Modal Logics for Mereotopological Relations

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1 Introduction
   • Previous work
     • Mereotopology
     • Our logic

2 Mereotopological structures
   • Contact algebras
   • Mereotopological structures
   • Representation theorem for mereotopological structures

3 MTML: modal logic for mereotopological relations
   • Syntax and semantics of MTML
   • Axiomatization and completeness theorem
   • Decidability

4 Concluding remarks

5 Adendum: Relations to modal syllogistics
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Previous work
1. Lutz and Wolter: Modal logics for topological relations

Carsten Lutz and Frank Wolter considered modal logics for topological relations. Standard frames for these logics are families of regular closed subsets of certain topological spaces, called REGIONS, equipped with the Egenhofer and Franzosa topological relations, known also as RCC-8 relations.
RCC-8 relations

- DC(a,b)
- EC(a,b)
- PO(a,b)
- TPP(a,b)
- TPP\(^{-1}\)(a,b)
- NTPP(a,b)
- NTPP\(^{-1}\)(a,b)
- EQ(a,b)

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Modal Logics for Mereotopological Relations
NOTE: All systems considered by Lutz and Wolter are UNDECIDABLE. The presented axiomatizations are not NORMAL, because they are based on additional rules of inference.

So, our aim is to find similar modal logics, based on reasonable relations between regions, which are DECIDABLE and have NORMAL AXIOMATIZATION.
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So, our aim is to find similar modal logics, based on reasonable relations between regions, which are DECIDABLE and have NORMAL AXIOMATIZATION.
2. Wakarelov: Modal logics for set relations

Standard frames for the logic of set relations are families of sets equipped with three basic relations between sets: inclusion, nonempty intersection, and non-full union. These relations are known also as the basic mereological relations: part-of, overlap, and its dual - underlap.
Basic mereological relations

- \( a \subseteq b \) part-of
- \( a \mathbin{\text{Ob}} b \) overlap
- \( a \mathbin{\text{\textasciitilde\text{Ob}}} b \) underlap

Basic mereological relations
NOTE:

the logic of set (or mereological) relations has a complete normal axiomatization and is **DECIDABLE**. Our aim is to extend this logic, **preserving decidability**, with some relations between sets of topological nature like **contact** and some of its derivatives like **interior part-of** and **dual contact**. Such relations are studied in a discipline called **MEREOTOPOLOGY**.
MEREOTOPOLOGY = MEREOLOGY + contact-like relations

MEREOLOGY = the theory of parts and wholes.

The basic relations of mereology are part-of, overlap, and underlap. Some of the mereotopological relations are contact (corresponding to overlap), interior part-of (corresponding to part-of), and dual contact (corresponding to underlap).
Mereotopology

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Let $a, b$ be regular closed subsets of some topological space $X$.

- **contact** $\ aCb \iff a \cap b \neq \emptyset$,
- **interior part-of** $\ a \ll b \iff a \subseteq \text{Int}(b)$,
- **dual contact** $\ a\hat{C}b \iff \text{Int}(a) \cup \text{Int}(b) \neq X$. 

*Topological definitions of some mereotopological relations*
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Some history of mereology and mereotopology

- **Mereology**: Lesniewski,
  Tarski: Mereology=theory of complete Boolean algebras. Standard examples of mereological structures are Boolean algebras of sets.

- **Mereotopology**: De Laguna, Whitehead (Pointfree theory of space),

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Mereotopology and some applied areas

- QSR (Qualitative Spatial Reasoning),
- KR (Knowledge Representation),
- GIS (Geographical Information Systems) and
- FO (Formal Ontology).

Natural language semantics.
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MTML is based on the so called mereotopological structures equipped with the following 6 relations:

- **Mereological relations**
  - part-of \( x \leq y \),
  - overlap \( xo y \),
  - underlap \( xO y \),

- **Mereotopological relations**
  - interior part-of \( x \ll y \),
  - contact \( xCy \),
  - dual contact \( x\hat{C}y \).

We present a complete and normal axiomatization of MTML and prove its decidability.
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Definition

By a *Contact Algebra* (CA) we will mean any system $B = (B, C) = (B, 0, 1, .., +, *, C)$, where $(B, 0, 1, .., +, *)$ is a non-degenerate Boolean algebra with a complement denoted by “*” and $C$ – a binary relation in $B$, called *contact* and satisfying the following axioms:

- (C1) $xCy \rightarrow x, y \neq 0$,
- (C2) $xCy \rightarrow yCx$,
- (C3) $xC(y + z) \leftrightarrow xCy$ or $xCz$,
- (C4) $x.y \neq 0 \rightarrow xCy$. 
Examples of contact algebras

Examples

(1). **Topological example**: the CA of regular closed sets. Let $X$ be an arbitrary topological space. A subset $a$ of $X$ is regular closed if $a = Cl(Int(a))$, where $Cl$ and $Int$ are the standard topological closure and interior operations in $X$. The set of all regular closed subsets of $X$ will be denoted by $RC(X)$.

Facts:

1. Regular closed sets with the operations $a + b = a \cup b$, $a \cdot b = Cl(Int(a \cap b))$, $a^* = Cl(X \setminus a)$, $0 = \emptyset$ and $1 = X$ form a Boolean algebra.

2. If we define the contact by $a C_X b$ iff $a \cap b \neq \emptyset$, then $RC(X)$ with the above contact is a contact algebra.
Examples of contact algebras

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Examples of contact algebras

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Examples of contact algebras

Examples

(2) **Non-topological example**, related to Kripke frames from modal logic.

Let \((X, R)\) be a reflexive and symmetric modal frame and let \(B(X)\) be the Boolean algebra of all subsets of \(X\). Define a contact \(C_R\) for \(a, b \in B(X)\) by

\[aC_R b \text{ iff } (\exists x \in a)(\exists y \in b)(xRy)\]

Then: \(B(X)\) equipped with the contact \(C_R\) is a contact algebra.

The above example is related to Galton’s adjacency spaces and discrete mereotopology.
Relational example

Non-topological contact

A  R  B
Representation theorems for contact algebras

Theorem

1. **Topological representation.** (Dimov, Vakarelov) Every contact algebra $B$ can be isomorphically embedded into the contact algebra $RC(X)$ over some topological space $X$.

Theorem

2. **Relational representation.** (Duentsch, Vakarelov) Every contact algebra $B$ can be isomorphically embedded into the contact algebra $B(W)$ over some reflexive and symmetric frame $(W, R)$. 
Theorem

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Mereotopological relations in contact algebras

We define the following relations in a given contact algebra $B$:

**Definition**

Mereological relations

- $xOy$ iff $x \cdot y \neq 0$, overlap
- $x\tilde{O}y$ iff $x^* \cdot y^* \neq 0$ iff $x + y \neq 1$, underlap, dual overlap,
- $x \leq y$ iff $x \cdot y^* = 0$, part-of, Boolean ordering.

Mereotopological relations

- $xCy$, contact,
- $x\tilde{C}y$ iff $x^* Cy^*$, dual contact
- $x \ll y$ iff $x\tilde{C}y^*$, interior part-of.
We define the following relations in a given contact algebra $B$:

**Definition**

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**Mereotopological relations**

- $xCy$, contact,
- $x\hat{C}y \iff x^* Cy^*$, dual contact
- $x \ll y \iff x\overline{Cy}^*$, interior part-of.
Lemma

1. The following 13 conditions are true for the mereological relations in any Boolean algebra $B$:

- $(\leq 0)$ $a \leq b$ and $b \leq a \rightarrow a = b$,
- $(\leq 1)$ $a \leq a$, $(\leq 2)$ $a \leq b$ and $b \leq c \rightarrow a \leq c$,
- $(O1)$ $aOb \rightarrow bOa$, $(O2)$ $aOb \rightarrow aOa$,
- $(O \leq)$ $aOb$ and $b \leq c \rightarrow aOc$, $(\bigO \leq)$ $a\bigO a \rightarrow a \leq b$,
- $(\hat{O}1)$ $a\hat{O}b \rightarrow b\hat{O}a$, $(\hat{O}2)$ $a\hat{O}b \rightarrow a\hat{O}a$,
- $(\hat{O} \leq)$ $c \leq a$ and $a\hat{O}b \rightarrow c\hat{O}b$, $(\bigO \leq)$ $b\bigO b \rightarrow a \leq b$,
- $(O\hat{O})$ $aOa$ or $a\hat{O}a$,
- $(\leq O\hat{O})$ $c\hat{O}a$ and $c\bigO b \rightarrow a \leq b$, 
Lemma

2. The following 17 conditions are true for the mereotopological relations in any contact Boolean algebra $B$:

- $(C) \ aCb \rightarrow bCa$, $(\hat{C}) \ a\hat{C}b \rightarrow b\hat{C}a$,
- $(CO1) \ aOb \rightarrow aCb$, $(\hat{C}O1) \ a\hat{O}b \rightarrow a\hat{C}b$,
- $(CO2) \ aCb \rightarrow aOa$, $(\hat{C}O2) \ a\hat{C}b \rightarrow a\hat{O}a$,
- $(\leq) \ aCb \text{ and } b \leq c \rightarrow aCc$, $(\hat{C} \leq) \ a\hat{C}b \text{ and } c \leq b \rightarrow a\hat{C}c$,
- $(\ll \leq 1) \ a \ll b \rightarrow a \leq b$, $(\ll \leq 2) \ a \leq b \text{ and } b \ll c \rightarrow a \ll c$,
- $(\ll \ll 3) \ a \ll b \text{ and } b \leq c \rightarrow a \ll c$, $(\ll O) \ a\hat{O}a \rightarrow a \ll b$,
- $(\ll \hat{O}) \ b\hat{O}b \rightarrow a \ll b$, $(\ll CO) \ aCb \text{ and } b \ll c \rightarrow aOc$,
- $(\ll \hat{C}O) \ c \ll a \text{ and } a\hat{C}b \rightarrow c\hat{O}b$,
- $(\ll \hat{C}O) \ c\hat{C}a \text{ and } c\hat{O}b \rightarrow a \ll b$. 
Definition of mereotopological structures

The relational system \( W = (W, \leq, O, \hat{O}, C, \hat{C}, \ll) \) is called:

- **mereotopological structure** if it satisfies the first-order conditions of Lemma 1 and Lemma 2;
- The fragment \( W = (W, \leq, O, \hat{O}) \) is called a **mereological structure**.
- **standard mereotopological structure** if there exists a contact algebra \( (B, C) \) such that \( W \subseteq B \) and the relations \( \leq, O, \hat{O}, C, \hat{C}, \ll \) have their standard definitions in \( (B, C) \);
- **relational mereotopological structure** if the algebra \( (B, C) \) is as in the relational example;
- **completely standard** if the algebra \( (B, C) \) is as in the topological example.
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Modal Logics for Mereotopological Relations
Theorem

1. Every mereotopological structure $\mathcal{W}$ can be isomorphically embedded into a relational mereotopological structure, hence $\mathcal{W}$ is isomorphic to a standard one.

Theorem

2. Every mereotopological structure is isomorphic to a completely standard one.

Proof.

Theorem 2 follows from theorem 1 and the topological representation theorem for contact algebras. Theorem 1 is proved by using Stone like techniques, generalizing the representation theory of distributive lattices.
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Filters and ideals in mereological structures - examples of facts

Definition

Let $\mathcal{W}$ be a mereological structure and $A$ be a subset of $\mathcal{W}$.

- $A$ is called a $\leq$-set if $(\forall x, y \in \mathcal{W})(x \in A \text{ and } x \leq y \rightarrow y \in A)$,
- $A$ is called a $\geq$-set if $(\forall x, y \in \mathcal{W})(x \in A \text{ and } x \geq y \rightarrow y \in A)$,
- $A$ is a filter if $A$ is a $\leq$-set and $(\forall x, y \in A)(xOy)$,
- $A$ is an ideal if $A$ is a $\geq$-set and $(\forall x, y \in A)(x\tilde{O}y)$,
- $A$ is a good filter if $A$ is a filter and $(\forall x, y \notin A)(x\tilde{O}y)$,
- $A$ is a good ideal if $A$ is an ideal and $(\forall x, y \notin A)(xOy)$.
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Filters and ideals in mereological structures - examples of facts

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Separation Lemma for filters and ideals

**Lemma**

Let $\mathcal{W} = (\mathcal{W}, \leq, O, \hat{O})$ be a mereological structure and

1. $F_0$ be a filter in $\mathcal{W}$,
2. $I_0$ be an ideal in $\mathcal{W}$,
3. and $F_0 \cap I_0 = \emptyset$.

Then there is a good filter $F$ and a good ideal $I$ such that

1. $F_0 \subseteq F$,
2. $I_0 \subseteq I$, and
3. $F \cap I = \emptyset$. 

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Modal Logics for Mereotopological Relations
Outline

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   - Previous work
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   - Our logic

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3. MTML: modal logic for mereotopological relations
   - Syntax and semantics of MTML
   - Axiomatization and completeness theorem
   - Decidability

4. Concluding remarks

5. Addendum: Relations to modal syllogistics

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Modal Logics for Mereotopological Relations
Syntax and semantics of MTML (Mereotopological Modal Logic)

Syntax

MTML has the following modal box operators: $[\leq], [\geq], [\ll], [\gg], [O], [\hat{O}], [C], [\hat{C}], [U]$, where $[U]$ is the universal modality. The corresponding diamond modality is denoted by $\langle R \rangle$ and defined as $\neg [\neg R]$.

Semantics

The standard semantics of MTML is the Kripke semantic over mereotopological structures. Notation: If $\nu$ is a valuation in $W$, then $A$ is true (false) at $x \in W$ will be denoted by $\nu(x, A) = 1$ ($\nu(x, A) = 0$). We adopt the standard truth definitions.
Syntax and semantics of MTML (Mereotopological Modal Logic)

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Semantics

The standard semantics of MTML is the Kripke semantic over mereotopological structures. Notation: If $v$ is a valuation in $W$, then $A$ is true (false) at $x \in W$ will be denoted by $v(x, A) = 1$ ($v(x, A) = 0$). We adopt the standard truth definitions.
Fact: The condition $(\leq 0)$ $a \leq b \& b \leq a \rightarrow a = b$ is modally undefinable.

So, we introduce nonstandard semantics for MTML replacing $(\leq 0)$ by the following modally definable consequences of it:

1. $(=1)$ $\overline{aO}a \& b \leq a \rightarrow a = b$,
2. $(=2)$ $\overline{aO}a \& a \leq b \rightarrow a = b$,
3. $(=3)$ $\overline{aO}c \& \overline{bO}c \& b \leq a \rightarrow a = b$. 
Equivalence of standard and nonstandard semantics of MTML

Lemma

Bulldozer Lemma.

Let $W$ be a nonstandard mereotopological structure. Then there exists a mereotopological structure $W'$ and a p-morphism $f$ from $W'$ to $W$.

Corollary

Standard and nonstandard semantics of MTML are equivalent.

Proof.

By a generalization of Segerberg’s Bulldozing construction.
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Modal Logics for Mereotopological Relations
Axiomatization of MTML

The axiomatics of the minimal polymodal logic

\((\text{Bool})\) All Boolean tautologies

\((K)\) \( [R](A \Rightarrow B) \Rightarrow ([R]A \Rightarrow [R]B) \),

Axioms for converse relations \( \langle \leq \rangle [\geq] A \Rightarrow A, \langle \geq \rangle [\leq] A \Rightarrow A, \langle \ll \rangle [\gg] A \Rightarrow A, \langle \gg \rangle [\ll] A \Rightarrow A, \)


Rules of inference Modus Ponens \( \text{MP} \ \frac{A, A \Rightarrow B}{B} \) and

Necessitation \( \text{N} \ \frac{A}{[R]A} \)

The additional axioms are Sahlqvist’s equivalents of the conditions of nonstandard semantics

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Modal Logics for Mereotopological Relations
Additional axioms

\[(A_{\leq 1}) \quad [\leq]A \Rightarrow A, \quad (A_{\leq 2}) \quad [\leq]A \Rightarrow [\leq][\leq]A, \quad (A_{O1})\]
\[\langle O \rangle[O]A \Rightarrow A,\]
\[(A_{\hat{O}1}) \quad \langle \hat{O} \rangle[\hat{O}]A \Rightarrow A, \quad (A_{O\leq}) \quad [O]A \Rightarrow [O][\leq]A,\]
\[(A_{\hat{O}\leq}) \quad [\hat{O}]A \Rightarrow [\hat{O}][\geq]A, \quad (A_{O\hat{O}}) \quad ([O]A \Rightarrow A) \lor ([\hat{O}]B \Rightarrow B),\]
\[(A_{\leq O\hat{O}}) \quad [O]A \land [\hat{O}]B \land \langle U \rangle([\leq]C \land \neg A) \Rightarrow [U](B \lor C),\]
\[(A_C) \quad \langle C \rangle[C]A \Rightarrow A, \quad (A_{\hat{C}}) \quad \langle \hat{C} \rangle[\hat{C}]A \Rightarrow A,\]
\[(A_{CO1}) \quad [C]A \Rightarrow [O]A, \quad (A_{\hat{C}O1}) \quad [\hat{C}]A \Rightarrow [\hat{O}]A,\]
\[(A_{CO2}) \quad \langle C \rangle\top \land [O]A \Rightarrow A, \quad (A_{\hat{C}O2}) \quad \langle \hat{C} \rangle\top \land [\hat{O}]A \Rightarrow A,\]
\[(A_{C\leq}) \quad [C]A \Rightarrow [C][\leq]A, \quad (A_{\hat{C}\leq}) \quad [\hat{C}]A \Rightarrow [\hat{C}][\geq]A,\]
\[(A_{\ll\leq 1}) \quad [\ll]A \Rightarrow [\ll\ll]A, \quad (A_{\ll\leq 2}) \quad [\ll\ll]A \Rightarrow [\leq][\ll\ll]A,\]
\[(A_{\ll\leq 3}) \quad [\ll\ll]A \Rightarrow [\ll\ll][\leq]A,\]
Additional axioms

\[(A ≪ O) \quad \neg A \land [O]A \land [≪]B \Rightarrow [U]B,\]
\[(A ≪ \hat{O}) \quad \neg A \land [\hat{O}]A \land [≫]B \Rightarrow [U]B,\]
\[(A ≪ CO) \quad [O]A \Rightarrow [C][≪]A, \quad (A ≪ \hat{C}O) \quad [\hat{O}]A \Rightarrow [\hat{C}][≫]A,\]
\[(A ≪ C\hat{O}) \quad [C]A \land [\hat{O}]B \land \langle U \rangle([≪]C \land \neg A) \Rightarrow [U](B \lor C),\]
\[(A ≪ \hat{C}O) \quad [O]A \land [\hat{C}]B \land \langle U \rangle([≪]C \land \neg A) \Rightarrow [U](B \lor C),\]
\[(A ≡_1) \quad \langle \leq \rangle([O]A \land \neg A \land B) \Rightarrow B, \quad (A ≡_2)\]
\[\langle \geq \rangle([\hat{O}]A \land \neg A \land B) \Rightarrow B,\]
\[(A ≡_3) \quad \langle U \rangle(B \land \neg C \land \langle \leq \rangle(A \land C)) \Rightarrow \langle O \rangle A \lor \langle \hat{O} \rangle B.\]
The following conditions are equivalent for any formula $A$:

(i) $A$ is a theorem of MTML,
(ii) $A$ is true in all non-standard mereotopological structures,
(iii) $A$ is true in all mereotopological structures,
(iv) $A$ is true in all standard and completely standard mereotopological structures.

Proof.

(i)$\leftrightarrow$(ii) – by generated canonical models.
(ii)$\leftrightarrow$(iii) – by the Bulldozer Lemma,
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Completeness theorem of MTML

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Yavor Nenov and D. Vakarelov
Modal Logics for Mereotopological Relations
Lemma

MTML do not possess fmp with respect to its standard semantics.

Proof.

Grzegorczyk formula

$$\leq ([\leq](p \Rightarrow [\leq]p) \Rightarrow p) \Rightarrow p$$

is true in all finite mereotopological structures (because they are finite partial orderings with respect to $\leq$) but that it is falsified in the generalized mereotopological structure $W = \{a, b\}$ in which the relations $\leq, O, \hat{O}, C, \hat{C}$ and $\ll$ coincide with $W^2$. 
Filtration of non-standard structures

**Theorem**

*MTML admits filtration with respect to its non-standard semantics and hence is decidable.*

**NOTE:** The filtration construction is quite complicate, because of the great number of interactive modalities. Part of the construction in the next slide.
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**NOTE:** The filtration construction is quite complicate, because of the great number of interactive modalities. Part of the construction in the next slide.
Example: the filtration construction for $\leq$

\[ |x| \leq' |y| \iff (\forall [\leq] A \in \Gamma) ((v(x, [\leq] A) = 1 \rightarrow v(y, [\leq] A) = 1) \& (v(y, [\geq] A) = 1 \rightarrow v(x, [\geq] A) = 1) \& (v(x, [\ll] A) = 1 \rightarrow v(y, [\ll] A) = 1) \& (v(y, [\gg] A) = 1 \rightarrow v(x, [\gg] A) = 1) \& (v(y, [O] A) = 1 \rightarrow v(x, [O] A) = 1) \& (v(x, [\hat{O}] A) = 1 \rightarrow v(y, [\hat{O}] A) = 1) \& (v(y, [C] A) = 1 \rightarrow v(x, [C] A) = 1) \& (v(x, [\hat{C}] A) = 1 \rightarrow v(y, [\hat{C}] A) = 1) \& (v(x, [O] \top) = 1 \rightarrow v(y, [O] \top) = 1) \& (v(y, [\hat{O}] \top) = 1 \rightarrow v(x, [\hat{O}] \top) = 1)) \]
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Summary

1. Abstract axioms for mereotopological structures and relational and topological representation theorem. Mereotopological structures have also an independent value as a first-order logic of some mereotopological relations free of the Boolean operations.

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Concluding remarks

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Open problems

1. Complexity results for MTML,

2. The axiom \((Con)\) \(aOb \land bOb \rightarrow a\hat{O}b \lor aCb\) is true in all models over connected topological spaces and is falsified for some non-connected spaces. Extend MTML for connected models and for models over Euclidean spaces.
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Mereotopological structures as a modal syllogistics

Let \((W, R)\) be a reflexive and symmetric frame and for \(a \subseteq W\) define the modal operations
\[
[R]a = \{x \in W : (\forall y \in W)(xRy \rightarrow y \in a)\}
\]
and
\[
\langle R \rangle a = \neg[R] \neg a.
\]
Then the mereotopological relations over \((W, R)\) have the following form, which suggests the corresponding syllogistic reading:

- \(a \leq b \iff a \subseteq b\), All \(a\) are \(b\),
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Thank you