. Lemma. Suppose \( \omega \) centralises \( W_J \), but not \( W_{J'} \) for any \( J' \) with \( J \subset J' \). Let \( I \subseteq J \), \( I \neq \emptyset \), and assume \( J \neq \{0, 1, \ldots, n-1\} \). Then \( M_J \) is a string group generated by involutions if and only if one of the following holds.

- \( J^C = \{k\} \) and \( I \subseteq \{k \pm 1\} \) and \( \omega \neq s_{k-1}, s_{k+1} \), or
- \( J^C = \{k\} \) and \( I = \{k\} \) and \( \omega \neq s_k \).

Proof. We need to check the order of \( (s_i \omega s_j \omega) \) whenever \( i \neq j \) and \( i \neq j \). If \( i, j \notin I \) the order is 2 and there is nothing to prove. Let \( i \in I \) and \( j \in J \), with \( i \neq j, j \pm 1 \). Note that \( i \in J \) also. If \( j \in I \), then \( (s_i \omega s_j \omega)^2 = (s_i s_j \omega^2)^2 = (s_i s_j)^2 = 1 \) as required. If on the other hand \( j \notin I \), then \( (s_i \omega s_j)^2 = s_i \omega s_j s_i \omega s_j = (s_i s_j)^2 \omega^2 = 1 \) also.

Now, let \( i \in I \) and \( j \notin J \), still with \( i \neq j, j \pm 1 \). In this case \( (s_i \omega s_j)^2 = s_i \omega s_j s_i \omega s_j = \omega s_i s_j s_i \omega s_j = \omega s_j \omega s_j = (\omega s_j)^2 \). This cannot be 1 since \( j \notin J \). Therefore, \( M_J \) will be a string group generated by involutions if and only if the case \( i \in I, j \notin J, i \neq j, j \pm 1 \) never occurs. This leads to the characterisation given in the statement of the theorem. \( \square \)

Assume the original polytope is non-degenerate, so that \( s_{k-1}s_k \) is never an involution. Suppose also that \( \omega = s_j \) for some \( j \). If \( j \neq 0, n-1 \), the only possibility is that \( J^C = \{j \pm 1\} \) and \( I = \{j\} \). However, \( s_j \omega \) is not an involution in this case. It follows that if \( \omega = s_j \), either \( j = 0, J = \{0, 2, 3, \ldots, n-1\} \) and \( I = \{2\} \), or \( j = n-1, J = \{0, 1, \ldots, n-3, n-1\} \), and \( I = \{n-3\} \). The traditional Petrie operation testifies that this construction does in fact work at times.

The above results characterise completely when \( M_I \) will be a string group generated by involutions. The authors regard it as beyond the scope of the article to completely characterise when \( M_I \) will be a C-group. This article therefore closes with some notes about a particular case of this construction which has proven of particular importance in the theory of thin flag regular geometries, and some statistics on how often the construction succeeds for a sample of small regular polytopes.

In [4] it was shown that there exist exactly six thin regular rank 4 geometries of the first Janko group \( J_1 \). Two of these are polytopes, one of type \( \{5, 3, 5\} \), the other of type \( \{5, 6, 5\} \). The
Table 2
Summary results of the new Petrial construction applied to “small” polytopes

<table>
<thead>
<tr>
<th></th>
<th>Rank 3</th>
<th>Rank 4</th>
<th>Rank 5</th>
<th>Rank 6</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of polytopes</td>
<td>5342</td>
<td>2513</td>
<td>325</td>
<td>2</td>
<td>8182</td>
</tr>
<tr>
<td>(</td>
<td>Z(H_0)</td>
<td>=1)</td>
<td>621</td>
<td>543</td>
<td>82</td>
</tr>
<tr>
<td>(</td>
<td>Z(H_0)</td>
<td>=2)</td>
<td>4721</td>
<td>1614</td>
<td>192</td>
</tr>
<tr>
<td>(</td>
<td>Z(H_0)</td>
<td>=4)</td>
<td>0</td>
<td>353</td>
<td>49</td>
</tr>
<tr>
<td>(</td>
<td>Z(H_0)</td>
<td>=8)</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Number of Petrials</td>
<td>4721</td>
<td>2694</td>
<td>353</td>
<td>0</td>
<td>7768</td>
</tr>
<tr>
<td>Non-polytopal Petrials</td>
<td>239</td>
<td>634</td>
<td>173</td>
<td>0</td>
<td>1046</td>
</tr>
<tr>
<td>Polytopal Petrials</td>
<td>4482</td>
<td>2060</td>
<td>180</td>
<td>0</td>
<td>6722</td>
</tr>
<tr>
<td>Self-Petrie polytopes</td>
<td>2299</td>
<td>1538</td>
<td>150</td>
<td>0</td>
<td>3987</td>
</tr>
<tr>
<td>Same size Petrials</td>
<td>2795</td>
<td>1631</td>
<td>156</td>
<td>0</td>
<td>4582</td>
</tr>
</tbody>
</table>

other geometries may be constructed from these two polytopes easily enough, using previously published methods. The importance of the Petrie-like construction given in this article was raised when it was discovered that it can be used to build the \(\{5, 6, 5\}\) from the \(\{5, 3, 5\}\). Specifically, let the \(\{5, 3, 5\}\) be given by the group \(\langle s_0, s_1, s_2, s_3 \rangle\). The polytope exists in a dual pair, one of the pair having dodecahedral facets, the other with hemidodecahedral facets—here, we assume the facets are hemidodecahedra, so that \(\omega = (s_1s_2s_3)^5\) is a central involution of \(\langle s_1, s_2, s_3 \rangle\), that is, \(J = \{1, 2, 3\}\). If we let \(I = \{1\}\) we obtain a string group generated by involutions \(\langle s_0, s_1 \omega, s_2, s_3 \rangle\). It is shown in [4] (with some difficulty) that the group is a C-group, in fact, the group of the polytope \(\{5, 6, 5\}\). Thus all the rank 4 geometries of \(J_1\) may be constructed from the \(\{5, 3, 5\}\) via various simple constructions.

The construction was also tested on a library of polytopes available to the authors, specifically, on all non-degenerate regular polytopes of rank 3 or more in [3], except for those whose groups have order \(256k\) for some \(k > 2\). For the polytopes tested, it was checked whether or not the group \(H_0 = \langle s_1, \ldots, s_{n-1} \rangle\) has a central involution \(\omega\), and for those which did, the group \(\langle s_0, s_1 \omega, s_2, \ldots, s_{n-1} \rangle\) was constructed, and tested for the intersection property. The results of this experiment are shown in Table 2. In some rank 4 and 5 cases, \(H_0\) had a centre of order 4 or even 8, leading to several possible choices for the central involution (the centre was always elementary abelian). The choice of central involution affects whether or not the construction is polytopal. This being the case, each possible choice was tested and contributed to the numerical summaries in Table 2.

The first row shows the number of polytopes tested, in each rank. The next rows show the size of the centres \(Z(H_0)\) of the groups of the vertex figure of each polytope. Since all the centres were elementary abelian, a centre of order \(n\) means there are \(n - 1\) different choices for \(\omega\), leading to, potentially, \(n - 1\) different polytopes.

The ‘Number of Petrials’ row shows how many different times the construction was in fact attempted. The next two rows shows how often the construction failed, and succeeded, to yield a well-defined polytope, that is, how many times the intersection property failed and succeeded for \(\langle s_0, s_1 \omega, s_2, \ldots, s_{n-1} \rangle\).

The construction could not be considered interesting if the constructed polytopes were always identical to the original. Thus, the number of self-Petrie polytopes were counted, and reported. Finally, the last shows the number of times the constructed polytope is the same 'size' as the original polytope, that is, the order of \(\langle s_0, s_1 \omega, s_2, \ldots, s_{n-1} \rangle\) is the same as that of \(\langle s_0, s_1, s_2, \ldots, s_{n-1} \rangle\).
It may be concluded that the Petrie-like construction analysed in this article is often a well-defined and interesting way to construct a polytope from another.

Acknowledgments

The authors would like to acknowledge the support of the “Fonds National de la Recherche Scientifique de Belgique” and the “Communauté Française de Belgique – Actions de Recherche Concertées”. They also thank Francis Buekenhout for his comments on a preliminary version of this article.

References