**Generalized projective synchronization in time-delayed systems: Nonlinear observer approach**

Dibakar Ghosh

Department of Mathematics, Dinabandhu Andrews College, Garia, Calcutta-700 084, India

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In this paper, we consider the projective-anticipating, projective, and projective-lag synchronization in a unified coupled time-delay system via nonlinear observer design. A new sufficient condition for generalized projective synchronization is derived analytically with the help of Krasovskii–Lyapunov theory for constant and variable time-delay systems. The analytical treatment can give stable synchronization (anticipatory and lag) for a large class of time-delayed systems in which the response system’s trajectory is forced to have an amplitude proportional to the drive system. The constant of proportionality is determined by the control law, not by the initial conditions. The proposed technique has been applied to synchronize Ikeda and prototype models by numerical simulation. © 2009 American Institute of Physics. [DOI: 10.1063/1.3054711]

In 1990, Pecora and Carroll\(^1\) first observed the synchronization of chaotic dynamical systems, and chaos synchronization has become a topic of great interest. Different kinds of synchronization phenomena have been discovered, such as complete synchronization, phase synchronization, lag synchronization, generalized synchronization, and so on. While most studies focused on these types of chaos synchronization in various fields, little attention has been paid to projective synchronization. In 1998, Gonzalez–Miranda\(^2\) observed that when chaotic systems exhibit invariance properties under a special type of continuous transformation, amplification and displacement of the attractor occurs. This degree of amplification or displacement obtained is smoothly dependent on the initial condition. Later, in 1999, Mainieri and Rehacek\(^3\) observed projective synchronization in partially linear three-dimensional chaotic systems. This paper proposes a new analytical treatment of stable projective synchronization (anticipating and lag) algorithm based on nonlinear observer design without the limitation of partial linearity. The scaling factor \(\alpha\) is determined by the control law, not the initial conditions. It is proved that nonlinear observer approach provides a unified treatment for synchronization in chaotic and hyperchaotic systems. Therefore, nonlinear observer approach provides a unified treatment for generalized projective synchronization of a large class of time-delayed systems with constant and variable time delay. Numerical simulations are proved to verify the effectiveness of the proposed scheme.

I. INTRODUCTION

Synchronization between two dynamical system has stimulated a wide range of research activity.\(^1\) The phenomena of synchronization in coupled systems have been extensively studied in the context of laser dynamics, electronics circuits, chemical, and biological systems.\(^4\) Over the past decade, following complete synchronization,\(^1,5\) several new types of synchronization have been found in interesting chaotic systems, such as generalized synchronization (GS),\(^6\) phase synchronization,\(^7\) lag synchronization,\(^8\) anticipatory synchronization,\(^9\) antiphase synchronization,\(^10\) etc.

Among all kinds of chaos synchronization, projective synchronization, characterized by a scaling factor that two systems synchronize proportionally, is one of the most interesting problems. Gonzalez–Miranda\(^2\) observed that when chaotic systems exhibit invariance properties under a special type of continuous transformation, amplification and displacement of the attractor occurs. This degree of amplification or displacement obtained is smoothly dependent on the initial condition. By this definition, complete synchronization is not a special case of projective synchronization. Mainieri and Rehacek,\(^3\) in 1999, called this type of synchronization projective synchronization, and declared that the two identical systems could be synchronized up to a scaling factor \(\alpha\). The scaling factor is a constant transformation between the synchronized variables of the master and slave systems. In Refs. 2 and 3, projective synchronization occurred because the response system had neutral stability, i.e., the largest conditional Lyapunov exponent was zero, so it followed a trajectory that was proportional to the drive system attractor with a scaling factor \(\alpha\) which was determined by initial conditions. Projective synchronization is not in the category of GS because the slave system of projective synchronization is not asymptotically stable. The response system attractor possesses the “same topological characteristic (such as Lyapunov exponents and fractal dimensions)” as the slave system attractor.\(^11\)

Recently, Hoang et al.\(^12\) proposed a new type of synchronization in multia delay feedback systems, which is called projective-anticipatory synchronization. This synchronization is a combination of projective and anticipatory synchronization, and the states of master and slave are related by \(\dot{y}(t) = b x(t + \tau)(\tau > 0)\), where \(a\) and \(b\) are nonzero real num-
bers. Projective-lag synchronization is defined as $\dot{y} = f(y, y_\tau) + u(x, y)$.

Projective synchronization is interesting because of its proportionality between the synchronized dynamical states. In applications to secure communications, this feature can be used to M-nary digital communication for achieving fast communication. Recently, Li et al. proposed generalized projective synchronization between two different chaotic systems using combination of active control and backstepping method. Very recently, Grassi and Miller introduced nonlinear observer design to and discrete-time systems via a linear observer. In Ref. 15, Grassi and Mascolo proposed nonlinear observer design to synchronize hyperchaotic systems. An observer is a dynamic system designed to be driven by the output of another dynamic system (plant) and having the property that the state of the observer converges to the state of the plant.

In the present paper, projective-anticipating, projective, and projective-lag synchronization in a unidirectional coupled time-delay system is investigated using nonlinear observer design. The proposed technique is based on nonlinear control theory and it can be successfully applied to a wide class of time delayed systems. Recently, chaotic time delay system has been suggested as a good candidates for secure communication. It is proved that low dimensional chaotic systems do not ensure a sufficient level of security for communications, as the associated chaotic attractors can be reconstructed with some effort and the hidden message can be retrieved by an attacker. In this regard, the time-delayed system received a lot of attention, i.e., $\dot{x} = f(x(t), x(t-\tau))$, where $\tau$ is a delay time. With increases in time delay $\tau$, the system becomes more complex, the number of positive Lyapunov exponents increases, and the system eventually transits to hyperchaos. In a recent study, it was discussed that a time-delayed system is still vulnerable for communication because time delay $\tau$ can be exposed by several measures: e.g., autocorrelation, filling factor, one step prediction error, average mutual information, etc. If the delay time $\tau$ is known, the time-delayed system becomes quite simple and the message encoded by the chaotic signal can be extracted by the common attack methods. From the above point of view, we can see that the study of projective-anticipating, projective, or projective-lag synchronization in a variable time-delay system is of high practical importance.

The organization of the remaining part is as follows: In Sec. II, some definitions and remarks used in this paper are presented. A general approach for projective synchronization for constant and variable time-delay systems via nonlinear observer are presented by Propositions 1 and 2. In Sec. III, two well-known systems (Ikeda and prototype) are considered to verify the effectiveness of the proposed scheme. Finally, conclusions are made in the last section.

II. NONLINEAR OBSERVER-BASED PROJECTIVE SYNCHRONIZATION SCHEMES

Consider the coupled time-delay system as

$$\dot{x} = f(x, x_\tau),$$

where $x, y \in \mathbb{R}^n, f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear vector field, $u(x, y)$ is the control term, and $x_\tau = x(t-\tau_1)$.

A definition of projective synchronization is presented below.

Definition 1: Let there be two chaotic time-delayed systems with initial conditions $x(0)$ and $y(0)$, whose trajectories are described by the time-delayed systems (1) and (2). We can state that systems (1) and (2) are synchronized in the sense of projective-anticipating, projective, or projective-lag according as $\tau < 0$, $\tau = 0$ or $\tau > 0$, respectively, if the dynamical behavior in which the amplitude of the masters state variable and that of the slave’s synchronizes up to a constant scaling factor $\alpha$, i.e.,

$$e(t) = (y(t) - \alpha x(t-\tau)) \rightarrow 0 \text{ as } t \rightarrow \infty,$$

where $e$ represents the synchronization error.

Definition 2: A projective observer is a dynamic system in which the state of the master is proportional by a constant scaling factor $\alpha$ with the state of the slave’s. Let the output of system (1) be $z = s(x, x_\tau)$. The dynamic system

$$\dot{y} = f(y, y_\tau) + g(z - s(y, y_\tau))$$

is then said to be nonlinear projective (anticipating, exact, or lag) observer of system (1) if its state $y \rightarrow \alpha x(t-\tau)$ as $t \rightarrow \infty$.

Moreover, system (4) is said to be global projective (anticipating, exact, or lag) observer of system (1) if $y \rightarrow \alpha x(t-\tau)$ as $t \rightarrow \infty$ for any initial condition $x(0)$ and $y(0)$.

The synchronization manifold of systems (1) and (2) is

$$y = \alpha x.$$  

Remark 1: System (4) is a global projective synchronization if the error system

$$\dot{e} = \dot{y} - \alpha \dot{x}(t-\tau)$$

$$= f(y, y_\tau) + g(z - s(y, y_\tau)) - \alpha f(x, x_\tau)$$

$$= f(e + \alpha x)x_\tau + \alpha x_\tau$$

$$+ g(s(x, x_\tau) - s(e + \alpha x)x_\tau + \alpha x_\tau) - \alpha f(x, x_\tau)$$

$$= \alpha h(e, e_\tau)$$

has a (globally) asymptotically stable equilibrium point for $e=0$.

Assumption: We consider the dynamic (1) in the form

$$\dot{x} = Ax + Bp(x_\tau),$$

where $p(x)$ is a nonlinear function of $x$, characterizing the system, e.g., $p(x)=\sin^2(x-x_0)$ for sine-squared model, $p(x)=\alpha x/(1+x^2)$ for the Mackey–Glass model, $p(x)=\pi b-\sin^2(x-x_0)$ for the Vallee model, $p(x)=\sin x$ for Ikeda model, etc.

Proposition 1: Given system (6), let $z = s(x, x_\tau)$ and $\dot{z} = p(x_\tau) + kx$ be the synchronizing signal ($k$ is the coupling strength) and let

$$\dot{y} = f(y, y_\tau) + u(x, y).$$

where $x, y \in \mathbb{R}^n, f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear vector field, $u(x, y)$ is the control term, and $x_\tau = x(t-\tau_1)$.

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$$e(t) = (y(t) - \alpha x(t-\tau)) \rightarrow 0 \text{ as } t \rightarrow \infty,$$

where $e$ represents the synchronization error.

Definition 2: A projective observer is a dynamic system in which the state of the master is proportional by a constant scaling factor $\alpha$ with the state of the slave’s. Let the output of system (1) be $z = s(x, x_\tau)$. The dynamic system

$$\dot{y} = f(y, y_\tau) + g(z - s(y, y_\tau))$$

is then said to be nonlinear projective (anticipating, exact, or lag) observer of system (1) if its state $y \rightarrow \alpha x(t-\tau)$ as $t \rightarrow \infty$ for any initial condition $x(0)$ and $y(0)$.

The synchronization manifold of systems (1) and (2) is

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Remark 1: System (4) is a global projective synchronization if the error system

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has a (globally) asymptotically stable equilibrium point for $e=0$.

Assumption: We consider the dynamic (1) in the form

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Proposition 1: Given system (6), let $z = s(x, x_\tau)$ and $\dot{z} = p(x_\tau) + kx$ be the synchronizing signal ($k$ is the coupling strength) and let

$$\dot{y} = f(y, y_\tau) + u(x, y).$$
be a function in Eq. (4). Projective synchronization then occurs if

$$\begin{align*}
g(z - s(y, y_{\tau_1})) &= B[s(ax_{\tau}, ax_{\tau+\tau_1}) - s(y, y_{\tau_1})] \quad (7)
\end{align*}$$

under the condition $\mu_2 = \alpha \mu_1$.

**Proof:** We consider the coupled system as

$$\begin{align*}
\dot{x} &= Ax + \mu_1 Bp(x_{\tau_1}), \\
\dot{y} &= Ay + \mu_2 Bp\left(\frac{y_{\tau_1}}{\alpha}\right) + g(z - s(y, y_{\tau_1})).
\end{align*}$$

The error dynamic is, then,

$$\begin{align*}
\dot{e} &= \dot{y} - \dot{x}(t - \tau) \\
&= Ay + \mu_2 Bp\left(\frac{y_{\tau_1}}{\alpha}\right) + g[s(ax_{\tau}, ax_{\tau+\tau_1}) - s(y, y_{\tau_1})] \\
&\quad - \alpha Ax_{\tau} - \alpha \mu_1 Bp(x_{\tau+\tau_1}) \\
&= A\varepsilon + \mu_2 Bp\left(\frac{y_{\tau_1}}{\alpha}\right) - \alpha \mu_1 Bp(x_{\tau+\tau_1}) + B[p(ax_{\tau+\tau_1}) \\
&\quad + ax_{\tau} - p(y_{\tau_1}) - ky].
\end{align*}$$

According to Eq. (5), the function $p(y_{\tau_1}/\alpha)$ can be written as $p(x_{\tau+\tau_1})$; then

$$\begin{align*}
\dot{e} &= (A - kB)e + \mu_2 Bp(x_{\tau+\tau_1}) - \alpha \mu_1 Bp(x_{\tau+\tau_1}) \\
&\quad - Bp'(ax_{\tau+\tau_1})e(t - \tau_1) \\
&= -(kB - A)e - Bp'(ax_{\tau+\tau_1})e(t - \tau_1),
\end{align*}$$

where

The error dynamics can be written as

$$\begin{align*}
\dot{e} &= -r(t)e + s(t)e(t - \tau),
\end{align*}$$

where $r(t) = kB - A$ and $s(t) = -Bp'(ax_{\tau+\tau_1})$.

Consider a positive definite Lyapunov functional of the form

$$V(t) = \frac{1}{2}e^2(t) + \zeta(t) \int_{-\tau}^{0} e^2(t + \zeta) d\zeta,$$

where $\zeta(t) > 0$ for any time.

The solution $e = 0$ is stable if the derivative of the functional $V(t)$ along the error equation, i.e.,

$$\dot{V}(t) = -r(t)e^2 + s(t)e\zeta(t) + \zeta(t)e^2 - \zeta(t)\zeta^2,\frac{e^2(t)}{2},$$

is negative. $\dot{V}$ will be negative if $4(r - \zeta(t)) > s^2$ and $r > \zeta(t) > 0$. The asymptotic stability condition for $e = 0$ is given for

$$r(t) > \frac{|s(t)|}{\sqrt{1 - \zeta^2(t)}} \quad \text{with} \quad \zeta(t) = \frac{|s(t)|}{2\sqrt{1 - \zeta^2(t)}}.$$
successfully applied to a wide class of time-delay systems with constant and variable time delay.

III. ILLUSTRATIVE EXAMPLES

Two examples will be used to illustrate the effectiveness of the obtained results. We consider the two well-known chaotic time-delay systems—Ikeda system and prototype delay system—and their numerical simulations are performed.

A. Example 1: Ikeda system with constant and variable time delay

To verify the above analytic calculation in numerically, we consider unidirectional coupled Ikeda system\(^\text{27}\) as

\[
\dot{x} = -ax + \mu_1 m_1 \sin(x_{\tau_1}),
\]

\[
\dot{y} = -ay + \mu_2 m_2 \sin\left(\frac{y_{\tau_1}}{\alpha}\right) + m_1 [\sin(\alpha x_{\tau_1}) + k \alpha x - \sin(y_{\tau_1}) - ky].
\]

(14)

Physically, \(x\) is the phase lag of the electric field across the resonator, \(a\) is the relaxation coefficient for the dynamical variable, and \(m_1\) is the laser intensity injected into the system. \(\tau_1\) is the round-trip time of the light in the resonator or feedback delay time in the coupled systems.\(^\text{30}\) The Ikeda model was introduced to describe the dynamics of an optical bistable resonator and is well known for delay-induced chaotic behavior.\(^\text{30}\)

System (13) is chaotic for the set of parameter values\(^\text{17}\) \(a=1.8, m_1=6.0, \) and \(\tau_1=2.0.\) For \(\tau=0, \mu_1=2.0, \mu_2=3.0,\) \(\alpha=1.5, \) and \(k=2,\) which satisfied the necessary and sufficient condition (8) for exact projective synchronization is shown in Fig. 1(a), and the corresponding projective synchronization manifold is in Fig. 1(b). The phase angle between the

![Graph](image-url)

FIG. 1. (a) Projective synchronization of \(x(t)\) (solid line) and \(y(t)\) (dotted line) for \(\alpha=1.5.\) (b) Corresponding projective synchronization manifold. (c) Antiphase projective synchronization of \(x(t)\) (solid line) and \(y(t)\) (dotted line) for \(\alpha=-1.5.\) (d) Corresponding projective synchronization error.
synchronized chaotic attractors is zero, with an identical pattern. For the opposite sign of scaling factor \( \alpha = -1.5 \), we see antiphase projective synchronization in Fig. 1(c). In Fig. 1(d), the antiphase projective synchronization error is shown. Figure 2(a) shows the attractor of the drive system (13), whereas for \( \alpha = 1.5 \), Fig. 2(b) shows that the attractor of the response system (14) has been scaled by 1.5.

For \( \tau = 3 \), projective-lag synchronization occurs. The time series of \( x(t) \) and \( y(t) \) are shown in Fig. 3(a) and the corresponding projective-lag synchronization manifold is shown in Fig. 3(b). In Fig. 3(a), it is shown that the lag time between drive and response is 3.0. The relation between the states of drive and response is \( y(t) = 1.5x(t-3) \). At this position, for opposite sign of scaling factor (\( \alpha = -1.5 \)), the antiphase projective-lag synchronization with antiphase pattern, and difference \( \pi \) of phase angles of the synchronized trajectories, are shown in Fig. 3(c). In Fig. 3(d), antiphase projective-lag synchronization error is shown. At the same time, it is observed that the amplitudes of drive state \( x(t) \) and response state \( y(t) \) are related by the relation \( y(t) = 1.5x(t-3) \), and that the slope of this line is \(-1.5\).

If \( \tau = -3 < 0 \), one can observe the projective-anticipating synchronization for \( k = 2.0 \) and other parameters as before. In Fig. 4(a), we can observe that the driven system anticipates the driver. At the same time, the amplitude of \( x(t+3) \) and \( y(t) \) are related by \( y(t) = 1.5x(t+3) \). Figure 4(b) shows their corresponding manifold. It is also clear that the slope of this manifold is 1.5.

For time delay modulation, we set the parameter values as \( \tau_0 = 2.0, a_1 = 0.5, \omega_1 = 0.02, \tau_0 = 3.0, a_0 = 0.2, \) and \( \omega_0 = 0.1 \). At this parameter values feedback delay varies from 1.5 to 2.5 and coupling delay from 3.2 to 3.54. Figure 5(a) shows the lag synchronization between Eqs. (13) and (14) for coupling \( k = 2.5 \), which satisfied the necessary and sufficient condition (12). For \( \alpha = 1.5 \), the projective-lag synchronization is shown in Fig. 5(b). For opposite sign of \( \tau_0 = -3.0, a_0 = -0.2 \), we get anticipatory synchronization in Fig. 5(c). In Fig. 5(d), projective-anticipating synchronization is shown for \( \alpha = 1.5 \).

### B. Example 2: Prototype system with constant and variable time delay

In this example, we consider the projective synchronization of two unidirectional coupled prototype systems. The delay differential equation describing the drive system is

\[
\dot{x} = \delta x(t - \tau_1) - \epsilon [x(t - \tau_1)]^3, \tag{15}
\]

where \( \delta \) and \( \epsilon \) are positive system parameters, and \( \tau_1 \) is the time delay. This system is used to observe self-oscillations in the shipbuilding industry. The chaotic behavior of system (15) for constant time delay are studied by Ucar.\(^\text{30}\) The multistability of periodic orbits and fixed points present in this system for higher value of delay. The system is in a chaotic state for the parameter values \( \delta = 1.0, \epsilon = 1.0, \text{ and } \tau_1 = 1.6 \). In Ref. 16, we replaced the time-delay parameter \( \tau_1 \) as a function of time instead of constant delay as in Eq. (11). The system (15) is chaotic for the parameter values \( \tau_{10} = 1.6, a_1 = 0.26, \) and \( \omega_1 = 0.8 \). The drive and response systems are written as
\[
\dot{x} = \mu_1 (\delta x_{t+\tau} - \varepsilon x_{t+\tau}^3),
\]

\[
\dot{y} = \mu_2 \left( \frac{\delta y_{t+\tau}}{\alpha} - \frac{\varepsilon y_{t+\tau}^3}{\alpha} \right) + \alpha \delta x_{t+\tau} - \alpha^3 \varepsilon x_{t+\tau}^3 + k \alpha x_{t-\tau} - \delta y_{t-\tau} + \varepsilon y_{t-\tau}^3 - ky.
\]

When coupling delay \( \tau = 0 \), projective synchronization is achieved, whereas for \( \alpha = 2.0 \), exact projective synchronization is observed, and for \( -2.0 \), antiphase projective synchronization is obtained. In Fig. 6(a), the chaotic attractor of system (15) is reported. For \( \alpha = 2.0 \), exact projective synchronization occurs and the attractor of the response system (17) has been scaled by twice than that the attractor of drive system (16) [Fig. 6(b)]. Antiphase projective synchronization with twice the scale for \( \alpha = -2.0 \) is shown in Fig. 6(c). Similar results are obtained for other values of scaling factor \( \alpha \).

For \( \tau_0 = 2.0 \), \( a_0 = 0.5 \), and \( \omega_0 = 0.02 \), i.e., \( \tau(t) > 0 \), Fig. 7(a) shows the time history of the exact projective-lag synchronization between the drive and response systems for \( \alpha = 1.0 \) and \( k = 6 \). For scaling factor \( \alpha = 2.0 \), we get projective-lag synchronization, which is shown in Fig. 7(b). We choose the parameter values \( \tau_0 = -2.0 \) and \( a_0 = -0.5 \), so that \( \tau(t) < 0 \). In Fig. 7(c), projective-anticipating synchronization between Eqs. (16) and (17) are shown. It is proved that the projective synchronization (anticipating and lag) are shown for any values of \( \alpha \).
FIG. 4. (a) Projective-anticipating synchronization of $x(t)$ (solid line) and $y(t)$ (dotted line) for $\alpha=1.5$. (b) Corresponding projective-anticipating synchronization manifold.

FIG. 5. For variable time delay, (a) lag synchronization of $x(t)$ (solid line) and $y(t)$ (dotted line) for $k=2.5$, (b) projective-lag synchronization when $\alpha=1.5$, (c) anticipatory synchronization of $x(t)$ (solid line) and $y(t)$ (dotted line), and (d) projective-anticipating synchronization between systems (13) and (14).
IV. CONCLUSIONS

In conclusion, we have analytically calculated the necessary and sufficient conditions for projective-anticipating, projective, and projective-lag synchronization of general class of time-delay system with constant and modulated delay time. We derived that the transition of projective-anticipating, projective, and projective-lag synchronization manifold can be achieved by adjusting the sign of coupling delay. Compared with other works, our work has the following advantages. (a) The early works of projective synchronization reported that the projective synchronization was usually observed only in the coupled partially linear systems. Although in previous studies projective synchronization occurred because the response system had neutral stability, i.e., the largest conditional Lyapunov exponent was zero, so it followed a trajectory that was proportional to the drive system attractor. The constant of proportionality is dependent on the initial condition. We realized stable projective synchronization (i.e., all the conditional Lyapunov exponents are negative) in time-delayed systems with constant and modulated delay times without the limitation of partial linearity. The scaling factor is determined by the control law, not the initial conditions. (b) The method can be easily implemented using simple analog circuit in practical communication applications. (c) This method is systematic and can be applied to a wide class of time-delayed systems with constant and modulated delay time. In application to secure communications, projective synchronization can be used to extend binary digital to M-nary digital communication for achieving fast communication, so the research on projective synchronization has important theory significance and important value. Furthermore, projective synchronization has been applied in the research of secure communication due to the unpredictability of the scaling factors. The effectiveness and feasibility of our method have been verified by computer implementation.

FIG. 6. Chaotic attractors of (a) drive system (16) and response system (17) for (b) $\alpha=2.0$ and (c) $\alpha=-2.0$.

FIG. 7. (a) Exact projective synchronization of $x(t)$ (solid line) and $y(t)$ (dotted line) for $\alpha=2.0$. (b) Projective-lag synchronization for $\alpha=2.0$. (c) Projective-anticipating synchronization for $\alpha=2.0$. 
simulation of Ikeda and prototype systems, which can also be used for other time-delayed systems.

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