Many valued lattices and their representations

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Abstract

This paper presents an investigation of many valued lattices from the point of view of enriched category theory. For a bounded partially ordered set \( P \), the conditions for \( P \) to become a lattice can be postulated as existence of certain adjunctions. Reformulating these adjunctions, by aid of enriched category theory, in many valued setting, two kinds of many valued lattices, weak \( \Omega \)-lattices and \( \Omega \)-lattices, are introduced. It is shown that the notion of \( \Omega \)-lattices coincides with that of lattice fuzzy orders of Bělohlávek; and the notion of weak \( \Omega \)-lattices coincides with that of vague lattices of Demirci.

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1. Introduction

Partial order is an important mathematical structure and is useful in many areas. Because of its usefulness, many people have endeavored to extend basic notions in order theory to the many valued setting since the inception of fuzzy sets in 1965, see [2–4,6,9,13,20,24] for example.

A lattice is a partially ordered set in which any two elements have both a supremum and an infimum. A classic text for the theory of lattices is Birkhoff [5]. A lattice can be described in many equivalent ways. We list several of these characterizations here. At first we need some notations from order theory.

A subset \( U \) in a partially ordered set \( L \) is called an upper set if \( x \in U \) and \( x \leq y \) implies \( y \in U \). For each subset \( A \), let \( \uparrow A = \{ x \in L : x \geq a \text{ for some } a \in A \} \). Clearly, \( \uparrow A \) is the smallest upper set which contains \( A \). An upper set \( U \) is called finitely generated if \( U = \uparrow A \) for some finite set \( A \subset L \). Dually, one can define finitely generated lower sets.

Write \( FU(L) \) for the set of all the finitely generated upper sets in \( L \) and \( FL(L) \) for the set of all the finitely generated lower sets in \( L \). Then both \( FU(L) \) and \( FL(L) \) become partially ordered sets under the inclusion order.

Given a partially ordered set \( L \) with a top element and a bottom element, the following conditions are equivalent:

(Lat1) Any two elements in \( L \) have both a supremum and an infimum.
(Lat2) Any finite subset of \( L \) have both a supremum and an infimum.
(Lat3) The diagonal \( d : L \to L \times L, \quad d(a) = (a, a) \), has both a left adjoint and a right adjoint.
(Lat4) The function \( y' : L \to FU(L)^{op} \) given by \( y'(x) = \uparrow \{ x \} \) has a right adjoint and the function \( y : L \to FL(L) \) given by \( y(x) = \downarrow \{ x \} \) has a left adjoint.

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An advantage of (Lat3) and (Lat4) is that these two conditions express the notion of lattice as categorical properties, i.e., existence of certain adjunctions.

Let \( \Omega = (\Omega, I, \ast) \) be a commutative unital quantale. That is to say, \( \Omega \) is a complete lattice, \( I \) is a fixed element in \( \Omega \), and \( \ast : \Omega \times \Omega \to \Omega \), called the tensor, is a commutative, associative binary operation such that (1) \( \ast \) is monotone on each variable; (2) \( I \) is a unit element for \( \ast \), i.e., \( \ast \ast I = I \) for every \( \ast \in \Omega \); and (3) for each \( \ast \in \Omega \), the monotone function \( \ast \ast (\ast) : \Omega \to \Omega \) has a right adjoint \( \ast \to (-) : \Omega \to \Omega \) in the sense that \( \ast \ast \beta \leq \gamma \iff \beta \leq \ast \to \gamma \). The binary function \( \to : \Omega \times \Omega \to \Omega \) is called the cotensor.

Due to the adjunction \( \ast \ast \beta \leq \gamma \iff \beta \leq \ast \to \gamma \) between the tensor and the cotensor, \( \Omega \) can be employed to play the role of the set of truth values of a many valued logic, with the tensor \( \ast \) and the cotensor \( \to \) playing roles of the logic connectives \textit{conjunction} and \textit{implication} [17].

A many valued preorder, precisely an \( \Omega \)-preorder [2–4,9,13,20,24], on a set \( X \) is a function \( P : X \times X \to \Omega \) such that (1) (reflexivity) \( P(x, x) \geq I \) for all \( x \in X \) and (2) (transitivity) \( P(x, y) \ast P(y, z) \leq P(x, z) \).

Once the notion of a many valued preorder is available, it is natural to extend the notion of lattice to the many valued setting. Indeed, generalizing condition (Lat1), Demirci proposed the concept of vague lattices in [9]; and generalizing condition (Lat2), Bělohlávek introduced the concept of lattice fuzzy orders in [2,4]. However, as we shall see, the notions of lattice fuzzy order and vague lattice are not equivalent.

The aim of this paper is to generalize the conditions (Lat3) and (Lat4) to the many valued setting. That means, we shall investigate many valued order structures from the viewpoint of category theory. The idea is as follows. From the view-point of category theory, a commutative unital quantale is a symmetric monoidal closed category and a many-valued preorder can be regarded as an \( \Omega \)-category, i.e., a category enriched over \( \Omega \). Hence, the study of many valued orders is a special case of the theory of enriched categories [7,15,17,24]. Consequently, categorical ideas and techniques can be applied to the study of many valued orders.

The contents are arranged as follows.

In Section 2, we review some basic notions in category theory which shall be needed in this article. For expositions of the general theory of enriched categories we refer to [15,17]. The categories enriched over a commutative unital quantale considered in this article is a special case of categories enriched over quantaloids. The monograph of Rosenthal [21] and the recent papers of Stubbe [22,23] provide a nice systematic treatment of these matters.

In Section 3, generalizing the conditions (Lat3) and (Lat4) to the many valued setting, we introduce the concepts of weak \( \Omega \)-lattices and \( \Omega \)-lattices, respectively. Basic properties of these two kinds of many valued lattices are studied. The results are analogous to those in Stubbe [23] for complete \( \Omega \)-categories.

In the last section, we show that the notion of \( \Omega \)-lattices coincides with that of lattice fuzzy orders of Bělohlávek [2–4]; and the notion of weak \( \Omega \)-lattices coincides with that of vague lattices of Demirci [9]. This establishes a link between enriched category theory with mathematics of fuzzy sets. For more on the relationship between category and fuzzy sets, see Gottwald [11,12] and the bibliographies therein.

### 2. \( \Omega \)-categories and \( \Omega \)-functors

In this section, we review some basic notions in category theory which shall be needed in this article.

The following proposition collects some basic properties of commutative unital quantales, which can be found in many places, for instance, [2,14].

**Proposition 2.1.** (1) \( 0 \ast x = 0 \); \( I \to x = x \); \( 0 \to x = 1 \).

\( (2) x \to \beta = \vee \{ \gamma : x \ast \gamma \leq \beta \} \).

\( (3) (x \to \beta) \ast (\beta \to \gamma) \leq (x \to \gamma) \).

\( (4) x \to (\beta \to \gamma) = (x \ast \beta) \to \gamma = \beta \to (x \to \gamma) \).

\( (5) (x \to \beta) \to \beta = x \to \beta \).

\( (6) x \ast \bigvee_{j \in J} \beta_j = \bigvee_{j \in J} x \ast \beta_j \).

\( (7) (\bigvee_{j \in J} x_j) \to \beta = \bigwedge_{j \in J} (x_j \to \beta) \).

\( (8) x \to (\bigwedge_{j \in J} \beta_j) = \bigwedge_{j \in J} (x \to \beta_j) \).

If the unit element \( I \) coincides with the greatest element 1 in \( \Omega \), we call \( \Omega \) an integral commutative quantale.
Definition 2.2 (Lawvere [17]). A category enriched over $\Omega$, or an $\Omega$-category, is a set $A$ together with a binary function $\text{hom}: A \times A \rightarrow \Omega$ such that

1. $I \leq \text{hom}(a, a)$ for every $a \in A$;
2. $\text{hom}(a, b) \ast \text{hom}(b, c) \leq \text{hom}(a, c)$ for all $a, b, c \in A$.

The set $A$ is called the underlying set of $(A, \text{hom}(\cdot, \cdot))$, and hom$(\cdot, \cdot)$ is called the hom functor.

We often write $A$ for $(A, \text{hom}(\cdot, \cdot))$, $A(\cdot, \cdot)$ for hom$(\cdot, \cdot)$, and $|A|$ for the underlying set.

Remark 2.3. Given an $\Omega$-category $A$, if we interpret $A(a, b)$ as the degree to which $a$ precedes $b$, then the condition $I \leq \text{hom}(a, a)$ is to require that $A$ be reflexive and the condition $\text{hom}(a, b) \ast \text{hom}(b, c) \leq \text{hom}(a, c)$ is the transitivity of order relation. Therefore, $\Omega$-categories can be studied as many-valued preordered sets. For this reason, an $\Omega$-category $(A, \text{hom})$ is also called an $\Omega$-preordered set and hom is also called an $\Omega$-preorder on the underlying set. In this paper, we emphasize on the many valued order aspect of $\Omega$-categories.

Definition 2.4 (Lawvere [17]). An $\Omega$-functor between $\Omega$-categories $A$ and $B$ is a function $f: |A| \rightarrow |B|$ such that $A(a, b) \leq B(f(a), f(b))$ for all $a, b \in A$.

We can also explain the condition $A(a, b) \leq B(f(a), f(b))$ as $a \leq b$ in $A$ implies that $f(a) \leq f(b)$ in $B$. Therefore, $\Omega$-functors between $\Omega$-categories can also be regarded as monotone functions between many-valued preordered sets. Write $\Omega$-Cats for the category of all $\Omega$-categories and $\Omega$-functors.

An $\Omega$-functor $f$ is called an $\Omega$-isometry if $A(a, b) = B(f(a), f(b))$ for all $a, b \in A$. Note that an $\Omega$-isometry is called a fully faithful functor in enriched category theory [15]. The term $\Omega$-isometry stems from generalized metric spaces. Generalized metric spaces are categories enriched over $([0, \infty]^\mathbb{R}, +, 0)$, which are taken as a prototype for enriched categories in Lawvere [17]. For quantale enriched categories, the term "$\Omega$-isometry" seems more suggestive than the categorical term "fully faithful functor". If an $\Omega$-isometry $f$ is also bijective on the underlying sets, it will be called an $\Omega$-isomorphism, or an isomorphism for short.

Two elements $x$ and $y$ in an $\Omega$-category $A$ are said to be isomorphic if $A(x, y) \geq I$ and $A(y, x) \geq I$. An $\Omega$-category $A$ is called antisymmetric if different elements in $A$ are always non-isomorphic.

In the following examples we list some methods to construct $\Omega$-categories. The aim is to fix notations. These methods are standard in category theory and it is hard to find where they appeared for the first time, so, we do not include any reference here.

Example 2.5. (1) (The canonical $\Omega$-category structure on $\Omega$) Let $\Omega(x, y) = x \rightarrow y$. Then, by Proposition 2.1, it is easy to check that $\Omega$ is an antisymmetric $\Omega$-category.

(2) (Discrete $\Omega$-categories) Given a set $X$ and $x, y \in X$, let $X(x, y) = I$ if $x = y$ and $X(x, y) = 0$ if $x \neq y$. Then $X$ becomes an $\Omega$-category. Such $\Omega$-categories are called discrete since that for any $\Omega$-category $B$, every function from $X$ to $B$ is an $\Omega$-functor.

(3) (Dual $\Omega$-category) Suppose $A$ is an $\Omega$-category. Let $A^\text{op}(a, b) = A(b, a)$ for all $a, b \in A$. Then $A^\text{op}$ is also an $\Omega$-category, called the opposite of $A$.

(4) (Subcategory) Let $A$ be an $\Omega$-category and $B$ is a subset of $A$. For all $x, y \in B$, let $B(x, y) = A(x, y)$. Then $B$ becomes an $\Omega$-category, called a full subcategory of $A$.

(5) (Product category) Suppose $\{A_i : i \in J\}$ is a family of $\Omega$-categories, the product of $\{A_i : i \in J\}$ in the category $\Omega$-Cats is given by

$$\prod_{i \in J} A_i(a, b) = \bigwedge_{i \in J} A_i(a_i, b_i), \quad a = (a_i)_{i \in J}, \quad b = (b_i)_{i \in J}.$$ 

(6) (Functor category) Given $\Omega$-categories $A$ and $B$, denote the set of all the $\Omega$-functors from $A$ to $B$ by $[A, B]$. Let $[A, B](f, g) = \bigwedge_{x \in A} B(f(x), g(x))$ for all $f, g \in [A, B]$. Then $[A, B]$ becomes an $\Omega$-category, called the functor category from $A$ to $B$.

Suppose that $A$ is an $\Omega$-category. We define a binary relation $\leq$ on the underlying set of $A$ as follows: $a \leq b$ if and only if $A(a, b) \geq I$. It is easy to see that $\leq$ is a preorder, i.e., a reflexive and transitive relation, on the underlying
set of \( A \). For each \( \Omega \)-category, we write \( A_0 \) for the preordered set \((A, \leq)\). In this way, we obtain a forgetful functor
\((-)_0: \Omega-\text{Cats} \longrightarrow \text{PrOrd}\) from the category \( \Omega-\text{Cats} \) of \( \Omega \)-categories to the category \( \text{PrOrd} \) of preordered sets. Clearly, \( A \) is antisymmetric if and only if \( A_0 \) is a partially ordered set.

Suppose \( P \) is a (classical) preordered set. Then a monotone function \( \phi : P \longrightarrow 2 \) is essentially an upper set in \( P \) and a monotone function \( \phi : P^{\text{op}} \longrightarrow 2 \) is essentially a lower set in \( P \). Thus, given an \( \Omega \)-category \( A \), \( \Omega \)-functors
\( \phi : A \longrightarrow \Omega \) shall also be called upper \( \Omega \)-subsets of \( A \) and \( \Omega \)-functors \( \phi : A^{\text{op}} \longrightarrow \Omega \) lower \( \Omega \)-subsets of \( A \). It should be noted that in enriched category theory, an \( \Omega \)-functor \( \phi : A^{\text{op}} \longrightarrow \Omega \) is usually called a weight [1,16] or a presheaf [22,23].

Given an \( \Omega \)-category \( A \) and \( a \in A \), let \( y(a)(x) = A(x,a) \) and \( y'(a)(x) = A(a,x) \) for all \( x \in A \). Then \( y(a) \) is a lower \( \Omega \)-subset of \( A \) and \( y'(a) \) is an upper \( \Omega \)-subset of \( A \).

**Lemma 2.6 (Yoneda lemma [15]).** (1) For all \( a \in A \) and \( \phi \in [A^{\text{op}}, \Omega] \), \([A^{\text{op}}, \Omega](y(a), \phi) = \phi(a)\). Hence, \( y : A \longrightarrow [A^{\text{op}}, \Omega] \) is an \( \Omega \)-isometry, called the Yoneda embedding.

(2) For all \( a \in A \) and \( \psi \in [A, \Omega] \), \([A, \Omega](y'(a), \psi) = \psi(a)\). Hence, \( y' : A \longrightarrow [A, \Omega]^{\text{op}} \) is an \( \Omega \)-isometry, called the co-Yoneda embedding.

**Definition 2.7 (Kelly [15] and Lawvere [17]).** We say that \( \psi \in [A, \Omega] \) is representable if \( \psi = y'(a) \) for some \( a \in A \). In this case, we also say that \( \psi \) is represented by \( a \). Dually, we say that \( \phi \in [A^{\text{op}}, \Omega] \) is representable if \( \phi = y(a) \) for some \( a \in A \).

**Definition 2.8 (Kelly [15] and Wagner [24]).** A pair of \( \Omega \)-functors \( f : A \longrightarrow B \) and \( g : B \longrightarrow A \) is said to be an \( \Omega \)-adjunction, if \( B(f(a), b) = A(a, g(b)) \) for all \( a \in A \) and \( b \in B \). In this case, we say that \( f \) is a left adjoint of \( g \) and \( g \) is a right adjoint of \( f \).

When \( \Omega = 2 \), \( \Omega \)-adjunctions reduce to Galois connections [10] between preordered sets. It is easily seen that if \( (f, g) \) is an \( \Omega \)-adjunction then the pair of monotone functions \( f : A_0 \longrightarrow B_0 \) and \( g : B_0 \longrightarrow A_0 \) forms a Galois connection. The following proposition shows that \( \Omega \)-adjunctions behave exactly in the manner as the Galois connections in lattice theory.

**Proposition 2.9 (Kelly [15]).** Suppose \( A \) and \( B \) are \( \Omega \)-categories and \( f : A \longrightarrow B \), \( g : B \longrightarrow A \) are two functions. Then the following conditions are equivalent:

(1) \((f, g)\) is an \( \Omega \)-adjunction.

(2) Both \( f \) and \( g \) are \( \Omega \)-functors and for all \( a \in A, b \in B \), \( A(a, g(f(a))) \geq I \) and \( B(f g(b), b) \geq I \).

(3) \( f \) is an \( \Omega \)-functor and \( B(f(a), b) = A(a, g(b)) \) for all \( a \in A \) and \( b \in B \).

(4) \( g \) is an \( \Omega \)-functor and \( B(f(a), b) = A(a, g(b)) \) for all \( a \in A \) and \( b \in B \).

These conditions imply that \( f(a) \) is isomorphic to \( f g f(a) \) for all \( a \in A \) and \( g(b) \) is isomorphic to \( g f g(b) \) for all \( b \in B \).

### 3. Many-valued lattices

In this section, we reformulate the conditions (Lat3) and (Lat4) for lattices in the many valued setting, then obtain two kinds of many-valued lattices, \( \Omega \)-lattice and weak \( \Omega \)-lattice. The relationship between these two kinds of many valued lattices and lattice fuzzy orders, vague lattices shall be discussed in the next section.

#### 3.1. \( \Omega \)-lattices

Let \( X \) be a set. As usual, a function \( \lambda : X \longrightarrow \Omega \) is called an \( \Omega \)-subset of \( X \). The support set of \( \lambda \) is the subset \([x \in X : \lambda(x) \neq 0]\). \( \lambda \) is said to be of finite support if its support set is finite.

Suppose that \( \mu : A \longrightarrow \Omega \) is an \( \Omega \)-subset of \( A \). For each \( x \in A \), let
\[
\uparrow \mu(a) = \bigvee_{y \in A} \mu(y) \ast A(y, x) = \bigvee_{y \in A} \mu(y) \ast y'(y)(x)
\]
\[ \downarrow \mu(x) = \bigvee_{y \in A} \mu(y) \ast A(x, y) = \bigvee_{y \in A} \mu(y) \ast y(y)(x). \]

Then \( \uparrow (\downarrow \mu, \text{resp.}) \) is the smallest upper (lower, resp.) \( \Omega \)-subset of \( A \) bigger than or equal to \( \mu \) under the pointwise order, called the upper (lower, resp.) \( \Omega \)-subset generated by \( \mu \).

A lower \( \Omega \)-subset (or a weight) \( \phi : A^{\text{op}} \rightarrow \Omega \) is said to be finitely generated if \( \phi = \downarrow f \) for some \( f \in \Omega^A \) with finite support. An upper \( \Omega \)-subset \( \phi \) of \( A \) is said to be finitely supported if \( \phi = \uparrow \mu \) for some \( \mu \in \Omega^A \) with finite support. Denote \( FL(A) \) (\( FU(A) \), resp.) the set of finitely generated lower (upper, resp.) \( \Omega \)-subsets of \( A \).

It should be warned that the meaning of the term \textit{finitely generated} here is different from the use of this term in enriched category theory [8].

\( FL(A) \) and \( FU(A) \) become antisymmetric \( \Omega \)-categories by inheriting the categorical structures of \( [A^{\text{op}}, \Omega] \) and \( [A, \Omega] \). Since for every \( a \in A \), \( y(a) = \downarrow I_a, y'(a) = \uparrow I_a \), where \( I_a : A \rightarrow \Omega \) is a finite \( \Omega \)-subset given by \( I_a(x) = 1 \) if \( x = a \) and \( I_a(x) = 0 \) if \( x \neq a \), the codomain of the Yoneda embedding \( y : A \rightarrow [A^{\text{op}}, \Omega] \) and that of the co-Yoneda embedding \( y' : A \rightarrow [A, \Omega]^{\text{op}} \) can be restricted to \( FL(A) \) and \( FU(A)^{\text{op}} \), respectively.

**Definition 3.1.** Let \( A \) be an \( \Omega \)-category. Then we say that

1. \( A \) is finitely cocomplete if \( y : A \rightarrow FL(A) \) has a left adjoint;
2. \( A \) is finitely complete if \( y' : A \rightarrow FU(A)^{\text{op}} \) has a right adjoint;
3. \( A \) is cocomplete if the Yoneda embedding \( y : A \rightarrow [A^{\text{op}}, \Omega] \) has a right adjoint;
4. \( A \) is complete if the co-Yoneda embedding \( y' : A \rightarrow [A, \Omega]^{\text{op}} \) has a left adjoint.

In order to characterize (finitely) complete (cocomplete) \( \Omega \)-categories, we need the following.

**Definition 3.2** (Wagner [24]). Given an \( \Omega \)-category \( A \) and \( \mu \in \Omega^A \), define \( \text{lb}(\mu), \text{ub}(\mu) \in \Omega^A \) as follows: for all \( x \in A \),

\[ \text{lb}(\mu)(x) = \bigwedge_{y \in A} (\mu(y) \rightarrow A(x, y)), \quad \text{ub}(\mu)(x) = \bigwedge_{y \in A} (\mu(y) \rightarrow A(y, x)). \]

For \( \mu \in \Omega^A \), the condition \( \text{lb}(\mu)(x) = \bigwedge_{y \in A} \mu(y) \rightarrow A(x, y) \) is a many valued interpretation of the statement \( x \in \text{lb}(\mu) \iff (\forall y \in A, y \in \mu \Rightarrow x \leq y) \). Thus, \( \text{lb}(\mu) \) can be regarded as the set of lower bounds of \( \mu \) in \( A \), \( \text{lb}(\mu)(x) \) as the degree that \( x \) is a lower bound of \( \mu \). Dually, \( \text{ub}(\mu) \) can be regarded as the set of upper bounds of \( \mu \). It is easy to check that \( \text{lb}(\mu) \in [A^{\text{op}}, \Omega] \), \( \text{ub}(\mu) \in [A, \Omega] \).

**Proposition 3.3** (Isbell conjugation [18,24]). Suppose \( A \) is an \( \Omega \)-category. Then both \( \text{ub} : [|A|, \Omega] \rightarrow [|A|, \Omega]^{\text{op}} \) and \( \text{lb} : [|A|, \Omega]^{\text{op}} \rightarrow [|A|, \Omega] \) are \( \Omega \)-functors, and \( \text{ub} \) is a left adjoint of \( \text{lb} \).

**Corollary 3.4.** Suppose \( A \) is an \( \Omega \)-category. Then for any \( a \in \Omega, \mu \in \Omega^A \) and \( F \subset \Omega^A \),

1. \( \text{lb}(\bigvee F) = \bigwedge_{\mu \in F} \text{lb}(\mu) ; \quad \text{ub}(\bigvee F) = \bigwedge_{\mu \in F} \text{ub}(\mu) \).
2. \( \text{lb} \circ \text{ub} (\mu) \geq \mu ; \quad \text{ub} \circ \text{lb} (\mu) \geq \mu \).
3. \( \text{lb} \circ \text{ub} \circ \text{lb} (\mu) = \text{lb}(\mu) ; \quad \text{ub} \circ \text{lb} \circ \text{ub} (\mu) = \text{ub}(\mu) \).
4. For any \( x \in A \), \( \text{lb}(y'(x)) = y(x), \text{ub}(y(x)) = y'(x) \).

**Definition 3.5** (Wagner [24]). An \( \Omega \)-subset \( \phi : A \rightarrow \Omega \) is said to have a supremum if there is an element \( \text{sup} \phi \in A \) such that \( \text{ub}(\phi) = y(\text{sup} \phi) \), or equivalently, for all \( a \in A \),

\[ A(\text{sup} \phi, a) = \text{ub}(\phi)(a) = \bigwedge_{x \in A} (\phi(x) \rightarrow A(a, x)). \]

\( \phi : A \rightarrow \Omega \) is said to have an infimum if there is an element \( \text{inf} \phi \in A \) such that \( \text{lb}(\phi) = y(\text{inf} \phi) \), or equivalently, for all \( a \in A \),

\[ A(a, \text{inf} \phi) = \bigwedge_{x \in A} (\phi(x) \rightarrow A(a, x)). \]
The intuition of the above definition is as follows: sup $\phi \in A$ is the supremum of $\phi$ if for all $a \in A$, sup $\phi \leq a \iff \forall x \in A(x \in \phi \Rightarrow x \leq a)$. Infimum of an $\Omega$-subset can be explained in a similar way. Clearly, sup $\phi$ and inf $\phi$ are unique up to isomorphism.

In terms of weighted limits in enriched category theory [7,15], for any function $\phi : |A| \to \Omega$ (i.e. an $\Omega$-subset), sup $\phi$ is the colimit of the identity functor $A \to A$ weighted by $\downarrow \phi$. And inf $\phi$ is the limit of the identity functor $A \to A$ weighted by $\uparrow \phi$.

**Theorem 3.6** (Stubbe [22]). Suppose $A$ is an $\Omega$-category. Then the following conditions are equivalent:

1. $A$ is cocomplete.
2. Each $\Omega$-subset of $A$ has a supremum.
3. Each $\Omega$-subset of $A$ has an infimum.
4. $A$ is complete.

The following conclusion is an extension of the above theorem to the finite case. However, it should be remarked that the equivalence “complete $\iff$ cocomplete” does not hold in the finite case.

**Theorem 3.7.** Let $A$ be an $\Omega$-category. Then $A$ is finitely cocomplete if and only if each finite $\Omega$-subset has a supremum; $A$ is finitely complete if and only if each finite $\Omega$-subset has an infimum.

**Proof.** We prove the first part of the theorem and the second part is dual.

Sufficiency: For any $\downarrow \mu \in FL(A)$, let $d(\downarrow \mu) = \sup(\mu)$. Firstly we show that $d : FL(A) \to A$ is an $\Omega$-functor. That is to say, for any $\downarrow \mu, \downarrow v \in FL(A)$,

$$FL(A)(\downarrow \mu, \downarrow v) \leq A(\sup \mu, \sup v).$$

At first, it is easy to check that $\downarrow v \leq y(\sup v)$. Hence,

$$FL(A)(\downarrow \mu, \downarrow v) = \bigwedge_{x \in A} \left( \downarrow \mu(x) \to \downarrow v(x) \right)$$

$$= \bigwedge_{x \in A} \left( \bigvee_{y \in A} (\mu(y) \ast A(x, y)) \to \downarrow v(x) \right)$$

$$= \bigwedge_{x \in A} \left( \bigwedge_{y \in A} (\mu(y) \to (A(x, y) \to \downarrow v(x))) \right)$$

$$\leq \bigwedge_{y \in A} \left( \mu(y) \to \bigwedge_{x \in A} (A(x, y) \to A(x, \sup v)) \right)$$

$$= \bigwedge_{y \in A} (\mu(y) \to A(y, \sup v))$$

$$= A(\sup \mu, \sup v).$$

Secondly, for all $y \in A$,

$$FL(A)(\downarrow \mu, y(y)) = [A^{\text{op}}, \Omega](\downarrow \mu, y(y))$$

$$= \bigwedge_{x \in A} (\downarrow \mu(x) \to A(x, y))$$

$$= \bigwedge_{x \in A} \left( \bigvee_{z \in A} (\mu(z) \ast A(x, z) \to A(x, y)) \right)$$

$$= \bigwedge_{z \in A} \bigwedge_{x \in A} (\mu(z) \ast A(x, z) \to A(x, y)).$$
\[ = \bigwedge_{z \in A} \left[ \mu(z) \to \left( \bigwedge_{x \in A} A(x, z) \to A(x, y) \right) \right] \]
\[ = \bigwedge_{z \in A} (\mu(z) \to A(z, y)) \]
\[ = A(\text{sup } \mu, y) \]
\[ = A(d(\downarrow \mu), y). \]

So, \(d\) is a left adjoint of \(y\).

**Necessity:** Suppose the left adjoint of \(y\) is \(d\). Then it is easy to see that for any \(\mu\) of finite support, \(d(\downarrow \mu)\) is the supremum of \(\mu\). \(\square\)

**Definition 3.8.** A join \(\Omega\)-semilattice is a finitely cocomplete, antisymmetric \(\Omega\)-category; a meet \(\Omega\)-semilattice is a finitely complete, antisymmetric \(\Omega\)-category; an \(\Omega\)-lattice is an \(\Omega\)-category which is simultaneously a join \(\Omega\)-semilattice and a meet \(\Omega\)-semilattice.

**Proposition 3.10.** Let \(A\) be an \(\Omega\)-category. Then

1. The following conditions are equivalent:
   (a) \(A\) is tensored;
   (b) For all \(z \in \Omega\), \(x \in A\), \(z \otimes x \) has a supremum;
   (c) For all \(z \in \Omega\), \(x \in A\), the \(\Omega\)-subset \(\mathcal{A}_x\), given by \(\mathcal{A}_x(z) = z \iff z = x\) and \(\mathcal{A}_x(0) = 0\) if \(z \neq x\), has a supremum.
2. The following conditions are equivalent:
   (a') \(A\) is cotensored;
   (b') For all \(z \in \Omega\), \(x \in A\), \(z \oslash y(x) \) has an infimum;
   (c') For all \(z \in \Omega\), \(x \in A\), the \(\Omega\)-subset \(\mathcal{A}_x\), given by \(\mathcal{A}_x(z) = z \iff z = x\) and \(\mathcal{A}_x(0) = 0\) if \(z \neq x\), has an infimum.

Clearly, every finitely cocomplete \(\Omega\)-category is tensored and every finitely complete \(\Omega\)-category is cotensored. Hence each \(\Omega\)-lattice is both tensored and cotensored.

In an \(\Omega\)-category \(A\), if the supremum and infimum of the \(\Omega\)-subset \(\mu \equiv 0\) exist, then for any \(a \in A\), \(A(\text{sup } 0, a) = \bigwedge_{x \in A}(0 \to A(x, a)) = 1\), \(A(0, \text{inf } 0) = \bigwedge_{x \in A}(0 \to A(a, x)) = 1\). Thus, sup \(0\) and inf \(0\) are, respectively, the bottom and the top element of \(A_0\). An \(\Omega\)-category \(A\) is said to be bounded if both sup \(0\) and inf \(0\) exist. In this case, we write \(\bot\) for sup \(0\) and \(\top\) for inf \(0\).

**Proposition 3.11 (Stubbe [23]).** Suppose \(A\) is a bounded antisymmetric \(\Omega\)-category.

(i) If \(A\) is tensored, then the tensor \(\otimes : \Omega \times A_0 \to A_0\) satisfies:
   1. \(0 \otimes x = \bot, I \otimes x = x\).
   2. \(A(\alpha \otimes x, y) = \alpha \to A(x, y)\). Hence in \(A_0\), \(\alpha \otimes x \leq y \iff \alpha \leq A(x, y)\).
   3. \((\alpha \star \beta) \otimes x = \alpha \otimes (\beta \otimes x)\).

(ii) If \(A\) is cotensored, then the cotensor \(\oslash : \Omega \times A_0 \to A_0\) satisfies:
   1. \(0 \oslash y = \top, I \oslash y = y\).
   2. \(A(x, \alpha \oslash y) = \alpha \to A(x, y)\). Hence in \(A_0\), \(x \leq \alpha \oslash y \iff \alpha \leq A(x, y)\).
   3. \((\alpha \star \beta) \oslash y = \alpha \oslash (\beta \oslash y)\).

(iii) If \(A\) is both tensored and cotensored, then for any \(\alpha \in \Omega\), \(\alpha \otimes (-) : A_0 \to A_0\) is a left adjoint of \(\alpha \oslash (-) : A_0 \to A_0\).
The following propositions show that the supremum and infimum in an \( \Omega \)-lattice \( A \) can be completely described by the lattice structure of \( A_0 \) and the tensor and cotensor of \( A \). They are extensions of corresponding results in [23] for complete \( \Omega \)-lattices.

**Proposition 3.12.** (1) If \( A \) is a join \( \Omega \)-semilattice. Then for any finite \( \Omega \)-subset \( \lambda \),

\[
\sup \lambda = \bigvee_{x \in A} (\check{\lambda}(x) \otimes x).
\]

(2) If \( A \) is a meet \( \Omega \)-semilattice. Then for any finite \( \Omega \)-subset \( \lambda \),

\[
\inf \lambda = \bigwedge_{x \in A} (\check{\lambda}(x) \rightarrow x).
\]

**Proof.** (1) For any \( b \in A \),

\[
A \left( \bigvee_{x \in A} (\check{\lambda}(x) \otimes x), b \right) = A \left( \bigvee_{x \in A, \check{\lambda}(x) \neq 0} (\check{\lambda}(x) \otimes x), b \right)
= \bigwedge_{x \in A, \check{\lambda}(x) \neq 0} A(\check{\lambda}(x) \otimes x, b)
= \bigwedge_{x \in A, \check{\lambda}(x) \neq 0} (\check{\lambda}(x) \rightarrow A(x, b))
= \bigwedge_{x \in A} (\check{\lambda}(x) \rightarrow A(x, b)).
\]

Therefore, \( \sup \lambda = \bigvee_{x \in A} (\check{\lambda}(x) \otimes x) \).

(2) Similar to (1). \( \square \)

**Proposition 3.13.** Suppose that \( A \) is a tensored and cotensored, antisymmetric \( \Omega \)-category. Then \( A \) is an \( \Omega \)-lattice if and only if \( A_0 \) is a lattice.

**Proof.** Only the sufficiency needs be shown. For any finite set \( \mu \in \Omega^A \), we show that the supremum of \( \mu \) is given by \( \sup \mu = \bigvee_{x \in A} (\mu(x) \otimes x) \). By the property of tensor, for any \( \alpha \in \Omega \), \( y \in A \),

\[
\alpha \leq A \left( \bigvee_{x \in A} (\mu(x) \otimes x), y \right) \iff \alpha \otimes \bigvee_{x \in A} (\mu(x) \otimes x) \leq y
\iff \bigvee_{x \in A} \alpha \otimes (\mu(x) \otimes x) \leq y
\iff \forall x \in A, \alpha \ast \mu(x) \leq A(x, y)
\iff \forall x \in A, \alpha \leq \mu(x) \rightarrow A(x, y)
\iff \alpha \leq \bigwedge_{x \in A} \mu(x) \rightarrow A(x, y).
\]

Therefore, \( A(\bigvee_{x \in A} (\mu(x) \otimes x), y) = \bigwedge_{x \in A} \mu(x) \rightarrow A(x, y) \), which means \( \bigvee_{x \in A} (\mu(x) \otimes x) \) is the supremum of \( \mu \). Similarly, the infimum of \( \mu \) is given by \( \inf \mu = \bigwedge_{x \in A} (\mu(x) \rightarrow x). \) \( \square \)

We are going to discuss the representation of \( \Omega \)-lattices. It was already observed by Stubbe [23] that complete \( \Omega \)-lattices are essentially \( \Omega \)-modules in the category of complete lattices and join-preserving functions. That is to say, a complete \( \Omega \)-lattice can be represented as a complete lattice \( A \) together with a binary operation \( \otimes: \Omega \times A \rightarrow A \)
which satisfies the following conditions:

1. \( 0 \otimes x = \perp, I \otimes x = x; \)
2. \( x \otimes (\bigvee_{t \in T} x_t) = \bigvee_{t \in T} (x \otimes x_t); \)
3. \( (\bigvee_{t \in T} x_t) \otimes x = \bigvee_{t \in T} (x_t \otimes x); \)
4. \( (x \ast \beta) \otimes x = x \otimes (\beta \otimes x). \)

Suppose \( A \) is a join \( \Omega \)-semilattice, then \( A \) is tensored and \( A_0 \) has a bottom element \( \perp \). By Proposition 3.11, the tensor \( \otimes : \Omega \times A_0 \to A_0 \) satisfies

(i) \( 0 \otimes x = \perp, I \otimes x = x; \)
(ii) \( (\bigvee_{t \in T} x_t) \otimes x = \bigvee_{t \in T} (x_t \otimes x) \) for any subset \( \{x_t : t \in T\} \) of \( \Omega; \)
(iii) \( (x \ast \beta) \otimes x = x \otimes (\beta \otimes x). \)

Let \([A_0 \to A_0]\) denote the poset of all finite-join preserving functions from \( A_0 \) to itself under the pointwise order. \([A_0 \to A_0]\) has a bottom element which maps every element in \( A_0 \) to the bottom element in \( A_0 \). We also write \( \perp \) for the bottom element in \([A_0 \to A_0]\).

The tensor \( \otimes : \Omega \times A_0 \to A_0 \) induces a function \( F : \Omega \to [A_0 \to A_0] \) in the way that \( F(x) = x \otimes (-) \) for all \( x \in \Omega \). Clearly, we have that:

1. \( F(0) = \perp, F(I) = \text{id}_{A_0}; \)
2. \( F(\bigvee_{t \in T} x_t) = \bigvee_{t \in T} F(x_t) \) for any subset \( \{x_t : t \in T\} \) of \( \Omega; \)
3. \( F(x \ast \beta) = F(x) \circ F(\beta). \)

Conversely, we have the following.

**Theorem 3.14.** Suppose that \( A_0 \) is a join semilattice with a bottom element and \( F : \Omega \to [A_0 \to A_0] \) is a function satisfying the above conditions (1)–(3). For all \( x, y \in A_0 \), let

\[
A(x, y) = \bigvee\{z \in \Omega \mid F(z)(x) \leq y\}.
\]

Then \( A \) is a join \( \Omega \)-semilattice with underlying poset \( A_0 \).

**Proof.** First, we show that \( x \leq A(x, y) \iff F(x)(x) \leq y \).

Suppose that \( x \leq A(x, y) \). Let \( D = \{\beta \in \Omega \mid F(\beta)(x) \leq y\} \). Then \( A(x, y) = \bigvee D \). Appealing to (2) we obtain that \( F(x)(x) \leq F(\bigvee D)(x) = \bigvee_{\beta \in D} F(\beta)(x) \leq y \). The other implication is trivial.

Secondly, \( A \) is an antisymmetric \( \Omega \)-category.

(i) Reflexivity, i.e., \( A(x, x) \geq I \) for any \( x \in A \). This is trivial since \( F(I)(x) = x \).

(ii) Transitivity. In fact, for all \( x, y, z \in A \),

\[
A(x, y) \ast A(y, z) = (\bigvee\{z \mid F(z)(x) \leq y\}) \ast (\bigvee\{z \mid F(z)(y) \leq z\})
\]

\[
= \bigvee\{z \mid F(z)(x) \leq y, F(z)(y) \leq z\}
\]

\[
\leq \bigvee\{z \mid F(z)(x) \leq z\}
\]

\[
= A(x, z).
\]

(iii) Antisymmetry. If \( A(x, y) \geq I, A(y, x) \geq I \), then \( F(I)(x) = x \leq y \) and \( F(I)(y) = y \leq x \), which implies \( x = y \).

Thirdly, \( A \) is finitely cocomplete. Suppose that \( \mu \in \Omega^{A_0} \) is of finite support. We show that the supremum of \( \mu \) is given by \( \text{sup} \mu = \bigvee_{x \in A_0} F(\mu(x))(x) \). Since \( F(0) \) is the constant function with value the bottom of \( A_0 \), the join \( \bigvee_{x \in A_0} F(\mu(x))(x) \) always exists in \( A_0 \). For all \( x \in \Omega \), \( y \in A_0 \),

\[
x \leq A_{x \in A_0} \bigvee F(\mu(x))(x), y \iff F(x) \left( \bigvee_{x \in A_0} F(\mu(x))(x) \right) \leq y
\]

\[
\iff \bigvee_{x \in A_0} F(x)(F(\mu(x))(x)) \leq y
\]

\[
\iff \bigvee_{x \in A_0} F(x)(x) \ast \mu(x) \leq y
\]
For any \( x \in A_0 \), \( F(\alpha \ast \mu(x))(x) \leq y \)
\( \iff \forall x \in A_0, \ z \ast \mu(x) \leq A(x, y) \)
\( \iff \forall x \in A_0, \ x \leq \mu(x) \rightarrow A(x, y) \)
\( \iff \forall x \in A_0, \ x \leq \bigwedge_{x \in A_0} \mu(x) \rightarrow A(x, y) \).

Therefore,
\[
A \left( \bigvee_{x \in A_0} F(\mu(x))(x), y \right) = \bigwedge_{x \in A_0} (\mu(x) \rightarrow A(x, y)),
\]
which means that \( \sup \mu = \bigvee_{x \in A_0} F(\mu(x))(x) \).

Finally, it is clear that the underlying poset of \( A \) is \( A_0 \). \( \square \)

By the above theorem we see that a join \( \Omega \)-semilattice \( A \) can be represented as a (classical) join semilattice \( A_0 \) together with a function from \( \Omega \) to the poset of finite-join-preserving functions \( A_0 \rightarrow A_0 \), which satisfies the conditions (1)–(3).

Meet \( \Omega \)-semilattices can be represented in an analogous way. The details are left to the reader.

By comparing the representation of \( \Omega \)-semilattices and that of complete \( \Omega \)-lattices in [23], it seems that if we replace the semilattice by a lattice in the above theorem, then we would obtain a representation for \( \Omega \)-lattices. However, this is not the case, the situation for \( \Omega \)-lattices is slightly complicated. The reason lies in that finite completeness is not equivalent to finite cocompleteness, meanwhile completeness is equivalent to cocompleteness. In order to obtain a similar representation for \( \Omega \)-lattices, we need some more notations.

Suppose \( A \) is a lattice with top \( \top \) and bottom \( \bot \). Let \( \text{Adj}(A \rightarrow A) \) denote the set of pairs \((f, g)\) of functions \( f, g : A \rightarrow A \) such that \( f \) is a left adjoint of \( g \). If we also write \( \top \) (\( \bot \), resp.) for the constant function with value \( \top \) (\( \bot \), resp.), then \((\bot, \top) \in \text{Adj}(A \rightarrow A)\). Clearly, for any \((f_1, g_1), (f_2, g_2) \in \text{Adj}(A \rightarrow A)\), \( f_1 \leq f_2 \) if and only if \( g_1 \geq g_2 \). Let \((f_1, g_1) \leq (f_2, g_2) \) if \( f_1 \leq f_2 \). Then \( \text{Adj}(A \rightarrow A) \) becomes a join semilattice with \((\bot, \top)\) as the bottom element. For any \((f_1, g_1), (f_2, g_2) \in \text{Adj}(A \rightarrow A)\), if we define \((f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_1 \circ g_2)\), then \((\text{Adj}(A \rightarrow A), \leq, \circ, (\text{id}, \text{id}))\) becomes a partially ordered monoid [5].

If \( A \) is an \( \Omega \)-lattice, for all \( \alpha \in \Omega \), \( x, y \in A \), \( x \otimes x \leq y \iff x \leq x \rightarrow y \) by Proposition 3.11. This means \((\alpha \otimes (-), x \rightarrow (-)) \in \text{Adj}(A_0 \rightarrow A_0)\). Define \( H : \Omega \rightarrow \text{Adj}(A_0 \rightarrow A_0) \) by \( H(\alpha) = (\alpha \otimes (-), x \rightarrow (-)) \). Then \( H \) is a join-preserving monoid morphism, that is, \( H \) satisfies the following conditions:

1. \((\text{id}, \text{id})\) is a left adjoint of \((\text{id}, \text{id})\).
2. \((f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_1 \circ g_2)\)
3. \( H(\alpha \ast \beta) = H(\alpha) \circ H(\beta) \).

On the other hand, we have the following.

**Theorem 3.15.** Suppose that \( A_0 \) is a lattice with top element \( \top \) and bottom element \( \bot \) and \( H : \Omega \rightarrow \text{Adj}(A_0 \rightarrow A_0) \), \( H(\alpha) = (f_2, g_2) \), is a join-preserving monoid morphism. Let \( A(x, y) = \lor \{x \in \Omega | f_2(x) \leq y\} \). Then \( A \) becomes an \( \Omega \)-lattice with underlying lattice \( A_0 \).

The conclusion of the last theorem can be regarded as a representation of \( \Omega \)-lattices.

### 3.2. Weak \( \Omega \)-lattices

In this subsection, we present another way to extend the notion of “lattice” to the many valued setting.

**Definition 3.16.** A bounded antisymmetric \( \Omega \)-category \( A \) is called a weak \( \Omega \)-lattice if the diagonal \( d : A \rightarrow A \times A \), \( d(a) = (a, a) \) has both a left adjoint and a right adjoint.
Example 3.17. Every $\Omega$-lattice is a weak $\Omega$-lattice. Indeed, if $A$ is a join $\Omega$-semilattice, then $d : A \rightarrow A \times A$ has a left adjoint. In fact, let $f(x, y) = \sup(I_x \lor I_y)$. Then for any $z \in A$,
\[
A(f(x, y), z) = A(\sup(I_x \lor I_y), z) = \bigwedge_{w \in A} ((I_x \lor I_y)(w) \rightarrow A(w, z))
\]
\[
= A(x, z) \land A(y, z) = (A \times A)((x, y), d(z)).
\]
Dually, if $A$ is a meet $\Omega$-semilattice, then $d : A \rightarrow A \times A$ has a right adjoint.

Proposition 3.18. If $A$ is a bounded antisymmetric $\Omega$-category, then $A$ is a weak $\Omega$-lattice if and only if for any $x, y \in A$, both $A(x, -) \land A(y, -)$ and $A(-, x) \land A(-, y)$ are representable. And in a weak $\Omega$-lattice $A$, the element which represents $A(x, -) \land A(y, -)$ is the join of $x$ and $y$ in $A_0$, and the element which represents $A(-, x) \land A(-, y)$ is the meet of $x$ and $y$ in $A_0$.

Proof. For the first part we check for example that the diagonal function $d : A \rightarrow A \times A$ has a left adjoint if and only if $A(x, -) \land A(y, -)$ is representable for any $x, y \in A$.

If $f : A \times A \rightarrow A$ is a left adjoint of $d : A \rightarrow A \times A$, then for any $x, y, z \in A, A(f(x, y), z) = (A \times A)((x, y), d(z)) = A(x, z) \land A(y, z)$, which means that $f(x, y)$ represents $A(x, -) \land A(y, -)$.

Conversely, suppose that $f : A \times A \rightarrow A$ is a function such that $f(x, y)$ represents $A(x, -) \land A(y, -)$ for all $x, y \in A$. Then for any $x, y, z \in A, A(f(x, y), z) = A(x, z) \land A(y, z) = (A \times A)((x, y), d(z))$. Hence $f$ is a left adjoint of $d$ by Proposition 2.9.

For the second part of the conclusion, suppose that $z$ represents $A(x, -)\land A(y, -)$, i.e., $A(z, -) = A(x, -)\land A(y, -)$. Then $A(x, -) \land A(y, z) = A(z, x) \lor A(z, y)$ for all $x, y \in A$. Therefore, $x \leq y \leq z$. If $x \leq w, y \leq w$, then $A(x, w) \lor A(y, w) \leq A(w, w) \lor A(x, y)$, and $A(z, w) = A(x, w) \lor A(y, w) \lor I$. Therefore $z$ is the join of $x$ and $y$ in $A_0$. Similarly, one can show that the element which represents $A(-, x) \land A(-, y)$ is the meet of $x$ and $y$ in $A_0$.

Remark 3.19. If $A$ is a weak $\Omega$-lattice, then $A_0$ is a lattice. Hence, a weak $\Omega$-lattice is an $\Omega$-lattice if and only if it is tensored and cotensored. Thus, the difference between weak $\Omega$-lattices and $\Omega$-lattices is that in a weak $\Omega$-lattice, it is required that the supremum and infimum of each crisp finite subset exist, while in an $\Omega$-lattice, it is required that the supremum and infimum of each $\Omega$-subset of finite support exist.

Here is an example of weak $\Omega$-lattice which is not an $\Omega$-lattice.

Example 3.20. Suppose $L$ is a lattice with the bottom element $\bot$ and the top element $\top$. $\Omega = ([0, 1], *, 1)$, where $*$ is the Łukasiewicz t-norm. Precisely, $a, b \in [0, 1], a * b = \max(a + b - 1, 0), a \rightarrow b = \min(1, 1 - a + b)$. Let $L(x, y) = 1$ if $x \leq y$; otherwise let $L(x, y) = 0$. Then $L$ becomes an antisymmetric $\Omega$-category. Moreover, $L$ is a weak $\Omega$-lattice, since $L(x, -) \land L(y, -) = L(x \lor y, -)$ and $L(-, x) \land L(-, y) = L(-, x \lor y)$. However, $L$ is not finitely cocomplete. To see this, consider the $\Omega$-subset $\mu : L \rightarrow \Omega$ defined by $\mu(x) = 1/2$ if $x = \top$ and $\mu(x) = 0$ if $x \neq \top$. If $\sup \mu = z$, then for every $y \in L, L(z, y) = 1/2 \rightarrow L(z, y) = \min(1, 1/2 + L(\top, y))$. In particular, $L(z, \bot) = 1/2 \rightarrow L(z, \bot) = 1/2$, which is impossible.

Proposition 3.21 (Bělohlávek [4] for $\Omega$-lattices). Suppose $\Omega$ is an integral commutative quantale and $A$ is a weak $\Omega$-lattice. Then $A$ is uniquely determined by $E(-, -) = A(-, -) \land A^{op}(-, -)$ and $A_{[1]}$, where $A_{[1]}$ is the 1-cut of $A$, i.e., $A_{[1]} = \{(x, y)|A(x, y) = 1\}$.

Proof. The conclusion was proved in Bělohlávek [4] for lattice fuzzy orders which are exactly the $\Omega$-lattices as we shall see in the next section, we include here a simple proof for the general case. Since $A$ is a weak $\Omega$-lattice, for any $x, y \in A$, $A(-, x \land y) = A(-, x) \land A(-, y)$, where the meet $x \land y$ is taken in $A_0$. We proceed with two cases:

1. If $(x, y) \in A_{[1]}$, then $A(x, y) = 1$ and $A(y, x) = A(x, y) \land A(y, x) = A(x, y)$.

2. If $(x, y) \notin A_{[1]}$, then in $A_0$, $x \land y \leq x$. Therefore, $(x \land y, x) \in A_{[1]}$. Appealing to (1) we obtain that,
\[
A(x, y) = A(x, x) \land A(x, y) = A(x, x \land y) = E(x \land y, x).
\]
This completes the proof. □
**Remark 3.22.** The above proposition also holds for any join (meet) $\Omega$-semilattice.

**Remark 3.23.** A weight $\phi : A^{\text{op}} \to \Omega$ is said to be finitely generated if $\phi = \downarrow f$ for some $f \in \Omega^A$ of finite support; $\phi$ is said to be conical [15] if there is some $f \in \Omega^A$ such that $f(x)$ is either 0 or the unit element $I$ in $\Omega$ and that $\phi = \downarrow f$. Finitely generated conical weights are defined in an obvious way. Then it is not hard to verify that both the class $\Phi$ of finitely generated weights and the class $\Psi$ of finitely generated conical weights are examples of saturated class of weights in the sense of [1,16]. Then, $\Phi$-complete and $\Phi$-cocomplete antisymmetric $\Omega$-categories are exactly the $\Omega$-lattices; and $\Psi$-complete and $\Psi$-cocomplete antisymmetric $\Omega$-categories are exactly the weak $\Omega$-lattices.

4. Equivalents of many valued lattices

In this section, we show that an $\Omega$-lattice is exactly a lattice fuzzy order in the sense of Bělohlávek [2,4] and a weak $\Omega$-lattice is exactly a vague lattice introduced by Demirci [9].

An $\Omega$-equivalence on a set $A$ is an $\Omega$-preorder $E$ on $A$ such that for all $x, y \in A$, $E(x, y) = E(y, x)$. An $\Omega$-equivalence $E$ on $A$ is said to be an $\Omega$-equality if $E$ is antisymmetric.

**Definition 4.1 (Bělohlávek [4]).** Let $E$ be an $\Omega$-equivalence on a set $A$. A singleton of $(A, E)$ is a map $s : A \to \Omega$ such that:

1. There exists some $x \in A$ such that $s(x) \geq I$.
2. For all $x, y \in A$, $s(x) * s(y) \leq E(x, y)$.
3. For all $x, y \in A$, $s(x) * E(x, y) \leq s(y)$.

Suppose that $A$ is an $\Omega$-category and $E$ is an $\Omega$-equivalence on $|A|$. We say that $A$ is grounded on $E$ if for all $x, y \in A$:

1. $(E$-reflexivity) $E(x, y) \leq A(x, y) \land A(y, x)$.
2. $(E$-antisymmetry) $A(x, y) * A(y, x) \leq E(x, y)$.

Given an (antisymmetric) $\Omega$-category $A$, it is easy to check that $E(x, y) = A(x, y) \land A(y, x)$ is an $\Omega$-equivalence ($\Omega$-equality, resp.) on $|A|$ and that $A$ is grounded on $E$.

**Lemma 4.2.** Suppose that $A$ is an antisymmetric $\Omega$-category and $E$ is the $\Omega$-equality given by $E(x, y) = A(x, y) \land A(y, x)$ for all $x, y \in A$. Then the following conditions are equivalent.

1. $A$ is a meet $\Omega$-semilattice.
2. For every $\mu \in \Omega^A$ of finite support, there exists some $a \in A$ such that $\ell b(\mu) \land u b(\mu) = A(-, a) \land A(-, a)$.
3. For every $\mu \in \Omega^A$ of finite support, $\ell b(\mu) \land u b(\mu)$ is a singleton in $(A, E)$.
4. For every $\mu \in \Omega^A$ of finite support, there exists some $a \in A$ such that $\ell b(\mu)(a) \land u b(\mu)(a) \geq I$.

**Proof.** (1) $\Rightarrow$ (2): Suppose $A$ is a meet $\Omega$-semilattice and $\mu \in \Omega^A$ is finite. There exists $a \in A$ such that $\ell b(\mu) = A(-, a)$. Consequently,

\[ \ell b(\mu) \land u b(\mu) = A(-, a) \land A(a, -). \]

(2) $\Rightarrow$ (3): This is easy since $A(-, a) \land A(a, -)$ is a singleton in $(A, E)$.

(3) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): It is sufficient to show that $\ell b(\mu)$ is representable for every finite $\mu \in \Omega^A$. By assumption, there is some $a \in A$ such that $\ell b(\mu)(a) \land u b(\mu)(a) \geq I$. Since $\ell b(\mu)$ is a lower $\Omega$-subset, for all $x \in A$, $\ell b(\mu)(x) \geq \ell b(\mu)(a) * A(x, a) \geq A(x, a)$. On the other hand, since $u b(\mu)(a) = \bigwedge_{x \in A} \ell b(\mu)(x) \to A(x, a) \geq I$, we obtain that $\ell b(\mu)(x) \leq A(x, a)$ for all $x \in A$. Therefore, $\ell b(\mu) = A(-, a)$, i.e. $\ell b(\mu)$ is represented by $a$. \qed

**Definition 4.3 (Bělohlávek [4]).** A lattice fuzzy order is a pair $(A, E)$, where $A$ is an $\Omega$-category and $E$ is an $\Omega$-equality on $|A|$ such that:

1. $E(x, y) = A(x, y) \land A(y, x)$ for all $x, y \in A$.
2. For every $\mu \in \Omega^A$ of finite support, both $\ell b(\mu) \land u b(\mu)$ and $u b(\mu) \land \ell b(\mu)$ are singletons in $(A, E)$.
Proposition 4.4. Suppose that $A$ is an antisymmetric $\Omega$-category. $E$ is the $\Omega$-equality given by $E(x, y) = A(x, y) \land A(y, x)$ for all $x, y \in A$. Then the following conditions are equivalent:

1. $A$ is an $\Omega$-lattice.
2. $(A, E)$ is a lattice fuzzy order.

Therefore, the notion of $\Omega$-lattices coincides with that of lattice fuzzy orders. Let $A$ be an antisymmetric $\Omega$-category. Demirci [9] defined two fuzzy relations $\sqcap$ and $\sqcup$ on $A \times A$ as follows:

\[
\sqcap (x, y, z) = \text{lb}(I_x \lor I_y)(z) \land \text{ub} \circ \text{lb}(I_x \lor I_y)(z) = A(z, x) \land A(z, y) \land \left( \bigwedge_{w \in A} ((A(w, x) \land A(w, y)) \rightarrow A(w, z)) \right),
\]

\[
\sqcup (x, y, z) = \text{ub}(I_x \lor I_y)(z) \land \text{lb} \circ \text{ub}(I_x \lor I_y)(z) = A(x, z) \land A(y, z) \land \left( \bigwedge_{w \in A} ((A(x, w) \land A(y, w)) \rightarrow A(z, w)) \right).
\]

$\sqcap$ and $\sqcup$ can be regarded, respectively, as meet and join operation on $A$.

The following proposition says that a bounded antisymmetric $\Omega$-category is a weak $\Omega$-lattice if and only if it is a vague lattice in the sense of Demirci [9].

Proposition 4.5. Let $A$ be a bounded antisymmetric $\Omega$-category. The following conditions are equivalent:

1. $A$ is a weak $\Omega$-lattice.
2. For any $x, y \in A$, there exist some $u, v \in A$ such that $\sqcup(x, y, u) \geq I$ and $\sqcap(x, y, v) \geq I$.

Proof. (1) $\Rightarrow$ (2): This is easy since for all $x, y \in A$, $A(-, x) \land A(-, y)$ and $A(x, -) \land A(y, -)$ are both representable.

(2) $\Rightarrow$ (1): By assumption, for any $x, y \in A$, there is some $z \in A$ such that $\sqcap(x, y, z) \geq I$. That is,

\[
\sqcap(x, y, z) = A(z, x) \land A(z, y) \land \left( \bigwedge_{w \in A} ((A(w, x) \land A(w, y)) \rightarrow A(w, z)) \right) \geq I.
\]

Thus, $A(z, x) \land A(z, y) \geq I$ and $A(-, x) \land A(-, y) \leq A(-, z)$. Therefore, for any $w \in A$,

\[
A(w, z) \leq A(w, z) \neq A(z, x) \leq A(w, x),
\]

that means, $A(-, z) \leq A(-, x)$. Similarly, we have that $A(-, z) \leq A(-, y)$. Hence, $A(-, x) \land A(-, y) = A(-, z)$. This means that $A(-, x) \land A(-, y)$ is representable as desired.

The representability of $A(x, -) \land A(y, -)$ can be verified in a similar way. Therefore, $A$ is a weak $\Omega$-lattice. $\square$

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