ZEROS OF CLASSICAL ORTHOGONAL POLYNOMIALS
OF A DISCRETE VARIABLE

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Abstract. In this paper we obtain sharp bounds for the zeros of classical orthogonal polynomials of a discrete variable, considered as functions of a parameter, by using a theorem of A. Markov and the so-called Hellmann-Feynman theorem. Comparisons with previous results for zeros of Hahn, Meixner, Kravchuk and Charlier polynomials are also presented.

1. Introduction

The behavior of zeros of the classical continuous orthogonal polynomials has been studied extensively, mainly because of their beautiful electrostatic interpretation and their important role as nodes of Gaussian quadrature formulae. The behavior of zeros of classical orthogonal polynomials of a discrete variable is of interest in many applications ranging from the least-squares method of approximation, queueing theory, lengths of weakly increasing sub-sequences of random words, totally asymmetric simple exclusion process to cross-directional control on paper machines. Despite the fact that Chebyshev emphasized the importance of the zeros of orthogonal polynomials of a discrete variable in 1855, results on monotonicity and limits of their zeros have been obtained only recently in. In the present paper we establish very sharp estimates for these zeros.

Since most of the classical orthogonal polynomials depend on parameters, one of the natural problems which arises is to study the zeros of these polynomials, considered as functions of the parameter. A natural tool to do so, is to employ a beautiful result of Andrei Markov which says that the monotonicity of the zeros of these polynomials with respect to the parameter depends on the monotonicity of the logarithmic derivative of the weight function, with respect to the parameter, considered as a function of the variable in the interval of orthogonality. Another powerful tool to investigate the monotonic dependence of zeros of orthogonal polynomials is the so-called Hellmann-Feynman theorem which provides information about the behavior of the eigenvalues of the Jacobi matrix associated with the orthogonal polynomials, and these, as it is well known,
coincide with the zeros of the polynomials. In this paper we make extensive use of these powerful methods to investigate the behavior of the zeros of four families of classical orthogonal polynomials of a discrete variable, those of Charlier, Meixner, Kravchuk and Hahn. This allows us to provide new and sharp bounds for their zeros.

We adopt the definitions and notations of these families of classical orthogonal polynomials of a discrete variable given in [23]:

- **Charlier polynomials**
  \[ C_n(x; a) = {}_2F_0 \left( \begin{array}{c} -n, -x \\ -a \end{array} \right), \]
  whose orthogonality property is
  \[ \sum_{k=0}^{\infty} \frac{a^k}{k!} C_n(k; a)C_m(k; a) = a^{-n}e^a n! \delta_{mn}, \quad a > 0. \]

- **Meixner polynomials**
  \[ M_n(x; \beta, c) = {}_2F_1 \left( \begin{array}{c} -n, -x \\ \beta \end{array} \right), \]
  whose orthogonality property is
  \[ \sum_{k=0}^{\infty} \frac{(-\beta)_k}{k!} M_n(k; \beta, c)M_m(k; \beta, c) = \frac{c^{-r}r!}{(\beta)^r(1-c)\beta} \delta_{mn}, \quad \beta > 0, \quad 0 < c < 1. \]

- **Kravchuk polynomials**, defined by
  \[ K_n(x; p, N) = {}_2F_1 \left( \begin{array}{c} -n, -x \\ -N, \frac{1}{p} \end{array} \right), \quad n = 0, 1, \ldots, N, \]
  whose orthogonality property is
  \[ \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} K_n(k; p, N)K_m(k; p, N) \]
  \[ = \frac{(-1)^n n!}{(-N)_n} \left( \frac{1-p}{p} \right)^n \delta_{mn}, \quad 0 < p < 1. \]

- **Hahn polynomials**
  \[ Q_n(x; \alpha, \beta, N) = {}_3F_2 \left( \begin{array}{c} -n, -x, n + \alpha + \beta + 1 \\ -N, \alpha + 1 \end{array} \right), \quad n = 0, 1, \ldots, N, \]
  whose orthogonality property is
  \[ \sum_{k=0}^{N} \binom{\alpha+k}{k} \binom{\beta+N-k}{N-k} Q_n(k; \alpha, \beta, N)Q_m(k; \alpha, \beta, N) \]
  \[ = \frac{(-1)^n(n+\alpha+\beta+1)_n(\beta+1)n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_nN!} \delta_{nm}, \quad \alpha > -1, \quad \beta > -1. \]

The paper is organized as follows. In Section 2 we furnish the necessary background and theoretical results which will be employed in the investigation. The remaining sections are devoted to more specific and precise results concerning monotonicity and sharp limits for the zeros of the four aforementioned families.
of orthogonal polynomials. The analysis starts with Charlier and Meixner orthogonal polynomials, for which the support of orthogonality is unbounded. In these two cases we provide bounds for all the zeros, and we compare our results with the best known limits in Sections 3 and 4, respectively. Moreover, results on the extreme zeros of Kravchuk and Hahn polynomials are also provided in Sections 5 and 6. We also compare the results obtained in this paper with other limits, exemplifying in several tables the sharpness of our results.

2. Basic tools

Let \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) be the set of nonnegative integers and let \( \mathcal{N} \) be \( \mathbb{N}_0 \) or a finite subset of the form \( \{0, 1, \ldots, N\} \). Let \( \{p_n(x, \tau)\}_{n=0}^\infty \) be a parametric sequence of orthonormal polynomials of a discrete variable, with respect to the positive weight function \( \omega(x; \tau) \), for \( \tau \in (\tau_1, \tau_2) \). That is, for every \( n \in \mathcal{N} \) and for every \( \tau \in (\tau_1, \tau_2) \), \( p_n(x, \tau) \) is a polynomial of degree exactly \( n \) and

\[
\sum_{k \in \mathcal{N}} \omega(k; \tau)p_n(k, \tau)p_m(k, \tau) = \delta_{nm} \quad \text{whenever } m, n \in \mathcal{N}.
\]

It suffices that the function \( \omega(x; \tau) \) is defined only in \( \mathcal{N} \) but in most cases \( \omega \) is well defined for all real values of \( x \) in the convex hull \( \mathcal{K} \) of \( \mathcal{N} \). In what follows we shall suppose that \( \omega \) obeys this property. It is well known that every \( p_n(x, \tau) \) has \( n \) real distinct zeros which belong to \( \mathcal{K} \) \( \mathbb{R} \).

We begin with the first tool we shall apply in our studies. It is probably the first result ever proved which deals with monotonicity of zeros of orthogonal polynomials and was established by Andrey Markov in 1886. For a more general version of Markov’s Theorem we refer the reader to Ismail’s book [17, Theorem 7.1.1, p. 204]. An idea for a completely different proof, without use of Gaussian quadrature, which also allows a slight generalization, can be found in [10]. Here we state the result in a form appropriate for our needs in this paper.

**Theorem A.** For every \( \tau \in (\tau_1, \tau_2) \), let the polynomials \( \{p_n(x, \tau)\}_{n=0}^\infty \) obey the orthogonality relation (2.1) with respect to the positive weight function \( \omega(x; \tau) \) and assume that it has continuous first derivative \( \omega_\tau(x; \tau) \) with respect to \( \tau \), for every \( \tau \in (\tau_1, \tau_2) \) and \( x \in \mathcal{K} \). Assume further that the series

\[
\sum_{k \in \mathcal{N}} k^j \omega_\tau(k; \tau), \quad j = 0, \ldots, 2n - 1,
\]

converge uniformly for \( \tau \) in every compact subset of \( (\tau_1, \tau_2) \). Then the zeros of \( p_n(x, \tau) \) are increasing (decreasing) functions of \( \tau \in (\tau_1, \tau_2) \) if \( \partial(\log \omega(x; \tau))/\partial \tau \) is an increasing (decreasing) function of \( x \) in \( \mathcal{K} \).

Consider the parametric sequence \( \{s_k(x; \tau)\}_{k=0}^\infty \) of orthogonal polynomials. It is well known that it satisfies the three term recurrence relation

\[
s_0(x; \tau) = 1, \quad s_1(x, \tau) = (x - \beta_0(\tau))/\gamma_0(\tau)
\]

\[
x s_k(x; \tau) = \gamma_k(\tau)s_{k+1}(x; \tau) + \beta_k(\tau)s_k(x; \tau) + \delta_k(\tau)s_{k-1}(x; \tau), \quad k = 1, 2, \ldots
\]

with \( \gamma_k(\tau)\delta_{k+1}(\tau) > 0 \).
We associate with this recurrence relation the \( n \times n \) matrix
\[
H_n(\tau) = \begin{pmatrix}
\beta_0(\tau) & \gamma_0(\tau) & 0 & 0 & \ldots & 0 \\
\delta_1(\tau) & \beta_1(\tau) & \gamma_1(\tau) & 0 & \ldots & 0 \\
0 & \delta_2(\tau) & \beta_2(\tau) & \gamma_2(\tau) & \ldots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \delta_{n-2}(\tau) & \beta_{n-2}(\tau) & \gamma_{n-2}(\tau) \\
0 & 0 & 0 & 0 & \delta_{n-1}(\tau) & \beta_{n-1}(\tau)
\end{pmatrix}.
\]

However, we can obtain a symmetric matrix when we consider the corresponding sequence \( \{p_k(x, \tau)\}_{k=0}^\infty \) of parametric orthonormal polynomials which are given by
\[
p_0(x, \tau) = 1, \quad p_k(x, \tau) = \left( \frac{\gamma_0 \gamma_1 \cdots \gamma_{k-1}}{\delta_1 \delta_2 \cdots \delta_k} \right)^{1/2} s_k(x, \tau)
\]
and they satisfy the three term recurrence relation
\[
(2.2) \quad xp_k(x, \tau) = a_k(\tau)p_{k+1}(x, \tau) + b_k(\tau)p_k(x, \tau) + a_{k-1}(\tau)p_{k-1}(x, \tau),
\]
where
\[
a_k(\tau) = \sqrt{\gamma_k(\tau)\delta_{k+1}(\tau)} \quad \text{and} \quad b_k(\tau) = \beta_k(\tau) \quad k \geq 0.
\]
Observe that the zeros of the polynomial \( p_n(x, \tau) \) coincide with those of \( s_n(x, \tau) \) and they are also the eigenvalues of the Jacobi matrix
\[
(2.3) \quad J_n = J_n(\tau) = \begin{pmatrix}
b_0(\tau) & a_0(\tau) & 0 & 0 & \ldots & 0 \\
a_0(\tau) & b_1(\tau) & a_1(\tau) & 0 & \ldots & 0 \\
0 & a_1(\tau) & b_2(\tau) & a_2(\tau) & \ldots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & a_{n-3}(\tau) & b_{n-2}(\tau) & a_{n-2}(\tau) \\
0 & 0 & 0 & 0 & a_{n-2}(\tau) & b_{n-1}(\tau)
\end{pmatrix}.
\]

Moreover, if \( \lambda_j = \lambda_j(\tau) \) is a zero of \( p_n(x, \tau) \) and
\[
\mathcal{P}_j = (p_0(\lambda_j; \tau), p_1(\lambda_j; \tau), \ldots, p_{n-1}(\lambda_j; \tau))^T,
\]
then
\[J_n \mathcal{P}_j = \lambda_j \mathcal{P}_j.
\]

Let us denote by \( J'_n = J'_n(\tau) \) the Jacobi matrix whose entries are the derivatives of the corresponding entries of \( J_n(\tau) \). We shall formulate the so-called Hellmann-Feynman theorem (see [17]) in a form which will be convenient for our needs.

**Theorem B.** For every zero \( \lambda_j(\tau) \) of \( p_n(x, \tau) \) we have
\[
\lambda'_j(\tau) = \frac{\mathcal{P}_j^T J'_n \mathcal{P}_j}{\mathcal{P}_j^T \mathcal{P}_j} = \frac{\sum_{k=0}^{n-1} b'_k p_k^2(\lambda_j; \tau) + 2 \sum_{k=0}^{n-2} a'_k p_k(\lambda_j; \tau) p_{k+1}(\lambda_j; \tau)}{\sum_{k=0}^{n-1} p_k^2(\lambda_j; \tau)}.
\]

Furthermore, if the sum in the numerator in the latter expression is positive, then the zeros \( \lambda_j(\tau) \) of \( p_n(x, \tau) \) are increasing functions of \( \tau \). This happens if \( J'_n \) is a positive definite matrix.

We need also the following corollary of the Perron-Frobenius theorem [16] Theorems 8.4.4 and 8.4.5]:
Theorem C. Let $H_n(\tau)$ be an $n \times n$ tridiagonal matrix with positive off-diagonal elements. If the entries of $H_n(\tau)$ are increasing functions of $\tau$, then the largest eigenvalue of $H_n(\tau)$ is an increasing function of $\tau$.

It is a well-known property of zeros of orthogonal polynomials with respect to a positive Borel measure that they are all real, distinct, belong to the convex hull of the support of the measure with respect to which the polynomials are orthogonal. Let the zeros of $C_n(x; a)$, $M_n(x; \beta, c)$, $K_n(x; p, N)$ and $Q_n(x; \alpha, \beta, N)$ be denoted by $c_{n,j}(a)$, $m_{n,j}(\beta, c)$, $\kappa_{n,j}(p, N)$ and $q_{n,j}(\alpha, \beta, N)$, respectively, all arranged in decreasing order, so that

$$0 < c_{n,n}(a) < c_{n,n-1}(a) < \cdots < c_{n,1}(a) < \infty, \quad a > 0,$$

$$0 < m_{n,n}(\beta, c) < m_{n,n-1}(\beta, c) < \cdots < m_{n,1}(\beta, c) < \infty, \quad \beta > 0, \quad c \in (0, 1),$$

$$0 < \kappa_{n,n}(p, N) < \kappa_{n,n-1}(p, N) < \cdots < \kappa_{n,1}(p, N) < N, \quad p \in (0, 1), \quad n \leq N,$$

and for $\alpha, \beta > -1$, $n \leq N$,

$$0 < q_{n,n}(\alpha, \beta, N) < q_{n,n-1}(\alpha, \beta, N) < \cdots < q_{n,1}(\alpha, \beta, N) < N.$$

Moreover, the zeros of any pair of orthogonal polynomials of consecutive degrees interlace. The following theorem summarizes known properties of the zeros of classical orthogonal polynomials of a discrete variable.

Theorem 2.1. Let $\{G_n(x)\}$ be any of the four sequences of classical orthogonal polynomials of a discrete variable (Charlier, Meixner, Kravchuk and Hahn). Then, for any admissible values of the parameters, the distance between every two consecutive zeros of $G_n(x)$ is greater than one, so that the zeros of $G_n(x)$ interlace with those of $G_n(x + 1)$ and $G_n(x - 1)$. Moreover, for every $n, j \in \mathbb{N}$ with $1 \leq j \leq n$,

- $c_{n,j}(a)$ are increasing functions of $a$, for $a \in (0, \infty)$;
- $m_{n,j}(\beta, c)$ are increasing functions of both $\beta \in (0, \infty)$ and $c \in (0, 1)$;
- $\kappa_{n,j}(p, N)$ are increasing functions of the parameter $p \in (0, 1)$ and increasing functions of $N$ in the sense that $\kappa_{n,j}(p, N) < \kappa_{n,j}(p, N + 1)$ whenever $N \geq n$.
- $q_{n,j}(\alpha, \beta, N)$ are increasing functions of $\alpha \in (-1, \infty)$, decreasing functions of $\beta \in (-1, \infty)$ and increasing functions of $N$, that is, $q_{n,j}(\alpha, \beta, N) < q_{n,j}(\alpha, \beta, N + 1)$ provided $N \geq n$.

Proof. The first statements of the theorem follow immediately from a nice result of Krasikov and Zarkh [28, Theorem 1]. It states that if the real polynomial $p(x)$, with only distinct real zeros $a < \zeta_1 < \zeta_2 < \cdots < \zeta_n < b$ satisfies the difference equation

$$p(x + 1) = 2A(x)p(x) - B(x)p(x - 1)$$

and $B(x) > 0$ for $x \in (a, b)$, then $\zeta_{j+1} - \zeta_j > 1$ for every $j = 1, \ldots, n - 1$. The classical orthogonal polynomials of a discrete variable are solutions of difference equations of the above form (see formulae (1.12.5), (1.9.5), (1.10.5) and (1.5.5) in [28]). It is straightforward to verify that the corresponding coefficients $B(x)$ are all positive in the convex hulls of the set of orthogonality of the corresponding $G_n(x)$.

The monotonicity of the zeros of Charlier, Meixner, Kravchuk and Hahn polynomials with respect to the parameters $a, p, \beta, c, \alpha$ and $\beta$, can be proved via Theorem A by computing the logarithmic derivative of the weight function and establishing its monotonicity with respect to the variable. We omit the straightforward technical details.
The monotonicity of the zeros $\kappa_{n,j}(p,N)$ of Kravchuk’s polynomials with respect to $N$, that is, the inequality $\kappa_{n,j}(p,N) < \kappa_{n,j}(p,N + 1)$, was established by L. Chihara and D. Stanton [8]. The monotonicity of the zeros of Hahn polynomials $q_{n,j}(\alpha,\beta,N)$ with respect to $N$ have been proved in [30, Theorem 6].

Below we illustrate the dependence of the zeros of the classical orthogonal polynomials of a discrete variable on the parameters. The figures show the zeros of fifth degree polynomials as functions of the corresponding parameters.

![Figure 2.1. Zeros of $C_5(x; a)$.](image1)

![Figure 2.2. Zeros of $M_5(x; \beta, 0.7)$.](image2)

![Figure 2.3. Zeros of $M_5(x; 10, c)$.](image3)

![Figure 2.4. Zeros of $K_5(x; p, 30)$.](image4)

![Figure 2.5. Zeros of $K_5(x; 0.7, N)$.](image5)

![Figure 2.6. Zeros of $Q_5(x; 10, 2, N)$.](image6)
Finally, let us mention that in the present paper we obtain bounds for the zeros of classical orthogonal polynomials of a discrete variable by using appropriate limit relations with classical orthogonal polynomials. Thus, in what follows we use the following standard definitions and notations of the classical orthogonal polynomials of Jacobi, Laguerre and Hermite [23]:

\[ P_n^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)_n}{n!} \binom{-n, n + \alpha + \beta + 1}{\alpha + 1} \frac{1 - x}{2}, \]

\[ L_n^{(\alpha)}(x) = \frac{(\alpha + 1)_n}{n!} \binom{-n}{\alpha + 1} x, \]

\[ H_n(x) = (2x)^n \binom{-n/2, -(n - 1)/2}{-1/2} \left( -\frac{1}{x^2} \right). \]

Their zeros are denoted by \( x_{n,j}^{(\alpha,\beta)}, x_{n,j}^{(\alpha)} \) and \( h_{n,j}, j = 1, \ldots, n \), respectively, and are supposed to be arranged in decreasing order. In other words, we have

\[-1 < x_{n,n}^{(\alpha,\beta)} < \cdots < x_{n,1}^{(\alpha,\beta)} < 1, \]
\[0 < x_{n,n}^{(\alpha)} < \cdots < x_{n,1}^{(\alpha)}, \]

and
\[ h_{n,n} < \cdots < 0 < h_{n,[n/2]} < \cdots < h_{n,1}. \]

3. CHARLIER POLYNOMIALS

Motivated by the limit relation [23] (2.12.1]

\[ \lim_{a \to \infty} (2a)^{\frac{1}{2}} C_n(a + x\sqrt{2a}; a) = (-1)^n H_n(x), \]

which implies
\[ (c_{n,j}(a) - a) / \sqrt{2a} \to h_{n,j} \text{ as } a \to \infty \]

(3.1)

and
\[ (c_{n,j}(a) - a - n + 1) / \sqrt{2a} \to h_{n,j} \text{ as } a \to \infty, \]

(3.2)

we obtain a result concerning the monotonicity, with respect to \( a \), of the quantities that appear on the left-hand sides of the latter two limit relations.

**Theorem 3.1.** Let \( n \in \mathbb{N} \) and \( a > 0 \). Then, for every \( j, 1 \leq j \leq n \), the functions \( (c_{n,j}(a) - a) / \sqrt{2a} \) decrease and \( (c_{n,j}(a) - a - n + 1) / \sqrt{2a} \) increase for \( a \in (0, \infty) \). Moreover, the inequalities

\[ a + \sqrt{2a} h_{n,j} \leq c_{n,j}(a) \leq a + n + \sqrt{2a} h_{n,j}, \]

hold for \( j = 1, \ldots, n \).
Proof. Since the polynomials satisfy the three term recurrence relation
\[-xC_n(x; a) = aC_{n+1}(x; a) - (n + a)C_n(x; a) + nC_{n-1}(x; a),\]
then the polynomials defined by \( \tilde{C}_n(x; a) = (-1)^n(2a)^{\frac{n}{2}}C_n(a + x\sqrt{2a}; a) \), whose zeros are \( (c_{n,j}(a) - a)/\sqrt{2a} \), satisfy
\[x\tilde{C}_n(x; a) = \frac{1}{2}\tilde{C}_{n+1}(x; a) + \frac{n}{\sqrt{2a}}\tilde{C}_n(x; a) + n\tilde{C}_{n-1}(x; a),\]
so that the corresponding orthonormal polynomials satisfy a recurrence relation of the form \([2.2]\) with \( \tilde{a}_k(a) = \sqrt{(k + 1)/2} \) and \( \tilde{b}_k(a) = k/\sqrt{2a} \). Since \( \tilde{a}'_k(a) = 0 \) for all \( k \) and \( \tilde{b}'_k(a) = -k(2a)^{-3/2} \) for \( k = 0, \ldots, n - 1 \), then \( \tilde{J}_n'(a) \) is a diagonal matrix whose entries are all negative. The Hellmann-Feynman theorem shows that the quantities \( (c_{n,j}(a) - a)/\sqrt{2a} \) are decreasing functions of \( a \). Since they converge to \( h_{n,j} \), then \( (c_{n,j}(a) - a)/\sqrt{2a} \geq h_{n,j} \). Therefore,
\[
(3.3) \quad a + \sqrt{2ah_{n,j}} \leq c_{n,j}(a).
\]

The polynomials \( \hat{C}_k(x; a) = (-1)^k(2a)^{\frac{k}{2}}C_k(a + n - 1 + x\sqrt{2a}; a) \), whose zeros are exactly \( (c_{n,j}(a) - a - n + 1)/\sqrt{2a} \), satisfy the three term recurrence relation
\[x\hat{C}_k(x; a) = \frac{1}{2}\hat{C}_{k+1}(x; a) - \frac{n - k - 1}{\sqrt{2a}}\hat{C}_k(x; a) + k\hat{C}_{k-1}(x; a),\]
and their corresponding orthonormal polynomials obey a three term recurrence relation of the form \([2.2]\) with \( \hat{a}_k(a) = \sqrt{(k + 1)/2} \) and \( \hat{b}_k(a) = (-n + k + 1)/\sqrt{2a} \). Since \( \hat{a}'_k(a) = 0 \) and \( \hat{b}'_k(a) = (n - k - 1)/(2a)^{3/2} > 0 \) for \( k = 0, \ldots, n - 1 \), then the Hellmann-Feynman theorem implies that \( (c_{n,j}(a) - a - n + 1)/\sqrt{2a} \) are increasing functions of \( a \). This, together with \([3.2]\), yields the inequality
\[
(3.4) \quad c_{n,j}(a) \leq a + n - 1 + \sqrt{2ah_{n,j}}. \]

Now we are in position to use various results concerning limits of zeros of Hermite polynomials to obtain explicit bound for the zeros of the Charlier polynomials. We begin with a result which concerns upper limits for half of the zeros of Charlier polynomials and lower limits for the other half. In order to this we use the fact that the zeros of Hermite polynomials are symmetric with respect to the origin and the following observation, given in [24 Eq. (1.5)]:
\[h_{n,j} \leq \sqrt{2n - 2} \cos \left( \frac{j - 1}{n - 1} \pi \right), \quad j = 1, \ldots, [n/2].\]

Then we immediately obtain

**Corollary 3.1.** Let \( n \in \mathbb{N}, n \geq 2 \) and \( a > 0 \). Then the inequalities
\[c_{n,j}(a) \leq a + n - 1 + 2\sqrt{(n - 1)a} \cos \left( \frac{j - 1}{n - 1} \pi \right)\]
and
\[c_{n,n+1-j}(a) \geq a - 2\sqrt{(n - 1)a} \cos \left( \frac{j - 1}{n - 1} \pi \right),\]
hold for \( j = 1, \ldots, [n/2] \). In particular, the zeros of the Charlier polynomials satisfy the inequalities
\[a - 2\sqrt{(n - 1)a} \leq c_{n,j}(a) \leq a + n - 1 + 2\sqrt{(n - 1)a}, \quad j = 1, \ldots, n.\]
Observe that the last inequality is obtained by an upper limit for the largest zero \( h_{n,1} \) of the Hermite polynomial. Such sharp bounds are known and two of the best ones are given in [11, Theorem 2] and by Szegö [32, (6.32.6)]. Theorem 2 in [11] states that

\[
(3.5) \quad h_{n,1}^2 \leq \frac{n^2 - \frac{3}{2}n + 2 + (n - 2)\sqrt{n^2 + n + 4}}{n + 4}
\]

for every \( n \in \mathbb{N} \) and the sharper estimate

\[
(3.6) \quad h_{n,1}^2 \leq \frac{n^2 - \frac{5}{2}n + \frac{15}{2} + (n - 3)\sqrt{n^2 + n + 10}}{n + 3},
\]

holds if \( n \) is an odd number. Then (3.5) implies:

**Corollary 3.2.** Let \( n \in \mathbb{N} \), \( n \geq 2 \), and \( a > 0 \). Then

\[
(3.7) \quad c_{n,j}(a) \leq a + n - 1 + \sqrt{a \frac{2n^2 - 3n + 4 + 2(n - 2)\sqrt{n^2 + n + 4}}{n + 4}},
\]

and

\[
(3.8) \quad c_{n,j}(a) \geq a - \sqrt{a \frac{2n^2 - 3n + 4 + 2(n - 2)\sqrt{n^2 + n + 4}}{n + 4}},
\]

hold for every \( j \) with \( 1 \leq j \leq n \).

We do not state explicitly the obvious refinements if we use (3.6) for odd values of \( n \).

Szegö’s limits are given in [32 (6.32.6)] by

\[
h_{n,1} \leq (2n + 1)^{1/2} - 6^{-1/3}(2n + 1)^{-1/6}i_1,
\]

where \( i_1 \) is the lowest zero of the Airy function [32 1.81] and, since \( 6^{-1/3}i_1 \geq 1.85575 \), then

\[
(3.9) \quad h_{n,1} \leq (2n + 1)^{1/2} - 1.85575(2n + 1)^{-1/6}.
\]

Then, using (3.4) we state

**Corollary 3.3.** Let \( n \in \mathbb{N} \), \( n \geq 2 \), and \( a > 0 \). Then

\[
c_{n,j}(a) \leq a + n - 1 + \sqrt{4n + 2a} - 1.85575 \sqrt{2a} (2n + 1)^{-1/6}
\]

and

\[
c_{n,j}(a) \geq a - \sqrt{4n + 2a} + 1.85575 \sqrt{2a} (2n + 1)^{-1/6},
\]

hold for every \( j \) with \( 1 \leq j \leq n \).

We remark that, when \( n \leq 8 \), the limits (3.5) and (3.6) are better than (3.9). Therefore, the same happens when we compare the results in Corollary 3.2 and Corollary 3.3.

A very interesting estimate which limits half of the zeros of Charlier polynomials is obtained by the following result for the zeros of Hermite polynomials, obtained in the same Theorem 2 in [11]. The latter states that

\[
(3.10) \quad h_{n,j}^2 \geq \frac{n^2 - \frac{3}{2}n + 2 - (n - 2)\sqrt{n^2 + n + 4}}{n + 4} \quad \text{for} \quad 1 \leq j \leq \lfloor n/2 \rfloor.
\]

Thus, we obtain:
Corollary 3.4. Let \( n \in \mathbb{N}, n \geq 2, \) and \( a > 0. \) Then
\[
c_{n,j}(a) \geq a + \sqrt{\frac{2n^2 - 3n + 4 - 2(n-2)\sqrt{n^2 + n + 4}}{n+4}}
\]
and
\[
c_{n,n+1-j}(a) \leq a + n - 1 - \sqrt{\frac{2n^2 - 3n + 4 - 2(n-2)\sqrt{n^2 + n + 4}}{n+4}},
\]
hold for \( j = 1, \ldots, [n/2]. \)

We illustrate our results with the figure below. It shows the zeros of \( C_5(x; a), \) in continuous line, and in dashed lines the extreme limits from Corollary 3.2 and the limits for the zeros “in the middle” from Corollary 3.4.

![Figure 3.1](image.png)

**Figure 3.1.** Zeros of \( C_5(x; a), \) limits for the extreme zeros from Corollary 3.2 and center limits from Corollary 3.4.

Having all these bounds for the zeros of Charlier polynomials, we compare them with the best limits known in the literature, obtained recently by Krasikov and Zarkh [26, Th.5]. There the authors prove that
\[
(3.11) \quad c_{n,1} < (\sqrt{a} + \sqrt{n})^2 - \frac{3a^{1/6}(\sqrt{a} + \sqrt{n})^{2/3}}{2^{2/3}n^{1/6}} \quad \text{and}
\]
\[
(3.12) \quad c_{n,n} > (\sqrt{a} - \sqrt{n})^2 + \frac{3a^{1/6}(\sqrt{a} - \sqrt{n})^{2/3}}{2^{2/3}n^{1/6}}
\]
under the restriction \( n < a. \) Therefore, our Corollaries 3.2 and 3.3 provide the first results which hold for all values of \( n \) and \( a. \) Let us compare our results with those in [26] when \( a \to \infty. \) It turns out that, if we fix \( n \in \mathbb{N}, n \geq 3, \) the bounds obtained in Corollary 3.3 are asymptotically sharper than (3.11) and (3.12). The vast number of numerical experiments we have performed show that when \( n \) is fixed, and \( a \) varies, the following happens: when \( a < n \) only the limits provided in our results hold. The estimates (3.11) and (3.12) are sharper when \( a \) is in an interval \((n, a_1).\) Finally, the estimates in Corollary 3.3 become sharper again for the remaining values of \( a \in (a_1, \infty). \) The numerical Table 3.1 below illustrate this phenomena.

Moreover, obviously Corollary 3.4 furnishes limits not only for the extreme zeros, but for half of the zeros of Charlier polynomials.

In the Table 3.1 we also provide the numerical values of the extreme zeros of \( C_5(x; a), \) and the limits, given in (3.12), (3.3), (3.4) and (3.11). Here and in what
follows the better lower and upper bounds are highlighted in gray; also, when all the bounds are out of the interval of orthogonality, no selection is made.

Table 3.1. Table with the extremal zeros of $C_5(x; a)$ and the limits (3.12), (3.8), (3.7) and (3.11)

<table>
<thead>
<tr>
<th>$a$</th>
<th>(3.12)</th>
<th>(3.8)</th>
<th>$c_{5.5}$</th>
<th>$c_{5.1}$</th>
<th>(3.7)</th>
<th>(3.11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.8067</td>
<td>0.0000</td>
<td>4.4330</td>
<td>5.0067</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>-1.4112</td>
<td>1.0000</td>
<td>14.2858</td>
<td>15.4112</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>2.8735</td>
<td>0.9332</td>
<td>21.8798</td>
<td>23.0668</td>
<td>22.6141</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>31.3089</td>
<td>29.7261</td>
<td>72.9684</td>
<td>74.2739</td>
<td>74.3490</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>72.4871</td>
<td>71.3284</td>
<td>131.3159</td>
<td>132.6716</td>
<td>133.1878</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>435.1245</td>
<td>435.8882</td>
<td>72.9684</td>
<td>74.2739</td>
<td>74.3490</td>
<td></td>
</tr>
</tbody>
</table>

4. MEIXNER POLYNOMIALS

The main result obtained in this section is motivated by the limit relation between Laguerre and Meixner polynomials [23 (2.9.2)]

$$\lim_{c \to 1} M_n \left( \frac{x}{1-c}; \alpha + 1, c \right) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}.$$  

Let us define new polynomials with shifted argument, as follows:

$$\tilde{M}_n(x; \alpha, c) := (-1)^n M_n \left( \sqrt{c} x + c(\alpha + 1); \alpha + 1, c \right),$$

$$\hat{M}_n(x; \alpha, c) := (-1)^n M_n \left( \sqrt{c} x + c(n + \alpha) + n - 1; \alpha + 1, c \right).$$

It is clear that the asymptotic formulae

$$\lim_{c \to 1} \tilde{M}_n(x; \alpha, c) = \frac{(-1)^n L_n^{(\alpha)}(x + \alpha + 1)}{L_n^{(\alpha)}(0)},$$

$$\lim_{c \to 1} \hat{M}_n(x; \alpha, c) = \frac{(-1)^n L_n^{(\alpha)}(x + \alpha + 2n - 1)}{L_n^{(\alpha)}(0)},$$

hold.

**Theorem 4.1.** Let $n \in \mathbb{N}$, $\beta > 0$ and $0 < c < 1$. Then $((1-c)m_{n,j}(\beta, c) - c\beta)/\sqrt{c}$ are decreasing functions of $c$ and $((1-c)m_{n,j}(\beta, c) - c\beta - n + 1)/\sqrt{c}$ are increasing functions of $c$, when $c \in (0, \infty)$. Moreover, the inequalities

$$\frac{\sqrt{c}}{1-c} \left( x_{n,j} - \beta(1 - \sqrt{c}) \right) \leq m_{n,j}(\beta, c),$$

$$m_{n,j}(\beta, c) \leq \frac{\sqrt{c}}{1-c} \left( x_{n,j} - \beta(1 - \sqrt{c}) + (n-1) \left( \frac{1-\sqrt{c}}{\sqrt{c}} \right)^2 \right),$$

hold for $j = 1, \ldots, n$.

**Proof.** The three term recurrence relation of $M_n(x; \beta, c)$ is

$$(c-1)xM_n(x; \beta, c) = c(n+\beta)M_{n+1}(x; \beta, c) - (n+(n+\beta)c)M_n(x; \beta, c) + nM_{n-1}(x; \beta, c).$$
After substitutions, we obtain the coefficients of the recurrence relation \((2.2)\) for the orthonormal polynomials, corresponding to \(\tilde{M}_n(x; \alpha, c)\) and \(\tilde{M}_n(x; \alpha, c)\). These coefficients are

\[
\tilde{a}_k(c) = \frac{1}{\tilde{a}_k'(c)} = \frac{1}{\sqrt{c} k(1+c)} \quad \text{and} \quad \tilde{b}_k(c) = -\frac{1}{\sqrt{c} k(1+c)}.
\]

The derivatives with respect to \(c\) are \(\tilde{a}_k(c) = \tilde{a}_k'(c) = 0\), \(\tilde{b}_k(c) = -k(1-c)/(2c^{3/2}) \leq 0\) and \(\tilde{b}_k'(c) = (n-1-k)(1-c)/(2c^{3/2}) \geq 0\). Therefore, by the Hellmann-Feynmann theorem, the zeros \(\{(1-c)m_{n,j}(\alpha + 1, c) - c(\alpha + 1)\}/\sqrt{c}\) of \(\tilde{M}_n(x; \alpha, c)\) are decreasing functions of \(c\) while the zeros \(\{(1-c)m_{n,j}(\alpha + 1, c) - c(\alpha + n) - n + 1\}/\sqrt{c}\) of \(\tilde{M}_n(x; \alpha, c)\) are increasing functions of \(c\), for \(c \in (0, 1)\).

Formulae \((4.1)\) and \((4.2)\) imply that

\[
\{(1-c)m_{n,j}(\alpha + 1, c) - c(\alpha + 1)\}/\sqrt{c} \to x_{n,j}(\alpha) - \alpha - 1
\]

and

\[
\{(1-c)m_{n,j}(\alpha + 1, c) - c(\alpha + 1) - n + 1\}/\sqrt{c} \to x_{n,j}(\alpha) - \alpha - 2n + 1, \quad c \to 1.
\]

The monotonic behavior of the functions on the left-hand sides shows that

\[
\{(1-c)m_{n,j}(\alpha + 1, c) - c(\alpha + 1)\}/\sqrt{c} \geq x_{n,j}(\alpha) - \alpha - 1
\]

and

\[
\{(1-c)m_{n,j}(\alpha + 1, c) - c(\alpha + 1) - n + 1\}/\sqrt{c} \leq x_{n,j}(\alpha) - \alpha - 2n + 1.
\]

These inequalities are equivalent to those in the statement of the theorem. \(\Box\)

Observe that the inequalities in Theorem \((4.1)\) coincide with those obtained by Ismail and Muldoon \([18, \text{Theorem 6.2}]\) using slightly different arguments. In fact, Ismail and Muldoon used the Hellmann-Feynmann theorem to obtain bounds for the zeros of the polynomials \(M_n(\sqrt{c}x/(1-c); \alpha + 1, c)\) and it suffices to ensure that the corresponding orthonormal polynomials are such that the coefficients \(a_k\) in the recurrence relation do not depend on \(c\). Then, they conclude, via an extension of the Hellmann-Feynmann theorem which ensures convexity of the zeros, that the derivatives obey certain inequalities and finally integrate these inequalities, having in mind the asymptotic values of these zeros when \(c\) converges to 1. Except for doing the transformation of Ismail and Muldoon, we performed additional translations of the zeros which guarantee the positive (negative) definiteness of the corresponding matrices \(J_n^l\) and obtain their result in a slightly different way.

Now we use limits for the zeros of the Laguerre polynomials to obtain explicit bounds for the zeros of the Meixner polynomials.

The best limits for the extreme zeros of \(L_n^{(\alpha)}(x)\), when \(\alpha\) is large, are given in \([25, \text{Theorem 1}]\) and read as follows:

\[
x_{n,1}(\alpha) < (\sqrt{n + \alpha + 1} + \sqrt{n})^2 - 3(\sqrt{n + \alpha + 1} + \sqrt{n})^{4/3} + 2
\]

and

\[
x_{n,n}(\alpha) > (\sqrt{n + \alpha + 1} - \sqrt{n})^2 + 3(\sqrt{n + \alpha + 1} - \sqrt{n})^{4/3}.
\]

They, together with Theorem \((4.1)\) yield
Corollary 4.1. Let $n \in \mathbb{N}$, $n \geq 2$, $0 < c < 1$ and $\beta > 0$. Then
\[
\frac{\sqrt{c}}{1 - c} \left( (\sqrt{\nu} - \sqrt{m})^2 + 3 \frac{(\sqrt{\nu} - \sqrt{m})^{4/3}}{4^{1/3}(n\nu)^{1/6}} - \beta(1 - \sqrt{c}) \right) < m_{n,j}(\beta, c)
\]
\[
< \frac{\sqrt{c}}{1 - c} \left( (\sqrt{\nu} + \sqrt{m})^2 - 3 \frac{(\sqrt{\nu} + \sqrt{m})^{4/3}}{4^{1/3}(n\nu)^{1/6}} + 2 - \beta(1 - \sqrt{c}) + (n - 1) \frac{(1 - \sqrt{c})^2}{\sqrt{c}} \right),
\]
hold for $1 \leq j \leq n$, where $\nu = n + \beta$.

The best limit for the largest zero if the Laguerre polynomials, when $n$ goes to infinity, is given by Szegö [32, (6.32.6)]:
\[
x_{n,1}(\alpha) \leq ((4n + 2\alpha + 2)^{1/2} - 6^{-1/3}(4n + 2\alpha + 2)^{-1/6}i_1)^2.
\]
Again, as in the previous section, since $6^{-1/3}i_1 \geq 1.85575$, then
\[
x_{n,1}(\alpha) \leq ((4n + 2\alpha + 2)^{1/2} - 1.85575(4n + 2\alpha + 2)^{-1/6})^2.
\]
Therefore, we have:

Corollary 4.2. Let $n \in \mathbb{N}$, $n \geq 2$, $0 < c < 1$ and $\beta > 0$. Then
\[
m_{n,j}(\beta, c) < \frac{\sqrt{c}}{1 - c} \left( ((4n + 2\beta)^{1/2} - 1.85575(4n + 2\beta)^{-1/6})^2 \right.
\]
\[
- \beta(1 - \sqrt{c}) + (n - 1) \frac{(1 - \sqrt{c})^2}{\sqrt{c}},
\]
hold for $1 \leq j \leq n$.

Still, we write the limits from [11, Theorem 1] that are good when $\alpha$ or $n$ is small. These limits are given by
\[
x_{n,n}(\alpha) > \frac{2n^2 + n(\alpha - 1) + 2(\alpha + 1) - 2(n - 1)\sqrt{n^2 + (n + 2)(\alpha + 1)}}{n + 2}
\]
and
\[
x_{n,1}(\alpha) < \frac{2n^2 + n(\alpha - 1) + 2(\alpha + 1) + 2(n - 1)\sqrt{n^2 + (n + 2)(\alpha + 1)}}{n + 2}.
\]
Therefore we obtain

Corollary 4.3. Let $n \in \mathbb{N}$, $n \geq 2$, $0 < c < 1$ and $\beta > 0$. Then
\[
\frac{\sqrt{c}}{1 - c} \left( \frac{2n^2 + n(\beta - 2) + 2\beta - 2(n - 1)\sqrt{n^2 + (n + 2)\beta}}{n + 2} \right) - \beta(1 - \sqrt{c}) \leq m_{n,j}(\beta, c) \leq \frac{\sqrt{c}}{1 - c} \left( \frac{2n^2 + n(\beta - 2) + 2\beta + 2(n - 1)\sqrt{n^2 + (n + 2)\beta}}{n + 2} \right.
\]
\[
- \beta(1 - \sqrt{c}) + (n - 1) \frac{(1 - \sqrt{c})^2}{\sqrt{c}}),
\]
hold for $1 \leq j \leq n$.

We illustrate our results with figures for $n = 5$. The Figure 4.1 shows the zeros of $M_5(x; \beta, 0.7)$ in relation to $\beta$, in continuous line, and the corresponding limits from Corollary 4.3 in dashed line. In the same way, the Figure 4.2 shows the zeros of $M_5(x; 10, c)$ in relation to $c$ and the corresponding limits from Corollary 4.3.
Krasikov and Zarkh [26, Theorem 7] proved that

\( m_{n,n}(\beta,c) > \mu_1 + \frac{3c^{1/6}\mu_1^{1/3}(\mu_1 + \beta)^{1/3}}{2^{2/3}(1-c)^{1/3}n^{1/6}(n+\beta)^{1/6}}, \) 

provided that \( n < \beta c / (1-c), \) and

\[ m_{n,1}(\beta,c) < \mu_2 - \begin{cases} 
3c^{1/6}\mu_2^{1/3}(\mu_2 + \beta)^{1/3} / (1-c)^{2/3}, & \mu_2 \leq \mu_1 + \sqrt{\mu_1(\mu_1 + \beta)}, \\
3c^{1/3}(\sqrt{\mu_1} + \sqrt{\mu_1 + \beta})^{2/3} / (1-c)^{2/3}, & \mu_2 > \mu_1 + \sqrt{\mu_1(\mu_1 + \beta)},
\end{cases} \]

where \( \mu_1 = (\sqrt{n} - \sqrt{c(n+\beta)})^2 / (1-c) \) and \( \mu_2 = (\sqrt{n} + \sqrt{c(n+\beta)})^2 / (1-c). \)

Let us compare asymptotically our results with the latter from [26]. For \( \beta \) and \( c \) fixed, the bounds (4.5) and (4.6) do not hold for large values of \( n \). If \( n \leq 6 \) and \( \beta \to \infty \), the bounds from Corollary 4.3 are sharper than (4.5) and (4.6), while if \( n \geq 6 \) the bounds (4.5) and (4.6) are sharper than those obtained in the above corollaries. For \( c \to 0 \), our results are better. For \( c \to 1 \), the lower bound from Corollary 4.1 and (4.5) are asymptotically equal. Numerical experiments show that if \( c \to 1 \), the upper estimates in the above corollaries, especially in Corollary 4.1, are sharper than (4.6) for most values of \( n \) and \( \beta \). The numerical tables below illustrate these phenomena.

### Table 4.1. Table with the extreme zeros of \( M_5(x;\beta,0.7) \) and the limits (4.5), (4.6), (4.3) and (4.4).

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>(4.5)</th>
<th>(4.3)</th>
<th>( m_{5,5} )</th>
<th>( m_{5,1} )</th>
<th>(4.4)</th>
<th>(4.6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0118</td>
<td>0.0218</td>
<td>31.2943</td>
<td>32.6834</td>
<td>49.5072</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2.5826</td>
<td>2.9145</td>
<td>3.4007</td>
<td>5.0590</td>
<td>52.6473</td>
<td>66.7876</td>
</tr>
<tr>
<td>20</td>
<td>21.4951</td>
<td>21.6617</td>
<td>23.1023</td>
<td>100.6055</td>
<td>103.9001</td>
<td>111.1347</td>
</tr>
<tr>
<td>100</td>
<td>163.3832</td>
<td>163.4497</td>
<td>167.4339</td>
<td>329.7718</td>
<td>335.4455</td>
<td>336.1549</td>
</tr>
<tr>
<td>500</td>
<td>993.2798</td>
<td>993.3690</td>
<td>1003.0964</td>
<td>1360.7678</td>
<td>1372.1928</td>
<td>1372.3100</td>
</tr>
</tbody>
</table>
Table 4.2. Table with the extreme zeros of $M_5(x; 10, c)$ and the limits (4.5), (4.6), (4.3) and (4.4).

<table>
<thead>
<tr>
<th>$c$</th>
<th>(4.5)</th>
<th>(4.3)</th>
<th>$m_{5.5}$</th>
<th>$m_{5.1}$</th>
<th>(4.4)</th>
<th>(4.6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.7950</td>
<td>0.0258</td>
<td>8.4283</td>
<td>9.1108</td>
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<td></td>
</tr>
<tr>
<td>0.3</td>
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<td>17.7329</td>
<td>18.6418</td>
<td>21.4749</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>2.3068</td>
<td>2.3280</td>
<td>3.0000</td>
<td>33.1590</td>
<td>34.5207</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>7.9671</td>
<td>8.2040</td>
<td>9.0673</td>
<td>68.2082</td>
<td>70.6912</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>37.4251</td>
<td>38.5348</td>
<td>40.9496</td>
<td>241.7715</td>
<td>249.9915</td>
<td></td>
</tr>
</tbody>
</table>

5. Kravchuk polynomials

The main results in this section are motivated by the relation

\[
(5.1) \quad \lim_{N \to \infty} \sqrt{\frac{N}{n}} K_n(pN + x\sqrt{2p(1-p)N}; p, N) = \frac{(-1)^n H_n(x)}{\sqrt{2^n n! \left( \frac{p}{1-p} \right)^n}}
\]

between the Kravchuk polynomials with a proper argument and Hermite polynomials, (see [23, (2.10.2)]).

We provide detailed results concerning the extreme zeros of $K_n(x; p, N)$.

**Theorem 5.1.** Let $n, N \in \mathbb{N}$, with $n \leq N$ and $0 < p < 1$.

(i) Let $0 < p \leq 1/2$. Then \( \kappa_{n,1}(p, N) - pN - (1-2p)(n-1) \) \( \sqrt{N} \) is an increasing function of $N$ while \( \kappa_{n,1}(p, N) - pN \) \( \sqrt{N} \) is a decreasing function of $N$. Further, \( \kappa_{n,n}(p, N) - pN \) \( \sqrt{N} \) is a decreasing function of $N$ while \( \kappa_{n,n}(p, N) - pN - (1-2p)(n-1) \) \( \sqrt{N} \) is an increasing function of $N$. Moreover, the inequalities

\[
\kappa_{n,1}(p, N) \leq pN + (1-2p)(n-1) + \sqrt{2p(1-p)N} h_{n,1},
\]

\[
\kappa_{n,1}(p, N) \geq pN + \sqrt{2p(1-p)(N-n+1)} h_{n,1},
\]

\[
\kappa_{n,n}(p, N) \leq pN + (1-2p)(n-1) - \sqrt{2p(1-p)(N-n+1)} h_{n,1},
\]

\[
\kappa_{n,n}(p, N) \geq pN - \sqrt{2p(1-p)N} h_{n,1},
\]

hold.

(ii) Let $1/2 \leq p < 1$. Then \( \kappa_{n,1}(p, N) - pN \) \( \sqrt{N} \) is an increasing function of $N$ and \( \kappa_{n,1}(p, N) - pN + (2p-1)(n-1) \) \( \sqrt{N} \) is a decreasing function of $N$. Further, \( \kappa_{n,n}(p, N) - pN + (2p-1)(n-1) \) \( \sqrt{N} \) is a decreasing function of $N$ while \( \kappa_{n,n}(p, N) - pN \) \( \sqrt{N} \) is an increasing function of $N$. Moreover,

\[
\kappa_{n,1}(p, N) \leq pN + \sqrt{2p(1-p)N} h_{n,1},
\]

\[
\kappa_{n,1}(p, N) \geq pN - (2p-1)(n-1) + \sqrt{2p(1-p)(N-n+1)} h_{n,1},
\]

\[
\kappa_{n,n}(p, N) \leq pN - \sqrt{2p(1-p)(N-n+1)} h_{n,1},
\]

\[
\kappa_{n,n}(p, N) \geq pN - (2p-1)(n-1) - \sqrt{2p(1-p)N} h_{n,1}.
\]
Proof. Recall that Kravchuk’s polynomials $K_n(x; p, N)$ obey the three term recurrence relation

$$-xK_n(x; p, N) = p(N-n)K_{n+1}(x; p, N) - (p(N-n) + k(1-p))K_n(x; p, N) + n(1-p)K_{n-1}(x; p, N).$$

The corresponding orthonormal polynomials are $\sqrt{\binom{N}{n} \left( \frac{p}{1-p} \right)^n} K_n(x; p, N)$. Set

$$\tilde{K}_n(x; p, N) = (-1)^n \sqrt{\binom{N}{n} \left( \frac{p}{1-p} \right)^n} K_n(pN + x\sqrt{2p(1-p)}; p, N).$$

Thus, by (5.1) we have

$$\lim_{N \to \infty} \tilde{K}_n(x; p, N) = \frac{H_n(x)}{\sqrt{2^n n!}}.$$

The Jacobi matrix of the form (2.3), associated with $\tilde{K}_n(x; p, N)$ has entries

$$\tilde{a}_j(N) = \sqrt{\frac{j+1}{2} \left( 1 - \frac{j}{N} \right)} \quad \text{and} \quad \tilde{b}_j(N) = \frac{j(1-2p)}{\sqrt{2p(1-p)}}.$$

It is easy to see that the entries (5.3) are increasing functions of $N$ when $1/2 \leq p < 1$. Then, by the Perron-Frobenius theorem, the largest zero of $\tilde{K}_n(x; p, N)$, which is $\{\kappa_{n,1}(p, N) - pN\}/\sqrt{2p(1-p)}N$, is an increasing function of $N$, provided $1/2 \leq p < 1$. Then, by (5.3), we have $\{\kappa_{n,1}(p, N) - pN\}/\sqrt{2p(1-p)}N \leq h_{n,1}$.

For $n$ fixed we consider, for $j = 0, \ldots, n$, the polynomials

$$\tilde{K}_j\left(x + \frac{(n-1)(1-2p)}{\sqrt{2p(1-p)}}; p, N\right), \quad (-1)^j \tilde{K}_j(-x; p, N), \quad \text{for } 0 < p \leq 1/2$$

and

$$(-1)^j \tilde{K}_j\left(-x - \frac{(n-1)(2p-1)}{\sqrt{2p(1-p)}}; p, N\right) \quad \text{for } 1/2 \leq p < 1.$$

These polynomials satisfy recurrence relations of the form (2.2) with

$$a_j(N) = \sqrt{\frac{j+1}{2} \left( 1 - \frac{j}{N} \right)}$$

and $b_j(N)$ is, respectively,

$$\frac{(n-1-j)(2p-1)}{\sqrt{2p(1-p)}N}, \quad \frac{j(2p-1)}{\sqrt{2p(1-p)}N} \quad \text{and} \quad \frac{(n-1-j)(1-2p)}{\sqrt{2p(1-p)}N}.$$

Since these $a_j(N)$ and $b_j(N)$ are increasing functions of $N$, for $0 \leq j \leq n-1$, then by Theorem [C] the largest zeros of the polynomials (5.5) and (5.6) are increasing functions of $N$. This fact, together with (5.3), yields the inequalities

$$\frac{\kappa_{n,1}(p, N) - pN - (1-2p)(n-1)}{\sqrt{2p(1-p)}N} \leq h_{n,1},$$

$$-\frac{\kappa_{n,n}(p, N) + pN}{\sqrt{2p(1-p)}N} \leq -h_{n,n},$$

$$-\frac{\kappa_{n,n}(p, N) + pN - (2p-1)(n-1)}{\sqrt{2p(1-p)}N} \leq -h_{n,n}.$$
Using the fact that the zeros of Hermite polynomials satisfy $-h_{n,n} = h_{n,1}$, we can write these inequalities as in the statement of the theorem.

For $j = 0, \ldots, n$ and $n$ fixed, consider the polynomials
\begin{equation}
\tilde{K}_j\left(x\sqrt{1 - \frac{n-1}{N}}; p, N\right), \quad (-1)^j \tilde{K}_j\left(-x\sqrt{1 - \frac{n-1}{N}} + \frac{(n-1)(1-2p)}{\sqrt{2p(1-p)N}}; p, N\right),
\end{equation}
for $0 < p \leq 1/2$ and
\begin{equation}
\tilde{K}_j\left(x\sqrt{1 - \frac{n-1}{N}} - \frac{(n-1)(2p-1)}{\sqrt{2p(1-p)N}}; p, N\right), \quad (-1)^j \tilde{K}_j\left(-x\sqrt{1 - \frac{n-1}{N}} - \frac{(n-1-1)(1-2p)}{\sqrt{2p(1-p)N}}; p, N\right),
\end{equation}
for $1/2 \leq p < 1$. These polynomials satisfy recurrence relations of the form (2.2) with
\begin{equation}
a_j(N) = \frac{(j+1)(N-j)}{2(N-n+1)}, \quad \frac{(n-1-j)(1-2p)}{\sqrt{2p(1-p)(N-n+1)}}, \quad \frac{(n-1-j)(2p-1)}{\sqrt{2p(1-p)(N-n+1)}}, \quad \frac{k(2p-1)}{\sqrt{2p(1-p)(N-n+1)}},
\end{equation}
and $b_j(N)$ is, respectively,
\begin{equation}
\frac{j(1-2p)}{\sqrt{2p(1-p)(N-n+1)}}, \quad \frac{(n-1-j)(1-2p)}{\sqrt{2p(1-p)(N-n+1)}}, \quad \frac{(n-1-j)(2p-1)}{\sqrt{2p(1-p)(N-n+1)}}, \quad \frac{k(2p-1)}{\sqrt{2p(1-p)(N-n+1)}}.
\end{equation}

Since these $a_j(N)$ and $b_j(N)$, for $0 \leq j \leq n-1$, are the entries of the Jacobi matrix $n \times n$ of the form (2.3) and are decreasing functions of $N$, then by Theorem C, the largest zeros of the polynomials (5.7) are decreasing functions of $N$.

The limit (5.3) together with the fact that the largest zeros of the polynomials (5.7) are decreasing functions of $N$ imply the inequalities
\begin{align*}
\frac{\kappa_{n,1}(p, N) - pN}{\sqrt{2p(1-p)(N-n+1)}} & \geq h_{n,1}, \\
\frac{-\kappa_{n,n}(p, N) + pN - (n-1)(1-2p)}{\sqrt{2p(1-p)(N-n+1)}} & \geq -h_{n,n}, \\
\frac{\kappa_{n,1}(p, N) - pN + (n-1)(2p-1)}{\sqrt{2p(1-p)(N-n+1)}} & \geq h_{n,1}, \\
\frac{-\kappa_{n,n}(p, N) + pN}{\sqrt{2p(1-p)(N-n+1)}} & \geq -h_{n,n}.
\end{align*}

By the fact that $-h_{n,n} = h_{n,1}$ we can write these inequalities as the inequalities of the Theorem 5.1. □

We obtain sharp bounds for the zeros of Kravchuk polynomials using those for the largest zero of Hermite polynomial. The limit (3.5) from [11, Theorem 2] yields

**Corollary 5.1.** Let $n, N \in \mathbb{N}$, with $n \leq N$, and $0 < p < 1$. 

\[ \]
(i) If $0 < p \leq 1/2$, then
\[
\kappa_{n,1}(p, N) \leq pN + (1 - 2p)(n - 1) + \sqrt{2p(1-p)N(n^2 - \frac{3}{2}n + 2 + (n-2)\sqrt{n^2 + n + 4})}.
\]
\[
\kappa_{n,n}(p, N) \geq pN - \sqrt{2p(1-p)N(n^2 - \frac{3}{2}n + 2 + (n-2)\sqrt{n^2 + n + 4})} + \sqrt{2p(1-p)N((2n+1)^{1/2} - 1.85575(2n+1)^{-1/6})},
\]

(ii) If $1/2 \leq p < 1$, then
\[
\kappa_{n,1}(p, N) \leq pN + \sqrt{2p(1-p)N((2n+1)^{1/2} - 1.85575(2n+1)^{-1/6})},
\]
\[
\kappa_{n,n}(p, N) \geq pN - (2p - 1)(n - 1) - \sqrt{2p(1-p)N((2n+1)^{1/2} - 1.85575(2n+1)^{-1/6})}.
\]

The limits (3.9), due to Szegő’s [32, (6.32.6)], imply

**Corollary 5.2.** Let $n, N \in \mathbb{N}$, with $n \leq N$, and $0 < p < 1$.

(i) If $0 < p \leq 1/2$, then
\[
\kappa_{n,1}(p, N) \leq pN + (1 - 2p)(n - 1) + \sqrt{2p(1-p)N((2n+1)^{1/2} - 1.85575(2n+1)^{-1/6})},
\]
\[
\kappa_{n,n}(p, N) \geq pN - \sqrt{2p(1-p)N((2n+1)^{1/2} - 1.85575(2n+1)^{-1/6})} + \sqrt{2p(1-p)N((2n+1)^{1/2} - 1.85575(2n+1)^{-1/6})}.
\]

(ii) If $1/2 \leq p < 1$, then
\[
\kappa_{n,1}(p, N) \leq pN + \sqrt{2p(1-p)N((2n+1)^{1/2} - 1.85575(2n+1)^{-1/6})},
\]
\[
\kappa_{n,n}(p, N) \geq pN - (2p - 1)(n - 1) + \sqrt{2p(1-p)N((2n+1)^{1/2} - 1.85575(2n+1)^{-1/6})}.
\]

In the following, we show figures for $n = 5$ with the limits from Corollary 5.1.

As throughout this paper the zeros are in continuous line and the limits in dashed line.
Now, let us compare our results with the sharper bounds found in the literature. Krasikov and Zarkh proved that \[ (5.12) \]
\[
-k_{n,n}(p, N) > \left( \sqrt{p(N-n)} + \sqrt{q} n \right)^2 + \frac{(pq)^{1/6} \left( \sqrt{p(N-n)} + \sqrt{q} n \right)^{2/3} (N - \left( \sqrt{p(N-n)} + \sqrt{q} n \right)^2)^{1/3}}{2^{2/3} n^{1/6} (N-n)^{1/6}},
\]
where \( q = 1 - p \), provided that \( n < pN \), and

\[ (5.13) \]
\[
k_{1,n}(p, N) < \left( \sqrt{p(N-n)} - \sqrt{q} n \right)^2 - \frac{(pq)^{1/6} \left( \sqrt{p(N-n)} - \sqrt{q} n \right)^{2/3} (N - \left( \sqrt{p(N-n)} - \sqrt{q} n \right)^2)^{1/3}}{2^{2/3} n^{1/6} (N-n)^{1/6}},
\]
where \( q = 1 - p \), provided that \( n < (1-p)N \).

Let us compare asymptotically, provided (5.12) and (5.13) hold. For a fixed value of \( p \) when \( N \) is large enough the bounds from Corollary 5.2 are sharper than (5.12) and (5.13); we remark that for \( n \leq 15 \) the bounds from Corollary 5.1 are the sharpest when \( N \to \infty \). When \( p \to 0 \) or \( p \to 1 \), the bounds from Corollaries 5.1 and 5.2 are sharper than (5.12) and (5.13) provided the latter hold. The numerical tables below confirm these observations.

**Table 5.1.** Table of the extreme zeros of \( K_5(x; p, 30) \) and the limits (5.12), (5.13) and limits from Corollary 5.1

<table>
<thead>
<tr>
<th>( p )</th>
<th>(5.12)</th>
<th>Cor. 5.1</th>
<th>( \kappa_{5,5} )</th>
<th>( \kappa_{5,1} )</th>
<th>Cor. 5.1</th>
<th>(5.13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0001</td>
<td>-0.1514</td>
<td>7.1 \times 10^{-16}</td>
<td>4.0129</td>
<td>4.1592</td>
<td>4.3045</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>2.5796</td>
<td>1.8035</td>
<td>3.1147</td>
<td>17.0013</td>
<td>17.7965</td>
<td>17.5276</td>
</tr>
<tr>
<td>0.5</td>
<td>6.9333</td>
<td>7.1479</td>
<td>7.4661</td>
<td>22.5339</td>
<td>22.8521</td>
<td>23.0667</td>
</tr>
<tr>
<td>0.7</td>
<td>12.4724</td>
<td>12.2035</td>
<td>12.9987</td>
<td>26.8853</td>
<td>28.1965</td>
<td>27.4204</td>
</tr>
<tr>
<td>0.9999</td>
<td>25.6955</td>
<td>25.8408</td>
<td>25.9871</td>
<td>30.0000</td>
<td>30.1540</td>
<td></td>
</tr>
</tbody>
</table>
Table 5.2. Table of the extreme zeros of $K_5(x; 0.3, N)$ and the limits (5.12), (5.13), (5.9) and (5.8).

<table>
<thead>
<tr>
<th>$N$</th>
<th>(5.12)</th>
<th>(5.9)</th>
<th>$\kappa_{5,5}$</th>
<th>$\kappa_{5,1}$</th>
<th>(5.8)</th>
<th>(5.13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>-1.4380</td>
<td>0.0075</td>
<td>4.7614</td>
<td>6.0380</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-1.1549</td>
<td>0.2593</td>
<td>7.5128</td>
<td>8.7549</td>
<td>8.2300</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>6.4055</td>
<td>5.093</td>
<td>25.1477</td>
<td>25.8907</td>
<td>25.7871</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>17.3408</td>
<td>16.8610</td>
<td>44.0398</td>
<td>44.7390</td>
<td>44.8992</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>120.1616</td>
<td>120.6203</td>
<td>121.8568</td>
<td>122.1080</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.3. Table of the extreme zeros of $K_5(x; 0.7, N)$ and the limits (5.12), (5.13), (5.11) and (5.10).

<table>
<thead>
<tr>
<th>$N$</th>
<th>(5.12)</th>
<th>(5.11)</th>
<th>$\kappa_{5,5}$</th>
<th>$\kappa_{5,1}$</th>
<th>(5.10)</th>
<th>(5.13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>-1.0380</td>
<td>0.2386</td>
<td>4.9925</td>
<td>6.4380</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1.7700</td>
<td>1.2451</td>
<td>2.2472</td>
<td>9.7407</td>
<td>11.1549</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>24.2129</td>
<td>24.1093</td>
<td>24.8523</td>
<td>43.0152</td>
<td>43.5945</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>55.1008</td>
<td>55.2610</td>
<td>55.9602</td>
<td>81.8957</td>
<td>82.6600</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>317.8920</td>
<td>319.0203</td>
<td>319.7040</td>
<td>378.1432</td>
<td>379.8384</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.4. Table of the extreme zeros of $K_5(x; 0.5, N)$ and the limits (5.12), (5.13), (5.11) and (5.10).

<table>
<thead>
<tr>
<th>$N$</th>
<th>(5.12)</th>
<th>(5.11)</th>
<th>$\kappa_{5,5}$</th>
<th>$\kappa_{5,1}$</th>
<th>(5.10)</th>
<th>(5.13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>-0.7056</td>
<td>0.0650</td>
<td>4.9350</td>
<td>5.7056</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.4666</td>
<td>1.0000</td>
<td>9.0000</td>
<td>9.5334</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>14.4814</td>
<td>14.8630</td>
<td>15.1226</td>
<td>34.8774</td>
<td>35.1370</td>
<td>35.5186</td>
</tr>
<tr>
<td>100</td>
<td>34.9994</td>
<td>35.6642</td>
<td>35.8725</td>
<td>64.1275</td>
<td>64.3358</td>
<td>65.0006</td>
</tr>
<tr>
<td>500</td>
<td>216.2194</td>
<td>217.9441</td>
<td>218.1282</td>
<td>281.8718</td>
<td>282.0559</td>
<td>283.7806</td>
</tr>
</tbody>
</table>

6. HAHN POLYNOMIALS

The results for the zeros of Hahn polynomials are motivated by the limit relation with the Jacobi polynomials [23 (2.5.1)]

$$
\lim_{N \to \infty} Q_n(Nx; \alpha, \beta, N) = \frac{P_n^{(\alpha, \beta)}(1 - 2x)}{P_n^{(\alpha, \beta)}(1)}.
$$

Theorem 6.1. Let $n$ and $N$ be positive integers with $n \leq N$ and the real parameters $\alpha$ and $\beta$ satisfy $\alpha, \beta > -1$. Then

$$
g_{n,1}(\alpha, \beta, N) + (1 + \alpha)/2
\frac{N + (\alpha + \beta + 2)/2}{N + (\alpha + \beta + 2)/2}
\quad \text{and} \quad
-g_{n,n}(\alpha, \beta, N) - (1 + \alpha)/2
\frac{N + (\alpha + \beta + 2)/2}{N + (\alpha + \beta + 2)/2}
+ \frac{n(n + \alpha)}{2n + \alpha + \beta}
$$

are increasing functions of $N$. Moreover, the inequalities

$$
q_{n,1}(\alpha, \beta, N) \leq \left( N + \frac{\alpha + \beta + 2}{2} \left( \frac{1 - x_{n,n}(\alpha, \beta)}{2} \right) - \frac{1 + \alpha}{2} \right).
$$


and
\begin{equation}
q_{n,n}^r(\alpha, \beta, N) \geq \left( N + \frac{\alpha + \beta + 2}{2} \right) \left( 1 - \frac{x_{n,1}}{2} \right) - \frac{1 + \alpha}{2}
\end{equation}
hold.

**Proof.** The polynomials \( Q_k(x; \alpha, \beta, N) \) satisfy the three term recurrence relation
\begin{equation}
-x Q_k(x; \alpha, \beta, N) = A_k Q_{k+1}(x; \alpha, \beta, N) - (A_k + C_k) Q_k(x; \alpha, \beta, N) + C_k Q_{k-1}(x; \alpha, \beta, N),
\end{equation}
where
\begin{align*}
A_k &= \frac{(k + \alpha + \beta + 1)(k + \alpha + 1)(N - k)}{(2k + \alpha + \beta + 1)(2k + \alpha + \beta + 2)}, \\
C_k &= \frac{k(k + \alpha + \beta + N + 1)(k + \beta)}{(2k + \alpha + \beta)(2k + \alpha + \beta + 1)}.
\end{align*}

Set
\begin{equation}
\tilde{Q}_k(x; \alpha, \beta, N) = (-1)^k Q_k \left( N + \frac{\alpha + \beta + 2}{2} \right) x - \frac{1 + \alpha}{2}; \alpha, \beta, N \right).
\end{equation}

Then the above limit relation immediately yields
\begin{equation}
\lim_{N \to \infty} \tilde{Q}_k(x; \alpha, \beta, N) = \frac{(-1)^k P_k^{(\alpha, \beta)}(1 - 2x)}{P_k^{(\alpha, \beta)}(1)}.
\end{equation}

Transforming (6.3) into a recurrence relation for the orthonormal polynomials \( \hat{Q}_k(x; \alpha, \beta, N) \), we see that the latter satisfy (2.2) with
\begin{align*}
\hat{a}_k(N) &= \frac{\sqrt{A_k C_{k+1}}}{N + (\alpha + \beta + 2)/2}, \\
\hat{b}_k(N) &= \frac{A_k + C_k + (1 + \alpha)/2}{N + (\alpha + \beta + 2)/2}.
\end{align*}

Observe that \( \hat{a}_k(N) \) and \( \hat{b}_k(N) \) are nonnegative for \( k = 0, \ldots, n-1 \). Furthermore, the derivatives of \( \hat{a}_k(N) \) with respect to \( N \) are nonnegative, while \( \hat{b}_k(N) \) do not depend on \( N \), so that their derivatives with respect to \( N \) vanish. Then, according to Theorem C, the largest zero of \( \hat{Q}_n(x; \alpha, \beta, N) \) is an increasing function of \( N \).

The largest zero of \( \hat{Q}_n(x; \alpha, \beta, N) \) is \( \{ q_{n,1}(\alpha, \beta, N) + (1 + \alpha)/2 \}/\{ N + (\alpha + \beta + 2)/2 \} \) and the largest zero of \( P_n^{(\alpha, \beta)}(1 - 2x) \) is \( \{ 1 - x_{n,n}(\alpha, \beta) \}/2 \). Then, by the preceding result and by the limit (6.4) we obtain
\begin{equation}
\frac{q_{n,1}(\alpha, \beta, N) + (1 + \alpha)/2}{N + (\alpha + \beta + 2)/2} \leq \frac{1 - x_{n,n}(\alpha, \beta)}{2}.
\end{equation}

This inequality implies the upper bound in the theorem.

Now, for \( n \) fixed, we work with the sequence of polynomials \( \hat{Q}_k(x; \alpha, \beta, N) \), \( k = 0, \ldots, n - 1 \), where
\begin{equation}
\hat{Q}_k(x; \alpha, \beta, N) = Q_k \left( -\left( N + \frac{\alpha + \beta + 2}{2} \right) \left( x - \frac{n(n + \alpha)}{2n + \alpha + \beta} \right) - \frac{1 + \alpha}{2}; \alpha, \beta, N \right).
\end{equation}

Observe that the transformation in the argument depends on \( n \). Then the following limit relation
\begin{equation}
\lim_{N \to \infty} \hat{Q}_k^{(\alpha, \beta)}(x; N) = \frac{P_k^{(\alpha, \beta)}(1 + 2x - \frac{2n(n + \alpha)}{2n + \alpha + \beta})}{P_k^{(\alpha, \beta)}(1)}
\end{equation}
holds. Again, straightforward manipulations in the recurrence relation for Hahn polynomials imply that the entries of the Jacobi matrix of the orthonormal polynomials, corresponding $\hat{Q}_k(x; \alpha, \beta, N)$ are

\[
\hat{a}_k = \sqrt{\frac{4(k+1)(1+k+\alpha)(1+k+\beta)(N-k)(2+k+N+\alpha+\beta)}{(1+2k+\alpha+\beta)(2+2k+\alpha+\beta)^2(3+2k+\alpha+\beta)(2N+2+\alpha+\beta)^2}}
\]

and

\[
\hat{b}_k = \frac{k(1+k)(n-k)(k(2n+\alpha+\beta)+(1+\alpha)(n+\alpha+\beta)+n(1+\beta))}{(2n+\alpha+\beta)(2+2k+\alpha+\beta)},
\]

for $k = 0, \ldots, n-1$.

We have that $\hat{a}_k = \hat{a}_k(N)$ and $\hat{b}_k$ are nonnegative for $n \geq 2$. Furthermore, $\hat{b}_k$ do not depend of $N$ and the derivatives of $\hat{a}_k(N)$ with respect to $N$ are nonnegative. Then, by the Perron-Frobenius theorem, the largest zero of $\hat{Q}_n^{(\alpha, \beta)}(x; N)$ is an increasing function of $N$.

The largest zero of $\hat{Q}_n(x; \alpha, \beta, N)$ is

\[
-q_{n,n}(\alpha, \beta, N) - \frac{(1+\alpha)/2}{n+\alpha+\beta} + \frac{n(n+\alpha)}{2n+\alpha+\beta}
\]

and the largest zero of $P_n^{(\alpha, \beta)}(1 + 2x - \frac{2n(n+\alpha)}{2n+\alpha+\beta})$ is

\[
\frac{x_{n,1}(\alpha, \beta) - 1}{2} + \frac{n(n+\alpha)}{2n+\alpha+\beta}.
\]

Then the preceding result and by the limit \([6,5]\) imply

\[
-q_{n,n}(\alpha, \beta, N) - \frac{(1+\alpha)/2}{n+\alpha+\beta} + \frac{n(n+\alpha)}{2n+\alpha+\beta} \leq \frac{x_{n,1}(\alpha, \beta) - 1}{2} + \frac{n(n+\alpha)}{2n+\alpha+\beta}.
\]

This inequality yields the upper bound in the statement of the theorem. \(\square\)

We use two recent results on limits of zeros of Jacobi polynomials. The first one was obtained in [11, Theorem 1] and states that

\[
x_{n,n}(\alpha, \beta) \geq \frac{B - 4(n-1)\sqrt{\Delta}}{A},
\]

\[
x_{n,1}(\alpha, \beta) \leq \frac{B + 4(n-1)\sqrt{\Delta}}{A},
\]

where

\[
B = (\beta - \alpha)((\alpha + \beta + 6)n + 2(\alpha + \beta)),
\]

\[
A = (2n + \alpha + \beta)(n(2n + \alpha + \beta) + 2(2 + \alpha + \beta)),
\]

\[
\Delta = n^2(n + \alpha + \beta + 1)^2 + (\alpha + 1)(\beta + 1)(n^2 + (\alpha + \beta + 4)n + 2(\alpha + \beta)).
\]

Thus, we obtain

**Corollary 6.1.** Let $n, N \in \mathbb{N}$, with $n \leq N$, $\alpha, \beta > -1$ and let $A, B$ and $\Delta$ be defined as above. Then

\[
q_{n,n}(\alpha, \beta, N) \geq \left(N + \frac{\alpha + \beta + 2}{2}\right) \left(\frac{A - B - 4(n-1)\sqrt{\Delta}}{2A}\right) - \frac{1 + \alpha}{2}
\]
and

\begin{equation}
q_{n,1}(\alpha, \beta, N) \leq \left( N + \frac{\alpha + \beta + 2}{2} \right) \left( \frac{A - B + 4(n - 1)\sqrt{A}}{2A} \right) - \frac{1 + \alpha}{2}.
\end{equation}

We employ the following limits for the zeros of the Jacobi polynomials, obtained by Krasikov [25, Theorem 2]:

\begin{align*}
x_{n,n}(\alpha, \beta) &\geq C + 3 \frac{(1 - C^2)^{2/3}}{(2R)^{1/3}}, \\
x_{n,1}(\alpha, \beta) &\leq D - 3 \frac{(1 - D^2)^{2/3}}{(2R)^{1/3}} + \frac{4(\alpha - \beta)(\alpha + \beta + 2)}{((2n + \alpha + \beta + 1)^2 + 2\alpha + 2\beta + 3)^{3/2}},
\end{align*}

where

\begin{align*}
R &= \sqrt{((2n + \alpha + \beta + 1)^2 - (\alpha - \beta)^2 + 2\alpha + 2\beta + 3)(4n^2 + 4n(\alpha + \beta + 1))}, \\
C &= -(R + (\alpha - \beta)(\alpha + \beta + 2))(2n + \alpha + \beta + 1)^2 + 2\alpha + 2\beta + 3)^{-1}, \\
D &= (R - (\alpha - \beta)(\alpha + \beta + 2))(2n + \alpha + \beta + 1)^2 + 2\alpha + 2\beta + 3)^{-1}.
\end{align*}

These yield:

\begin{enumerate}
\item \textbf{Corollary 6.2.} Let \( n, N \in \mathbb{N} \), with \( n \leq N \) and \( \alpha, \beta > -1 \). Let \( C, D \) and \( R \) be defined as above. Then,

\begin{align*}
q_{n,1}(\alpha, \beta, N) &\leq \left( N + \frac{\alpha + \beta + 2}{2} \right) \left( \frac{2R^{1/3} - 2CR^{1/3} - 3(1 - C^2)^{2/3}}{2(2R)^{1/3}} \right) - \frac{1 + \alpha}{2}, \\
q_{n,n}(\alpha, \beta, N) &\geq \left( N + \frac{\alpha + \beta + 2}{2} \right) \left( \frac{2R^{1/3} - 2DR^{1/3} + 3(1 - D^2)^{2/3}}{2(2R)^{1/3}} \right) - \frac{4(\alpha - \beta)(\alpha + \beta + 2)}{2((2n + \alpha + \beta + 1)^2 + 2\alpha + 2\beta + 3)^{3/2}} - \frac{1 + \alpha}{2}.
\end{align*}
\end{enumerate}

The figures below illustrate our results.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig1.png}
\caption{Zeros of \( Q_5(x;10,2,N) \) with the limits of the Corollary}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig2.png}
\caption{Zeros of \( Q_5(x;\alpha,10,30) \) with the limits from Corollary}
\end{figure}
Finally, we analyse these results, comparing them with the limits obtained by Krasikov and Zarkh \cite[Theorem 10]{Krasikov2001} who proved that, if

\[
V_1 = \sqrt{n(n + \alpha + \beta + 1) + (1 + \alpha)(1 + \beta)},
\]

\[
V_2 = \sqrt{n(n + \alpha + \beta + 1)((n + \alpha + \beta + 2)N - n(n + \alpha + \beta + 1))},
\]

\[
U_1 = N((n + \alpha + \beta + 2) - N) - 2n(n + \alpha + \beta + 1),
\]

\[
U_2 = (n + \alpha + \beta + 2)((n + \alpha + \beta + 2) - N) + 2n(n + \alpha + \beta + 1),
\]

then

\[
q_{n,n}(\alpha, \beta, N) < \frac{S_1 - 2V_1 V_2}{S_2} - \frac{3}{2S_2} \left( \frac{((\alpha - \beta)V_2 - U_1 V_1)^2((\alpha - \beta)V_2 - U_2 V_1)^2}{V_1 V_2 S_2} \right)
\]

provided that \( n(n + \alpha + \beta + 1) < (1 + \beta)N \), and

\[
q_{n,1}(\alpha, \beta, N) > \frac{S_1 - 2V_1 V_2}{S_2} + \frac{3}{2S_2} \left( \frac{((\alpha - \beta)V_2 + U_1 V_1)^2((\alpha - \beta)V_2 + U_2 V_1)^2}{5V_1 V_2 S_2} \right)
\]

provided that \( n \geq 5 \) and \( n(n + \alpha + \beta + 1) < (1 + \alpha)N \).

Obviously, these limits are valid only when, roughly speaking, \( n < \sqrt{(1 + \beta)N} \) or \( n < \sqrt{(1 + \alpha)N} \).

The results in the above corollaries are better than (6.8) and (6.9) when \( N \) is large enough. In particular, in the case \( \alpha = \beta = 0 \) which corresponds with the discrete Chebyshev polynomials in the terminology of \cite{Szeg} or Gram polynomials in the terminology of \cite{Gonchar2000}. When \( \alpha \to \infty \) the lower limit (6.8) is not applicable because the restriction \( n(n + \alpha + \beta + 1) < (1 + \beta)N \) is not satisfied and the upper limit (6.9) is better than the one given in (6.7) only for a very restricted part of the \((n, \beta)\) plane. Similarly, when \( \beta \to \infty \), the restriction \( n(n + \alpha + \beta + 1) < (1 + \beta)N \) fails to hold, so that the lower limit (6.8) cannot be used. In this case the upper limit (6.6) is better than (6.8) for almost all values of \( n \) and \( \alpha \). The tables below exemplify some of these questions.

### Table 6.1. Table of the extreme zeros of \( Q_5(x; 10, 2, N) \) and the limits (6.8), (6.9), (6.6) and (6.7).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( (6.8) )</th>
<th>( (6.6) )</th>
<th>( q_{5,5} )</th>
<th>( q_{5,1} )</th>
<th>( (6.7) )</th>
<th>( (6.9) )</th>
</tr>
</thead>
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<tr>
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<td>4.9975</td>
<td>5.8174</td>
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<td></td>
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<tr>
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<td>1.5604</td>
<td>9.9130</td>
<td>10.5330</td>
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<td>34.2895</td>
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<td>95.4133</td>
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<tr>
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<td>175.6412</td>
<td>182.5365</td>
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</table>
Table 6.2. Table of the extreme zeros of $Q_5(x; \alpha, 10, 30)$ and the limits (6.8), (6.9), (6.6) and (6.7).

<table>
<thead>
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<th>$\alpha$</th>
<th>(6.8)</th>
<th>(6.9)</th>
<th>$q_{5.5}$</th>
<th>$q_{5.1}$</th>
<th>(6.7)</th>
<th>(6.9)</th>
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<tr>
<td>-0.5</td>
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<td>19.0850</td>
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<td>5</td>
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<tr>
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<tr>
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<td>18.4064</td>
<td>21.3024</td>
<td>29.9415</td>
<td>32.1985</td>
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</tr>
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</table>

Table 6.3. Table of the extreme zeros of $Q_5(x; 10, \beta, 30)$ and the limits (6.8), (6.9), (6.6) and (6.7).

<table>
<thead>
<tr>
<th>$\beta$</th>
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<th>(6.9)</th>
<th>$q_{5.5}$</th>
<th>$q_{5.1}$</th>
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<th>(6.9)</th>
</tr>
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<tr>
<td>-0.5</td>
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</tbody>
</table>

Table 6.4. Table of the extreme zeros of Gram or discrete Chebyshev polynomials $Q_5(x; 0, 0, N)$ and the limits (6.8), (6.9), (6.6) and (6.7).

<table>
<thead>
<tr>
<th>$N$</th>
<th>(6.8)</th>
<th>(6.9)</th>
<th>$q_{5.5}$</th>
<th>$q_{5.1}$</th>
<th>(6.7)</th>
<th>(6.9)</th>
</tr>
</thead>
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<td></td>
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</table>

References


ZEROS OF CLASSICAL ORTHOGONAL POLYNOMIALS

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