Degenerate Equilibria at Infinity in the Generalized Brusselator

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Abstract—In this paper, we consider a polynomial differential system of \( p + q \) degree, which was given from a general multi-molecular reaction in biochemistry as a theoretical problem of concentration kinetics. We analyze qualitative properties of its equilibria at infinity, determining characteristic directions and the numbers of orbits which go towards or away from those equilibria in characteristic directions. In the analysis, the high degree of polynomials and the high degeneracy of equilibria at infinity make so much trouble that both the known blowing-up method and the normal sector method are not effective in some cases. Our difficulties are overcome by discussing a kind of angular regions, which extends the classic normal sectors to more general. © 2005 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Since chemical oscillations were observed in 1960s, more and more contributions [1–7] have been made in this field. In 1977, Prigogine set up a theory of dissipative structures. Based on his work, many results [8–12] are given to investigate dynamics of the basic Brusselator,

\[
\begin{align*}
\frac{dx}{dt} &= a - (b + 1)x + x^q y, \\
\frac{dy}{dt} &= bx - x^p y,
\end{align*}
\]

and its periodic perturbations in details. Later, along with consideration on multi-molecular reaction models [13,14], many attentions are also paid to the generalization of the Brusselator

\[
B(a, b; p, q) : \begin{cases} \\
\frac{dx}{dt} &= a - (b + 1)x + x^p y^q, \\
\frac{dy}{dt} &= bx - x^p y^q,
\end{cases}
\]  

(1.1)

where \( x, y \geq 0 \) are concentrates of two substances in the reaction, positive integers \( p, q \) are numbers of molecules of the two substances respectively, and \( a > 0, b > 0 \) are both parameters.

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related to kinetic constants. The systems $B(a, b; p, 1)$ and $B(a, b; 1, q)$ are discussed in [15,16].

The most general system $B(a, b; p, q)$ is discussed in [17], where the system is proved to have a unique equilibrium $S : \left(a, (b/a^{p-1})^{1/q}\right)$ in the first quadrant and at most two limit cycles can be produced from Hopf bifurcation at $S$.

Although local qualitative properties of the general system $B(a, b; p, q)$ are given in [17], its global dynamics still needs to be discussed. An interesting work is to give qualitative properties of its equilibria at infinity, which reflect the tendency of the products of substances $x, y$ growing in large amount. With the Poincaré transformation $x = 1/z$, $y = u/z$, equation (1.1) can be rewritten as

$$
\frac{du}{d\tau} = -u^{q} - u^{q+1} + b \frac{u^{p+q-1} + (b + 1) u z^{p+q-1} - a u z^{p+q},}{dz} = -u^{q} z + (b + 1) z^{p+q} - a z^{p+q+1},
$$

where $d\tau = \frac{dt}{z^{p+q}}$. System (1.2) has two equilibria $O(0, 0)$ and $A(-1, 0)$ but only the first one corresponds to an equilibrium at infinity in the first quadrant of the $x$-$y$ plane. Let $I_{x}$ denote the equilibrium since it is on the $x$-axis. With another Poincaré transformation $x = v/z$, $y = l/z$, equation (1.1) is changed into the form,

$$
\frac{dv}{d\tau} = v^{p} + v^{p+1} + a \frac{v z^{p+q} - (b + 1) v z^{p+q-1} - b v z^{p+q-1},}{dz} = v^{p} z - b v z^{p+q},
$$

where $d\tau = \frac{dt}{z^{p+q}}$. As shown in [18], we only need to study its equilibrium $O(0, 0)$, which corresponds to an equilibrium $I_{y}$ on the $y$-axis in the first quadrant of the $x$-$y$ plane. The main difficulty in discussing $I_{x}$ and $I_{y}$ is that their degeneracies are determined by the unspecified numbers $p$ and $q$ and so high, when $p$ and $q$ are large, that the blowing-up method (which decomposes a complicated equilibrium into several simple ones with the Briot-Bouquet’s transformation [18]) is not effective. In some cases, the normal sector method [18,19], also does not work because there is no way to construct an appropriate normal sector about a characteristic direction.

Our difficulties are overcome in this paper by discussing a kind of angular regions, which extends the classic normal sectors to more general. Such an angular region, allowing curves and orbits to be part of its boundary, may not be an angular neighborhood of the characteristic direction. Using this method we determine the characteristic directions and the numbers of orbits which go towards or away from the equilibria $I_{x}$ and $I_{y}$ at infinity.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure.png}
\caption{p \geq 2 and q \geq 2 for $I_{x}$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{p \geq 2 and q \geq 2 for $I_{y}$.}
\end{figure}

2. QUALITATIVE ANALYSIS OF EQUILIBRIUM $I_{x}$

**Theorem 1.** For positive integers $p$ and $q$, system (1.1) has a unique orbit entering $I_{x}$, which goes in the direction vertical to the $x$-axis, and a unique orbit leaving from $I_{x}$, which goes either in the direction of angle $\theta = \arccotn(1/b^{1/q})$ from the $x$-axis for $p = 1$ or in the direction of the $x$-axis for $p \geq 2$. There are no orbits connecting $I_{x}$ in other directions.
It suffices to discuss how many orbits of system (1.2) enter or leave O in the first quadrant of
the \( u-z \) plane as \( \tau \to \infty \). Our discussion is proceeded in the three cases.

Case 1: \( q = 1 \).
Case 2: \( p = 1 \) and \( q \geq 2 \).
Case 3: \( p \geq 2 \) and \( q \geq 2 \).

**Lemma 1.** For \( q = 1 \), system (1.2) has an orbit leaving \( O \) in the direction \( \theta = \arctan(1/b) \) as
\( p = 1 \) and in the direction \( \theta = \pi/2 \) as \( p \geq 2 \), an orbit entering \( O \) in \( \theta = 0 \) on the positive
half \( u \)-axis as \( p \) is an arbitrary positive integer, and no more orbits connect with \( O \) in the first
quadrant.

**Proof.** For \( p = q = 1 \), the equilibrium \((0,0)\) of system (1.2) is degenerate but not all eigenvalues
are zero (the nonzero one is \(-1\)). With the change of variables \((u, z) \mapsto (u_1, z)\), where \( u_1 = u-bz \),
system (1.2) is rewritten as

\[
\begin{align*}
\frac{dz}{d\tau} &= z^2 - u_1 z - az^3 := Z_1(u_1, z), \\
\frac{du_1}{d\tau} &= -u_1 + u_1 z - u_1^2 - au_1 z^2 := U_1(u_1, z).
\end{align*}
\]

Solving \( U_1(u_1, z) = 0 \), we get two implicit functions, \( u_1(z) = 0 \), \( u_1(z) = -1 + z - az^2 \). The second
function does not intersect a sufficiently small neighborhood of \( O \). Substituting the first function
into the first equation of (2.1), we get

\[
\frac{dz}{d\tau} = Z_1(0, z) = z^2 - az^3.
\]

For \( p \geq 2 \) and \( q = 1 \), system (1.2) is of the form,

\[
\begin{align*}
\frac{dz}{d\tau} &= -uz + (b+1)z^{p+1} - az^{p+2} := Z_2(u, z), \\
\frac{du}{d\tau} &= -u - u^2 + bz^p + (b+1)uz^p - az^{p+1} := U_2(u, z),
\end{align*}
\]

which is also degenerate at \( O \) with exactly a nonzero eigenvalue \(-1\). Since \( U_2(0,0) = 0 \), from
\( U_2(u, z) = 0 \), we can solve an implicit function

\[
u = \phi(z) := \frac{1}{2} \{(-az^{p+1} + (b+1)z^p - 1) + \sqrt{z(z)}\},
\]

where \( \zeta(z) := a^2z^{2p+2} - 2a(b+1)z^{p+1} + (b+1)^2z^{2p} + 2az^{p+1} + 2(b-1)z^p + 1 \). Clearly, \( \phi(0) = 0 \)
and \( \phi \) is analytic in a neighborhood of \( O \). It is not hard to verify that the derivative \( \phi'(0) \)
vanishes. Moreover, similar to the proof of Lemma 3 in [13, p. 199], we can prove that

\[
\phi^{(p)}(0) = \frac{1}{2} \left\{(b+1)A_p + \frac{1}{2} \left[2(b-1)A_p^p\right]\right\} = b!p!,
\]

\[
\phi^{(k)}(0) = 0, \quad \forall k \leq p - 1,
\]

when \( p \geq 2 \), where \( A_m^k \) denotes the numbers of permutation, i.e., \( A_m^k = m!/(m-k)! = m(m-1)\ldots(m-k+1) \). Thus, from (2.4) we get

\[
\phi(z) = \frac{1}{p!} \phi^{(p)}(0)z^p + \text{h.o.t.} = bz^p + \text{h.o.t.}
\]

in a neighborhood of \( O \), where h.o.t. denotes all terms with order \( \geq p + 1 \). Therefore, from the
first equation of (2.3) we see that

\[
\frac{dz}{d\tau} = Z_2(\phi(z), z) = z^{p+1} + \text{h.o.t.},
\]

in which the term \( z^{p+1} \) of the lowest degree plays the decisive role.
Either from (2.2) or (2.8), Theorem 7.1 in [18, Ch. 2] implies that $O$ is either a saddle-node (precisely, a saddle on the side of positive $z$ and a node on the side of negative $z$) for odd $p$ or a saddle for even $p$. Furthermore, the stable manifold of system (1.2) near $O$ is on the $u$-axis, i.e., $z = 0$. On the other hand, for $p = q = 1$ $u_1 = 0$ is a center manifold, so $u = bz$ is the orbit leaving from the origin in the first quadrant. For $p \geq 2$ and $q = 1$, there is either an unstable manifold or a center manifold tangent to the $z$-axis at $O$. Those invariant manifolds in the first quadrant imply our results explicitly.

**Lemma 2.** For $p = 1$ and $q \geq 2$, system (1.2) has an orbit leaving $O$ in the direction $\theta = \arctan(1/\sqrt{q})$, an orbit entering $O$ in $\theta = 0$ on the positive half $u$-axis, and no more orbits connect with $O$ in the first quadrant.

**Proof.** With the polar coordinates system (1.2), reduces to

\[
\frac{dr}{d\tau} = b r^q \sin^q \theta \cos \theta - r^q \cos^q \theta + (b + 1) r^{q + 1} \sin^q \theta \\
- r^{q + 1} \cos \theta - ar^{q + 2} \sin^{q + 1} \theta,
\]

\[
\frac{d\theta}{d\tau} = r^{q - 1} \sin \theta \cos^q \theta - br^{q - 1} \sin^{q + 1} \theta. \tag{2.9}
\]

As in [18], no orbits connect the equilibrium $O$ in directions other than characteristic directions, and for a characteristic direction $\theta = \theta_0$ of the system, there exists a sequence of points $P_n : (r_n, \theta_n)$, such that $\lim_{n \to \infty} \tan \theta_n = \lim_{n \to \infty} r \frac{d\theta}{dr} |_{(r, \theta)} = 0$. In order to find characteristic directions of (1.2), rewrite (2.9) as

\[
\frac{1}{r} \frac{dr}{d\theta} = \frac{H(\theta) + o(1)}{G(\theta) + o(1)}, \tag{2.10}
\]

near $O$, where $G(\theta) = \sin \theta \cos^q \theta - b \sin^{q + 1} \theta$, $H(\theta) = b \sin^q \theta \cos \theta - \cos^{q + 1} \theta$. Hence, a characteristic direction $\theta = \theta_0$ must satisfy the characteristic equation $G(\theta) = 0$, which obviously has exactly four roots: 0, $\arctan(1/\sqrt{q})$, $\pi$, $\pi + \arctan(1/\sqrt{b})$ in $[0, 2\pi)$. We only need to discuss the first two, which lie in the first quadrant.

$\theta_1 := 0$ is a simple root and satisfies $G'(0)H(0) < 0$. Theorem 3.7 in [18, Ch. 2] implies that system (1.2) in this case has a unique orbit going towards $O$ in the direction $\theta = \theta_1$ as $\tau \to \infty$. Actually, this orbit is on the positive half $u$-axis, i.e., $z(u) \equiv 0$ in the $u$-$z$ plane.

We cannot discuss orbits in $\theta = \theta_2 := \arctan(1/\sqrt{q})$ directly by the known theorems in [18, Ch. 2], because $H(\theta_2) = 0$ but the theorems in [18] require the condition that $H(\theta_2) \neq 0$. We will discuss by constructing an appropriate normal sector [18,19] around the characteristic direction.

From (2.9), we see that $\frac{d\theta}{dr} < 0$ as $\theta > \theta_2$ and $\frac{d\theta}{dr} > 0$ as $\theta < \theta_2$ near $\theta_2$. Thus, we can consider a sufficiently small angular neighborhood of $\theta_2$, denoted by $\triangle OAB := \{\theta : \theta_A \leq \theta \leq \theta_B\}$, where $\theta_A = \theta_2 - \vartheta$, $\theta_B = \theta_2 + \vartheta$, and $\vartheta > 0$ is small. Consider an open sector $\triangle OAB$ in the first quadrant, where the lengths of $OA$ and $OB$ are the same. From (2.9),

\[
\frac{1}{r} \frac{dr}{d\theta} = (b \sin^q \theta - \cos^q \theta)(\cos \theta + r) + r \sin^q \theta (1 - ar \sin \theta). \tag{2.11}
\]

Observe the second term. When $0 < r \leq 1/2a$ and $|\theta - \theta_2| \leq 1/2M$, where $M := q \max_{\theta_A \leq \theta \leq \theta_B} |\sin^{q-1} \theta|/|\sin^q \theta_2|$, we have

\[
\frac{r \sin^q \theta (1 - ar \sin \theta)}{r \sin^q \theta_2} = \frac{r (1 - ar \sin \theta)}{r (1 - ar \sin \theta_2)} \frac{\sin^q \theta_2 + (\theta - \theta_2) q \sin^{q-1} \xi}{\sin^q \theta_2}
\]

\[
= r \sin^q \theta_2 \left[1 + \frac{q \sin^{q-1} \xi (\theta - \theta_2)}{\sin^q \theta_2}\right] (1 - ar \sin \theta)
\]

\[
\geq r \sin^q \theta_2 (1 - M |\theta - \theta_2|) (1 - ar)
\]

\[
\geq \frac{1}{4} r \sin^q \theta_2 > 0,
\]

(2.12)
for a certain $\xi$ between $\theta$ and $\theta_2$. On the other hand, for such $r$ and $\theta$,

$$|(b \sin^q \theta - \cos^q \theta)(\cos \theta + r)| \leq \left(1 + \frac{1}{2a}\right)|b \sin^q \theta - \cos^q \theta|,$$

in the first term of (2.11). By continuity and the fact $b \sin^q \theta_2 - \cos^q \theta_2 = 0$, there exists a constant $\delta > 0$, such that

$$|b \sin^q \theta - \cos^q \theta| < \frac{(r \sin^q \theta_2)}{4(1 + 1/2a)}$$

as $|\theta - \theta_2| \leq \delta$. Thus, from (2.11)-(2.13), we see that

$$\frac{dr}{d\tau} \geq r^q \left\{\frac{1}{4}r \sin^q \theta_2 - \left(1 + \frac{1}{2a}\right)|b \sin^q \theta - \cos^q \theta|\right\} > 0,$$

when $0 < r \leq 1/2a$ and $|\theta - \theta_2| \leq \min\{1/2M, \delta\}$. Therefore, the sector $\triangle OAB$ where the length of $OA(=OB)$ is sufficiently small is a normal sector of type II (see the definition in [18,19]). By Lemma 3.2 in [18, Ch. 2], system (1.2) has orbits going away from $O$ in the direction $\theta = \theta_2$ as $\tau \to \infty$.

For indirect proof of the uniqueness of orbit in $\theta = \theta_2$, assume that there are two orbits $\overline{OQ}, \overline{OR}$ in the normal sector $\triangle OAB$, which both start from $O$ and intersect $AB$ at $S_1, S_2$, respectively. Let $\mathcal{L}$ be the boundary of the region $\mathcal{D}$ surrounded by the orbits $\overline{OS}_1, \overline{OS}_2$ and circular arc $\overline{S_1S_2}$. Let $(P,Q)$ denote the vector field of system (1.2) and $\bar{n}$ be the unit outer normal vector of $\mathcal{L}$.

Then,

$$\int_{\mathcal{L}} (P,Q) \cdot \bar{n} ds = \int_{\overline{S_1S_2}} (P,Q) \cdot \bar{n} ds > 0,$$

because $(P,Q) \cdot \bar{n} = 0$ on the segments $\overline{OS}_1$ and $\overline{OS}_2$ of orbits and the angle between $(P,Q)$ and $\bar{n}$ is less than $\pi/2$ on the arc $\overline{S_1S_2}$ as shown in last paragraph. However, by the Green's Formula,

$$\int_{\mathcal{L}} (P,Q) \cdot \bar{n} ds = \iint_{\mathcal{D}} \text{div} (P,Q) \ du \ dz$$

$$= \iint_{\mathcal{D}} \left\{-qu^{q-1} + (b + 1)(q + 2)z^q - (q + 2)u^q \right\} \ du \ dz < 0,$$

where $k$ is the slope of $OA$. In fact, $z < ku$ in $\mathcal{D}$ and

$$-qu^{q-1} + (b + 1)(q + 2)k^q u^q = -qu^{q-1} \left(1 - \frac{(b + 1)(q + 2)k^q u}{q}\right) < 0,$$

for sufficiently small $u > 0$. The contradiction between (2.15) and (2.16) implies that system (1.2) has a unique orbit in the direction $\theta = \theta_2$ in the first quadrant, which goes away from $O$ as $\tau \to \infty$.

The proof is completed. \  

**Lemma 3.** For $p \geq 2$ and $q \geq 2$, system (1.2) has an orbit leaving $O$ in the direction $\theta = \pi/2$, an orbit entering $O$ in $\theta = 0$ on the positive half $u$-axis, and no more orbits connect with $O$ in the first quadrant.
PROOF. With the same polar coordinates as in the proof of Lemma 2, system (1.2) can be represented as

\[ \frac{dr}{d\tau} = -r^q \cos^{q+1} \theta - r^{q+1} \cos^q \theta + br^{p+q-1} \sin^{p+q-1} \theta \cos \theta \\
+ (b + 1) r^{p+q} \sin^{p+q-1} \theta - ar^{p+q+1} \sin^{p+q} \theta, \tag{2.17} \]

\[ \frac{d\theta}{d\tau} = r^{q-1} \sin \theta \cos^q \theta - br^{p+q-2} \sin^{p+q} \theta, \]

which can be rewritten further in the form of (2.10), where

\[ G(\theta) = \sin \theta \cos^q \theta, \quad H(\theta) = -\cos^{q+1} \theta. \tag{2.18} \]

The characteristic equation \( G(\theta) = 0 \) has exactly four roots: 0, \( \pi/2 \), \( \pi \), 3\( \pi/2 \) in \([0, 2\pi)\) but it suffices to discuss \( \theta_1 = 0 \) and \( \theta_2 = \pi/2 \) in the first quadrant.

\( \theta_1 = 0 \) is a simple root, such that \( G'(0)H(0) < 0 \). By Theorem 3.7 in [18, Ch. 2], system (1.2) has a unique orbit going towards \( O \) in the direction \( \theta = \theta_1 \) as \( \tau \to \infty \). Actually, this orbit is on the positive half \( w \)-axis, i.e., \( z(u) \equiv 0 \) on the \( w \)-z plane.

In the direction \( \theta = \theta_2 \), \( G'(\theta_2) = H(\theta_2) = 0 \), which does not match any condition of a theorem in found references [18,19]. Then, we hope to construct normal sectors as done in the proof of Lemma 2. Unfortunately, the direction \( \theta = \theta_2 \) is on the positive \( z \)-axis, the boundary of the first quadrant, so that we have no way to construct a normal sector, an open subset in the first quadrant, to contain the direction. Moreover, even if we extend our scope to both the first quadrant and the second quadrant, as to be shown later, both the horizontal isocline \( \mathcal{H} \) and the vertical one \( \mathcal{V} \) are tangent to the \( z \)-axis at the origin \( O \), which implies that \( \frac{dz}{d\tau} \) will change its sign near the \( z \)-axis in the neighborhood of \( O \) and therefore \( \frac{dz}{d\tau} \) vanishes somewhere in the sector between the two isoclines. This also prevents us to construct a normal sector about \( \theta_2 \) as usual.

In what follows, we construct another kind of angular regions which allows curves and orbits to be boundaries of those regions. From \( \frac{dz}{d\tau} = 0 \) in (2.17), we obtain a line \( z = 0 \) and a curve,

\[ \mathcal{Q}_1 := \left\{ (u, z) \in \mathbb{R}_+^2 : u^q = bz^{p+q-1}, 0 < \sqrt{u^2 + z^2} < \ell \right\}, \]

in the \( u-z \) plane, where \( \ell > 0 \) is a sufficiently small constant. \( \frac{dz}{d\tau} > 0(< 0) \) below (above) \( \mathcal{Q}_1 \). \( \mathcal{Q}_1 \) is tangent to the \( z \)-axis at \( O \) in the first quadrant because \( p + q - 1 > q \). Furthermore, from \( \frac{dz}{d\tau} = 0 \) in (1.2), we obtain horizontal isoclines \( z = 0 \) and \( \mathcal{H} := \left\{ (u, z) \in \mathbb{R}_+^2 : u = \sqrt{(1 - az) + b z^{(p+q-1)/q}}, 0 < \sqrt{u^2 + z^2} < \ell \right\}, \]

which lies below \( \mathcal{Q}_1 \) because \( \sqrt{(1 - az) + b} > \sqrt{b} \) and, therefore, is also tangent to the \( z \)-axis at \( O \). From \( \frac{dz}{d\tau} = 0 \) in (1.2), we obtain vertical isoclines implicitly given by

\[ f(u, z) := bz^{p+q-1} + (b + 1) uz^{p+q-1} - au z^{p+q} - u^q - u^{q+1} = 0. \]

Notice that \( \hat{f}(u, \zeta) := b' \zeta + (b + 1) u \zeta - au z^{(p+q)/(p+q-1)} - u^q - u^{q+1} \) vanishes at \( (u, \zeta) = (0, 0) \) and its partial derivative with respect to \( \zeta \) does not equal 0 there. Thus, the implicit function theorem gives a unique function \( \zeta = \zeta(u) \), such that \( \zeta(0) = 0 \) and \( \hat{f}(u, \zeta(u)) = 0 \) near the origin. Hence, a vertical isocline

\[ \mathcal{V} := \left\{ (u, z) \in \mathbb{R}_+^2 : z = \left( \zeta(u) \right)^{1/(p+q-1)}, \sqrt{u^2 + z^2} < \ell \right\}, \]

is determined uniquely in the interior of the first quadrant, which is tangent to the \( z \)-axis at the origin, because it lies between \( \mathcal{Q}_1 \) and \( \mathcal{H} \), which is implied by the continuity of \( f \) and the fact
that \( \frac{du}{dz} |_{Q_1} = (1 - az)uz^{p+q-1} > 0, \frac{du}{dz} |_{H} = -(1 - az)uz^{p+q-1} < 0 \), i.e., \( f(u, z) > 0 \) on \( Q_1 \) and \( f(u, z) < 0 \) on \( H \).

Consider the sign of \( \frac{\partial}{\partial u} (\tilde{u}/\tilde{z}) \). From (1.2),

\[
\frac{\partial}{\partial u} \left( \frac{\tilde{u}}{\tilde{z}} \right) = \frac{K(u, z) + K_1(u, z)}{(\tilde{z})^2},
\]

where \( K(u, z) := (b+1)^2z^{2p+2q-1} - au^{q-1}z^{p+q+u^2q}z \) and \( K_1(u, z) := a^2z^{2p+2q+1} - 2a(b+1)z^{2p+2q+2au^2z^{p+q+1}} - 2au^qz^{p+q+1} - 2b(b+1)u^qz^{p+q+1} \). The sign of \( \frac{\partial}{\partial u} (\tilde{u}/\tilde{z}) \) is determined by the sign of \( K(u, z) \), because each term in \( K_1(u, z) \) is higher than a corresponding term in \( K(u, z) \). Moreover, the denominator vanishes only on the positive half u-axis, the positive half z-axis, and \( H \). Clearly, \( K(u, z) = 0 \) on either \( u = 0 \) or \( z = 0 \) or the curves

\[
Q_2 := \left\{ (u, z) \in \mathbb{R}^2_+ : u = \left( \frac{(b + 1)^2}{q} \right)^{1/(q-1)}z^{(p+q-1)/(q-1)} + \text{h.o.t.}, \ 0 < \sqrt{u^2 + z^2} < \ell \right\},
\]

\[
Q_3 := \left\{ (u, z) \in \mathbb{R}^2_+ : u = q^{1/(q+1)}z^{(p+q-1)/(q+1)} + \text{h.o.t.}, \ 0 < \sqrt{u^2 + z^2} < \ell \right\}.
\]

Moreover, \( Q_2 \) is tangent to the \( z \)-axis at \( O \). So is \( Q_3 \) for \( p \geq 3 \). \( Q_2 \) lies above \( Q_1 \) because \( \sqrt{b^2z^{(p+q-1)/q}} < \left( (b + 1)^2/q \right)^{1/(q-1)}z^{(p+q-1)/(q-1)} \) near the origin in the first quadrant. Furthermore, \( \left( \frac{\partial K}{\partial u} \right)(u, z) = qu^{q-2}z\left\{ -(q - 1)z^{p+q-1} + 2u^{q+1} \right\} \). We also see that \( \frac{\partial K}{\partial u} = 0 \) on either \( u = 0 \) or \( z = 0 \) or the curve

\[
Q_4 := \left\{ (u, z) \in \mathbb{R}^2_+ : u = \left( \frac{q-1}{2} \right)^{1/(q-1)}z^{(p+q-1)/(q-1)}, \ 0 < \sqrt{u^2 + z^2} < \ell \right\},
\]

which lies below \( H \) because \( \left( (q - 1)^2/2 \right)^{1/(q-1)}z^{(p+q-1)/(q-1)} > 4\sqrt{1 - az} + bz^{(p+q-1)/q} \) near the origin in the first quadrant and is tangent to the \( z \)-axis at \( O \) for \( p \geq 3 \). Moreover, \( Q_4 \) lies above \( Q_3 \).

Summarily, we obtain curves \( H, V, \) and \( Q_i, i = 1, \ldots, 4 \), which divide the first quadrant into seven open angular regions: \( \Delta \tilde{W} \tilde{O} \tilde{Q}_2 \), \( \Delta \tilde{W} O \tilde{Q}_1 \), \( \Delta \tilde{Q}_1 \tilde{V} \), \( \Delta \tilde{V} \tilde{O} \tilde{H} \), \( \Delta \tilde{H} \tilde{O} \tilde{Q}_4 \), \( \Delta \tilde{Q}_4 \tilde{O} \tilde{Q}_3 \), and \( \Delta \tilde{Q}_3 \tilde{O} \tilde{U} \). Let \( U = \{(u, z) \in \mathbb{R}^2_+ : z = 0, \ 0 < \sqrt{u^2 + z^2} < \ell \}, \ Z = \{(u, z) \in \mathbb{R}^2_+ : u = 0, \ 0 < \sqrt{u^2 + z^2} < \ell \} \).

Clearly,

\[
\frac{\partial}{\partial u} \left( \frac{\tilde{u}}{\tilde{z}} \right) > 0, \quad \text{as} \ (u, z) \in \Delta \tilde{W} \tilde{O} \tilde{Q}_2 \cup \Delta \tilde{Q}_3 \tilde{O} \tilde{U},
\]

\[
\frac{\partial}{\partial u} \left( \frac{\tilde{u}}{\tilde{z}} \right) < 0, \quad \text{as} \ (u, z) \in \Delta \tilde{Q}_4 \tilde{O} \tilde{U} \cup \Delta \tilde{V} \tilde{O} \tilde{H} \cup \Delta \tilde{H} \tilde{O} \tilde{Q}_3.
\]

We first claim that there are no orbits approaching \( O \) in the region \( \Delta \tilde{W} \tilde{O} \tilde{Q}_2 \). This region is an open quasisector containing neither horizontal isolines nor vertical ones. Assume that an orbit \( \Gamma : u = \tilde{u}(z) \) connects with \( O \) in \( \Delta \tilde{W} \tilde{O} \tilde{Q}_2 \). By the continuity and tangency of \( \Gamma \) at \( O \), for small \( \ell > 0 \) there is a small positive constant \( c_1 < \ell \), such that \( \Gamma \subset \Delta \tilde{W} \tilde{O} \tilde{Q}_2 \) and \( \frac{d\tilde{u}(z)}{dz} \leq \sigma_0/2 \), for \( 0 < z < c_1 \), where \( \sigma_0 < b/(b + 1)\ell \) is a positive constant. However, for \( 0 < z < c_1 \), by the mean value theorem, there exists a \( \xi \in (0, \tilde{u}(z)) \), such that

\[
\frac{d\tilde{u}(z)}{dz} = \frac{\tilde{u}(z)}{z} |_{(\tilde{u}(z), z)} = \left( \frac{\tilde{u}}{z} |_{(\tilde{u}(z), z)} - \frac{\tilde{u}}{z} |_{(0, z)} \right) + \frac{\tilde{u}}{z} |_{(0, z)} \]

\[
\geq \frac{\tilde{u}}{z} |_{(0, z)} > \frac{b}{(b + 1)\ell} > \sigma_0.
\]

Thus, our claim is proved by this contradiction.
Continue to discuss the region $\Delta Q_2 O V = \Delta Q_2 O Q_1 \cup \Delta Q_3 O V$, where $\frac{\partial}{\partial u} (u/z) < 0$, and claim that there is at most one orbit approaching $O$ in the region. Suppose that $\Gamma_1 : u = u_1(z)$ and $\Gamma_2 : u = u_2(z)$ are both orbits connecting with $O$ in the region. By continuity, there is a small positive constant $c_2 < \ell$, such that $\Gamma_1, \Gamma_2$ both lie in $\Delta Q_2 O V$ and that there is no intersection of the two curves for $0 \leq z \leq c_2$, except for $O$. Without loss of generality, we suppose that $u_2(c_2) > u_1(c_2)$. Then, $u_2(z) > u_1(z)$, for all $0 < z < c_2$. By the mean value theorem,

$$\frac{d}{dz} \left( u_2(z) - u_1(z) \right) = \frac{d}{dz} \left( u_2(z) - u_1(z) \right)_{u_1, z} = \frac{\partial}{\partial u} \left( \frac{u}{z} \right) \left( u_2 - u_1 \right) \leq 0,$$

where $\eta$ is a constant in $(u_1(z), u_2(z))$. This implies that $u_2(z) - u_1(z)$ is a nonincreasing function in $z \in (0, c_2]$ and therefore, $u_2(z) - u_1(z) \leq 0$ for $0 < z \leq c_2$ since $u_2(0) - u_1(0) = 0$. This contradicts that $u_2(z) > u_1(z)$.

Then, we discuss existence of orbits. We know that there are neither horizontal isoclines nor vertical isoclines in the interior of $\Delta Q_2 O V$. Moreover, $\frac{du}{dz} = b z^{p+q-1} + \text{h.o.t.} > 0$ on $Q_2$. By continuity, $\frac{du}{dz} > 0$ in $\Delta Q_2 O V$. On the other hand, $\frac{du}{dz} > 0$ in $\Delta Q_2 O V$ because this region lies above $\mathcal{H}$, i.e., $u < \sqrt{(1 - az) + b z^{p+q-1}/q}$. Therefore, $\frac{du}{dz} = \left( u + z^2 \right)/r > r$ in $\Delta Q_2 O V$, implying that the possible orbit of system (1.2) connecting with $O$ in this angular region must leave $O$ as $r \to +\infty$. We further observe $\Delta Q_2 Q_1$. Since $0 < u < \sqrt{(1 - az) + b z^{p+q-1}/q}$ in $\mathcal{A} \backslash \{O\}$, we see that $\frac{du}{dr} \geq u z^{p+q-1} (1 - az) \geq 0$ and $\frac{du}{r} = z^{p+q} (1 - az) > 0$, and therefore,

$$\frac{dr}{dz} = \frac{u \hat{u} + z \hat{z}}{r} > 0 \quad (2.21)$$

in $\mathcal{C}(\Delta Q_2 Q_1) \backslash \{O\}$. Thus, an orbit starting from the boundary $Z$ has to enter into the closure $\mathcal{C}(\Delta Q_2 Q_1)$. So does an orbit from another boundary part $Q_1$, because $\frac{dz}{dr} = 0$ in curve $Q_1$. We know the curves $Z, Q_1$ have the same endpoint $O$. Let $Z, Q_1$ denote the other endpoints of the curves, respectively. For arbitrary $M \in Z$, the orbit passing through $M$ must intersect $\overline{ZQ_1}$ at $P_M$ at a certain positive time by (2.21) and continuity because there are no equilibria in the region $\Delta Q_2 Q_1$. It follows that $P_0 = \lim_{M \to O} P_M \in \mathcal{C}(\overline{ZQ_1})$, as well as on $Q_1$ and another limit $Q_0 \in \mathcal{C}(\overline{ZQ_1})$, is obtained. Obviously, the segment $P_0 Q_0$ is nonempty and all orbits starting from it go towards $O$ as $r \to -\infty$. In particular, such orbit is unique if $P_0 = Q_0$. Thus, we have proved that system (1.2) has no orbits connecting with $O$ in $\Delta Q_2 Q_3$, at most one such orbit in $\Delta Q_2 O V$, and at least one in $\Delta Q_2 Q_1$. It implies that system (1.2) has a unique orbit in $\Delta Q_2 O V$, which actually leaves $O$ in $\Delta Q_2 Q_1$, as $r \to -\infty$.

Further, consider the region $\Delta Q_2 Q_3$ and claim that no orbits connect with $O$ in this region. For indirect proof, suppose that $\Gamma : u = \hat{u}(z)$ is such an orbit. Then $\hat{u}(0) = 0$. By continuity, there is a positive constant $c_3 < \ell$, such that $\{ \hat{u}(z) : 0 < z \leq c_3 \} \subset \Delta Q_2 Q_3$. Clearly, $c_3 > 0$ and $\hat{u}(c_3) > 0$. By the mean value theorem, there exists $0 < \xi < c_3$, such that

$$\frac{d\hat{u}}{dz}(\xi) = \frac{\hat{u}(c_3)}{c_3} > 0. \quad (2.22)$$

On the other hand, it is obvious that $\frac{d\hat{u}}{dz} = z^{p+q-1} (-1 + az) < 0$ in $\mathcal{H}$ and $\frac{d\hat{u}}{dz} > 0$ in $\mathcal{V}$ because $u < \sqrt{(1 - az) + b z^{p+q-1}/q}$. By continuity, $\frac{d\hat{u}}{dz} < 0$ and $\frac{d\hat{u}}{dz} > 0$, i.e., $\frac{d\hat{u}}{dz} < 0$ in $\Delta Q_2 Q_3$, because there are neither horizontal isoclines nor vertical ones in the region. This inequality contradicts to (2.22).

At last, we discuss $\Delta \mathcal{H} O \mathcal{U} = \Delta \mathcal{H} O \mathcal{Q}_3 \cup \Delta \mathcal{Q}_3 O \mathcal{U}$. From (2.20), $\frac{\partial}{\partial u} (u/z) < 0$ in the first subregion and $> 0$ in the second. Similar to $\Delta Q_2 O V$ discussed as above, we also see that system (1.2) has at most one orbit connecting with $O$ in $\Delta \mathcal{H} O \mathcal{Q}_3$. In $\Delta \mathcal{Q}_3 O \mathcal{U}$, similar to $\Delta Q_2 O V$ discussed as above, there are no orbits connecting with $O$ because $\hat{u}/\hat{z} = (1/z) + \text{h.o.t.} \geq 1/\ell$
on $Q_3$. It is worth mentioning that for $p = 2$, we have no need to discuss $\Delta Q_3 O U$ because $Q_3$ is a line then and not tangent to the $z$-axis. In order to see the possibility of the orbit in $\Delta Q_3 U$, we investigate a larger region $\Delta H O U$. Clearly, $H < 0$ in $H$, which implies that orbits starting from $H$ all depart from $\Delta H O U$ as $\tau \to \infty$. Another boundary part $U$ is a segment of orbit which goes towards $O$ as $\tau \to \infty$. Similar to $\Delta Q_3 O$, discussed as above, in the closure $\text{cl}(\Delta H O U)$, there are either infinitely many orbits (which go towards $O$ as $\tau \to +\infty$) or a unique one (which must be the line $U$). It follows that there are either infinitely many such orbits or no such ones in $\Delta H Q_3 O$. Consequently, no orbits connect with $O$ in $\Delta H Q_3 U$. Since none of $H, V, Z$, and $Q_i$, $i = 1, \ldots, 4$ is an orbit of system (1.2), we see that there is exactly one orbit connecting with $O$ in the direction $\theta = \theta_2$, which actually leaves $O$ as $\tau \to +\infty$.

The proof of the lemma is completed.

Summarizing results in Lemmas 1–3, we obtain Theorem 1 immediately and omit the statement of its proof.

3. QUALITATIVE ANALYSIS OF EQUILIBRIUM $I_y$

THEOREM 2. For positive integers $p$ and $q$, system (1.1) has a unique orbit leaving $I_y$ in the direction vertical to the $y$-axis. There are no orbits connecting with $I_y$.

Similar to $I_x$, it suffices to discuss (1.3) in the first quadrant of the $v$-$z$ plane in the case of $p = 1$ and the case of $p \geq 2$.

LEMMA 4. For $p = 1$, system (1.3) has a unique orbit leaving $O$ in $\theta = 0$ on the positive half $v$-axis, and no more orbits connect with $O$ in the first quadrant.

PROOF. In this case, system (1.3) is degenerate but has a nonzero eigenvalue 1. The system can be rewritten as

\begin{align*}
\frac{dz}{d\tau} &= v z - b v z^{1+q} := Z_3 (v, z), \\
\frac{dv}{d\tau} &= v + v^2 + a z^{1+q} - (b+1) v z^q - b v^2 z^q := V (v, z).
\end{align*}

(3.1)

Note that $V(0,0) = 0$. From $V(v,z) = 0$, we obtain an implicit function,

\begin{equation}
v = \phi_1 (z) := \frac{\beta (z) + \sqrt{\alpha_1 (z)}}{\alpha (z)},
\end{equation}

where $\alpha (z) := 2(-bz^q + 1)$, $\beta (z) := (b+1)z^q - 1$, and $\alpha_1 (z) := 4abz^{2q+1} - (b+1)^2 z^{2q} - 4az^{q+1} - 2(b+1)z^q + 1$. Clearly, $\phi_1 (0) = 0$ and $\phi_1$ is analytic in a neighborhood of $O$. We calculate the derivative $\phi_1' (z) = \omega (z) / \alpha^2$, where $\omega (z) = I_1 + I_2 - I_3 - I_4$ and $I_1 := \alpha (z) \beta' (z)$, $I_2 := \alpha (z) s_1 (z) / (2s_1^{1/2})$, $I_3 := \alpha' (z) \beta (z)$, and $I_4 := \alpha' (z) s_1^{1/2}$, and check easily that $\phi_1' (0) = 0$. For $q = 1$, we further calculate $\phi_1'' (0) = -2a < 0$. Thus,

\begin{equation}
\phi_1 (z) = -az^2 + \text{h.o.t.},
\end{equation}

in a neighborhood of $O$. More generally, we claim that for $q \geq 2$,

\begin{equation}
\phi_1^{(k)} (0) = 0, \quad k = 2, \ldots, q, \quad \phi_1^{(q+1)} (0) = -a (q + 1)!. \tag{3.4}
\end{equation}

In fact, we can prove that

\begin{equation}
\phi_1^{(k)} (z) = \frac{\omega^{(k-1)} (z)}{\alpha^2} + \sum_{i=1}^{k-1} a_i \phi_1^{(i)} (z) \left( \frac{\alpha^2}{\alpha^2} \right)^{(k-1-i)} + \Xi (z; q, k),
\end{equation}

(3.5)
for $k \geq 2$ by induction, where

$$
\Xi(z; q, k) = \sum_{j=2}^{k-1} \frac{b_j}{\alpha^{2j}} \sum_{l_1, \ldots, l_j \text{ are positive integers}} c_{l_1, \ldots, l_j} \omega^{(l_1)}(\alpha^{2}) \cdots \omega^{(l_j)}(\alpha^{2^{j-1}}),
$$

(3.6)

$a_i, b_j,$ and $c_{l_1, \ldots, l_j}$ are real constants and $l_1, \ldots, l_j$ are positive integers. Consider $\omega^{(l)}(z)$ and notice that

$$
I_l^{(l)}(0) = 0, \quad l = 0, \ldots, q - 2, q, \quad I_l^{(q-1)}(0) = 2(b + 1)q!,
$$

(3.7)

Imitating the procedure in the proof of Lemma 3 in [13, (p. 199)], we obtain

$$
I_4^{(l)}(0) = 0, \quad l = 0, \ldots, q - 2, q, \quad I_4^{(q-1)}(0) = -2bq!.
$$

(3.8)

The calculation about $I_2$ is complicated relatively, because a square root appears in its denominator. Obviously, $I_2(0) = 0$ and $I_2'(0) = -4(b + 1)$ for $q = 2$ and $I_2(0) = 0$ for $q \geq 3$. By induction, we see that for $l \geq 2$,

$$
I_l^{(l)} = \sum_{i=0}^{l} d_l \mathcal{G}^{(l)}(z) \frac{1}{\mathcal{S}_1^{3/2}} + \sum_{j=0}^{l-2} c_j I_2^{(j)} \left[ \frac{\mathcal{S}_1}{\mathcal{S}_1} \right]^{(l-2-j)},
$$

(3.9)

where $\mathcal{G}(z) = \alpha(z) \mathcal{S}_1(z)/2$ and $d_l, c_j$ are constants. In particular, $d_l = 1$. We further calculate that $\mathcal{G}(0) = 0, i = 0, \ldots, q - 2, \mathcal{G}^{(q-1)}(0) = -2(b + 1)q!$, $\mathcal{G}^{(q)}(0) = -4a(q + 1)!$, and that $c_1(0) = 1, c_1^{(1)}(0) = 0, i = 1, \ldots, q - 1, c_1^{(q)}(0) = -2(b + 1)q!$. Thus, the recursive formula (3.9) gives $I_2^{(l)}(0) = 0, l = 0, \ldots, q - 2$, and

$$
I_2^{(q-1)}(0) = -2(b + 1)q!, \quad I_2^{(q)}(0) = -4a(q + 1)!.
$$

(3.10)

By the definition of $\omega(z)$, (3.7), (3.8), and (3.10)

$$
\omega^{(l)}(0) = 0, \quad l = 0, \ldots, q - 1, \quad \omega^{(q)}(0) = -4a(q + 1)!. \tag{3.11}
$$

Thus, it is ready to calculate $\phi_1^{(l)}(z)$, where we note that $\Xi(z; q, k) = 0$ for $k \leq q + 1$ because every term of $\Xi(z; q, k)$ includes a factor $\omega^{(l)}(z), 1 \leq l \leq k - 2$. The claimed result in (3.29) can be checked easily.

By (3.4) and (3.2),

$$
\phi_1(z) = \frac{1}{(q + 1)!} \phi_1^{(q+1)}(0) z^{q+1} + \text{h.o.t.} = -az^{q+1} + \text{h.o.t.}
$$

(3.12)

for $q \geq 2$ in a neighborhood of $O$. Together with (3.3), from the first equation of (3.1), we see that $\frac{d\phi}{dz} = Z_3(\phi_1(z), z) = -az^{q+2} + \text{h.o.t.}$, in which the term $-az^{q+2}$ of the lowest degree plays the decisive role. Since the decisive term has an coefficient $-a < 0$ and the eigenvalues of system (3.1) are 0 and 1, Theorem 7.1 in [18, Ch. 2] implies that $O$ is a saddle-node (precisely, a saddle on the side of positive $z$ and a node on the side of negative $z$) for even $q$ or a saddle for odd $q$. Furthermore, the unstable manifold of system (3.1) near $O$ is on the $v$-axis, i.e., $z \equiv 0$. On the other hand, since $\frac{d\phi}{dz} = az^{q+q} > 0$ in the positive half $z$-axis, either the stable manifold for odd $q$ or the center manifold for even $q$ on the side of positive $z$ lies in the second quadrant. These invariant manifolds imply our results explicitly.
LEMMA 5. For $p \geq 2$, system (1.3) has a unique orbit leaving $O$ in $\theta = 0$ on the positive half $v$-axis, and no more orbits connect with $O$ in the first quadrant.

PROOF. With the polar coordinates system, (1.3) reduces to

$$
\frac{dr}{d\tau} = r^p \cos^{p+1} \theta + r^{p+1} \cos^p \theta - br^{p+q} \sin^{p+q-1} \theta \cos^2 \theta \\
+ ar^{p+q} \sin^{p+q} \theta \cos \theta - br^{p+q+1} \sin^{p+q-1} \theta \cos \theta,
$$

(3.13)

and can be further rewritten in the form of (2.13), where

$$
G(\theta) = -\sin \theta \cos^p \theta, \quad H(\theta) = \cos^{p+1} \theta.
$$

From the characteristic equation $G(\theta) = 0$, we obtain exactly four roots $0, \pi/2, \pi, 3\pi/2$ in $[0, 2\pi)$ but it suffices to discuss $\theta_1 = 0$ and $\theta_2 = \pi/2$ in the first quadrant. $\theta_1 = 0$ is a simple root, such that $G'(\theta_1) \neq 0$. By Theorem 3.7 in [18, Ch. 2], system (1.3) has a unique orbit going away from $O$ in direction $\theta = \theta_1$ as $\tau \to \infty$. Actually, this orbit is on the positive half $v$-axis, i.e., $z(v) \equiv 0$ on the $v$-$z$ plane.

Consider $\theta_2 = \pi/2$. Notice that $G'(\theta_2) = H(\theta_2) = 0$. That is the case encountered in the proof of Lemma 3, where no normal sectors can be constructed about the direction $\theta_2$. In order to construct the same kind of angular region as in the proof of Lemma 3, we analyze horizontal isoclines and vertical ones of system (1.3) first. From $\frac{dz}{dt} = 0$ in (1.3) we obtain horizontal isoclines

$$
\{ (v, z) \in \mathbb{R}^2_+ : v = b/(p-1) z^{(p+q-1)/(p-1)}, \ 0 < v^2 + z^2 < \ell \},
$$

where $\ell > 0$ is a sufficiently small constant. $\mathcal{H}'$ is tangent to the $z$-axis at $O$ since $(p+q-1)/(p-1) > 1$. On the other hand, let

$$
\mathcal{Q}'_1 := \left\{ (v, z) \in \mathbb{R}^2_+ : v = \frac{az}{(2b+1)}, \ 0 < \sqrt{v^2 + z^2} < \ell \right\},
$$

which obviously lies below $\mathcal{H}'$. Let $\mathcal{Z}' := \left\{ (v, z) \in \mathbb{R}^2_+ : v = 0, \ 0 < \sqrt{v^2 + z^2} < \ell \right\}$. Then

$$
\frac{dv}{dt} \geq v^p + vz^{p+1} + \frac{az}{(2b+1)} z^{p+q-1} - (2b+1) vz^{p+q-1} = v^p + vz^{p+1} + \frac{az}{(2b+1)} z^{p+q-1} > 0
$$

(3.14)

in the region $\Delta \mathcal{Z}' \mathcal{Q}'_1$, which implies the nonexistence of vertical isoclines in this region. Since $\mathcal{Q}'_1$ is not tangent to the $z$-axis, in order to determine how many orbits connect with $O$ in the direction $\theta = \theta_2$ we only need to discuss the angular region $\Delta \mathcal{Z}' \mathcal{H}' \mathcal{Q}'_1$. This region is divided by $\mathcal{H}'$ into two open subregions $\Delta \mathcal{Z}' \mathcal{H}' \mathcal{Q}'_1$ and $\Delta \mathcal{H}' \mathcal{O} \mathcal{Q}'_1$.

We first claim that there are no orbits connecting with $O$ in $\Delta \mathcal{Z}' \mathcal{H}' \mathcal{Q}'_1$. In fact, $\frac{dz}{dt} < 0$ in $\Delta \mathcal{Z}' \mathcal{H}' \mathcal{Q}'_1$ since this region lies above $\mathcal{H}'$, i.e., $v < b/(p-1) z^{(p+q-1)/(p-1)}$. Together with (3.14), we have $\frac{dz}{dt} < 0$ in $\Delta \mathcal{Z}' \mathcal{H}' \mathcal{Q}'_1$. Such an inequality guarantees that the claimed result is true, as discussed in the region $\Delta \mathcal{Z}' \mathcal{O} \mathcal{H}'$ in the proof of Lemma 3.

In order to investigate the region $\Delta \mathcal{H}' \mathcal{O} \mathcal{Q}'_1$, we need to discuss the sign of $\frac{\partial}{\partial v'} (v'/z)$ in the region. From (1.3),

$$
\frac{\partial}{\partial v'} \left( \frac{\dot{v}}{\dot{z}} \right) = K'(v, z) + K'_1 (v, z)
$$

(3.15)
where \( K'(v, z) = v^2p - apv^{p-1}z^{p+q+1} + (p-1)v^p z^{p+q} + abz^{2p+2q} \) and \( K'_1(v, z) = -2bv^{p+1}z^{p+q} + \varepsilon^2z^{2p+2q-1} \). The sign of \( \frac{\partial K'}{\partial v}(v, z) \) is determined by the sign of \( K'(v, z) \) because each term in \( K'_1(v, z) \) is higher than a corresponding term in \( K_1(v, z) \) in \( A_\infty'0 \rightarrow \infty \).

For \( q = 1 \), it suffices to consider the subregion \( \Delta H_0' \Omega L_0 \) instead of \( \Delta H_0' \Omega Q_1' \), where \( L_0 := \{(v, z) \in \mathbb{R}^2_+ : v = \varepsilon_0z, 0 < \sqrt{v^2 + z^2} < \ell \} \) and \( \varepsilon_0 > 0 \) is a sufficiently small constant. Clearly, \( L_0 \) lies above \( Q_1' \). Notice that

\[
\frac{\partial K'}{\partial v}(v, z) = pv^{p-2}z \left\{ -a(p-1)z^{p+1} + (p-1)vz^p + 2v^{p+1} \right\} < 0,
\]

in \( \Delta H_0' \Omega L_0 \) because \( 0 < \frac{b}{(p-1)^{2p/(p+1)}} < v < \varepsilon_0z \) in the region and that \( K'(v, z)|_{\Omega'0} = (1-p)abz^{2p+2} + \text{h.o.t.} < 0 \). Thus, \( \frac{\partial K'}{\partial v}(v, z) < 0 \) in \( \Delta H_0' \Omega L_0 \) for sufficiently small \( \varepsilon_0 > 0 \). Using the same arguments as for the region \( \Delta Q_2' \Omega V \) in the proof of Lemma 3, we see that system (1.3) has at most one orbit connecting with \( O \) in \( \Delta H_0' \Omega L_0 \). Furthermore, to investigate the possibility of such an orbit in \( \Delta H_0' \Omega L_0 \), we calculate by (1.3) that \( \frac{dx}{dt} = \varepsilon_0z^{p+1}(\varepsilon_0^{p-1} - bz) > 0 \) and \( \frac{dz}{dt} = \varepsilon_0z^p[(1 + \varepsilon_0z)(\varepsilon_0^{p-1} - bz) + 2bz] > 0 \), which implies that \( \frac{dx}{dt} < \varepsilon_0 \) on \( L_0 \). Thus, orbits starting from \( L_0 \) all depart from \( \Delta Z_0' \Omega L_0 \) as \( t \rightarrow +\infty \). On the other hand, \( \frac{dx}{dt} > 0 \) in \( \Omega' \) by (3.14), which implies that orbits starting from \( \Omega' \) all enter into \( \Delta H_0' \Omega L_0 \) as \( t \rightarrow +\infty \). Similar to the regions \( \Delta Z_0' \Omega Q_1' \) and \( \Delta H_0' \Omega L_0 \) discussed in the proof of Lemma 3, the closure \( \Omega(\Delta H_0' \Omega L_0) \) contains either infinitely many or no orbits in connect with \( O \). This result, together with the “at most” result just obtained, implies that no orbits connect with \( O \) in \( \Delta H_0' \Omega L_0 \).

For \( q \geq 2 \), we see that \( \frac{\partial K'}{\partial v}(v, z) = pv^{p-2}z\{2v^{p+1} - a(p-1)z^{p+q} + (p-1)vz^{p+q} \} \). Clearly, \( \frac{\partial K'}{\partial v} = 0 \) on \( z = 0, v = 0 \) and the curve

\[
Q_2' := \{(v, z) \in \mathbb{R}^2_+ : v = \frac{(ap - a)}{2z}^{1/(p+1)}z^{(p+q)/(p+1)} + \text{h.o.t.}, 0 < \sqrt{v^2 + z^2} < \ell \}
\]

Note that \( Q_2' \) is tangent to the \( z \)-axis at \( O \) because \( (p + q)/(p + 1) > 1 \) and additionally since \( b^{1/(p-1)}z^{(p+q-1)/(p-1)} < ((ap - a)/2)^{1/(p+1)}z^{(p+q)/(p+1)} \) near the origin in the first quadrant. Therefore, in a small neighborhood of the origin \( O \), curve \( Q_2' \) lies below \( \Omega' \) and above \( Q_1' \) because \( Q_1' \) is not tangent to the \( z \)-axis at \( O \). Furthermore, we can see that

\[
K'(v, z)|_{Q_1'} = \left(\frac{a}{2b+1}\right)^{2p}z^{2p+1} + \text{h.o.t.} > 0,
\]

\[
K'(v, z)|_{Q_2'} = -\left(\frac{a}{2}\right)^{2p/(p+1)}(p+1)(p-1)^{(p-1)/(p+1)}z^{(2p^2+2pq+p+1)/(p+1)} + \text{h.o.t.} < 0,
\]

\[
K'(v, z)|_{Q_3'} = (1-p)abz^{2p+2q} + \text{h.o.t.} < 0,
\]

which imply by continuity and the monotonicity of \( K'(v, z) \) with respect to \( v \), that there exists a unique curve \( Q_3' : K'(v, z) = 0 \) in \( \Delta Q_2' \Omega Q_3' \) but no such ones in \( \Delta H_0' \Omega Q_2' \). Moreover, \( v > ((ap - a)/2)^{1/(p+1)}z^{(p+q)/(p+1)} \) on \( Q_3' \) because this curve lies below \( Q_2' \). Therefore, from \( K'(v, z) = 0 \), we can see that

\[
Q_3' := \{(v, z) \in \mathbb{R}^2_+ : v = (ap)^{1/(p+1)}z^{(p+q)/(p+1)} + \text{h.o.t.}, 0 < \sqrt{v^2 + z^2} < \ell \}
\]

Obviously, \( Q_3' \) is tangent to the \( z \)-axis. Summarily, the curves \( Q_2' \) and \( Q_3' \) divide \( \Delta H_0' \Omega Q_1' \) into three open angular regions \( \Delta H_0' \Omega Q_2', \Delta Q_2' \Omega Q_3', \) and \( \Delta Q_3' \Omega Q_1' \). Clearly,

\[
\frac{\partial}{\partial v}(\frac{v}{z}) < 0, \quad \text{as } (v, z) \in \Delta H_0' \Omega Q_3',
\]

\[
\frac{\partial}{\partial v}(\frac{v}{z}) > 0, \quad \text{as } (v, z) \in \Delta Q_2' \Omega Q_1'.
\]
Degenerate Equilibria at Infinity

Similar to the region $\Delta Q_3O\bar{V}$, discussed in the proof of Lemma 3, the subregion $\Delta \bar{H}\bar{O}Q_3$ contains, at most, one orbit of system (1.3), which connects with $O$. In the subregion $\Delta Q_3O\bar{Q}_1'$, using the same arguments as for $\Delta \bar{Z}OQ_3$ in the proof of Lemma 3, there are no orbits connecting with $O$ because $\dot{v}/\dot{z} = (1/z) + \text{h.o.t.} \geq 1/\ell$ on $\bar{Q}_3$. In order to observe the possibility of orbits connecting with $O$ in $\Delta \bar{H}\bar{O}Q_1' = \Delta \bar{H}\bar{O}Q_3 \cup \Delta Q_3O\bar{Q}_1'$, we calculate by (1.3) that

$$\frac{dz}{d\tau} = \frac{a}{2b+1} z^{p+1} \left\{ \left( \frac{a}{2b+1} \right)^{p-1} - bz^q \right\} > 0,$$

$$\frac{dv}{d\tau} = \frac{a}{2b+1} z^p \left\{ 1 + \frac{a}{2b+1} z \left( \left( \frac{a}{2b+1} \right)^{p-1} - bz^q \right) + 2bz^q \right\} > 0,$$

on $\bar{Q}_1'$ and therefore, $\frac{dz}{d\theta} < (2b+1)/a$ on $\bar{Q}_1'$, which implies that orbits starting from $\bar{Q}_1'$ all depart from $\Delta \bar{H}\bar{O}Q_1'$ as $\tau \to +\infty$. Similar to $\Delta \bar{H}\bar{O}Q_3$ discussed as above, the closure $\text{cl}(\Delta \bar{H}\bar{O}Q_1')$ contains either infinitely many orbits or no orbits which connect with $O$. Consequently, no orbits connect with $O$ in $\Delta \bar{H}\bar{O}Q_1'$.

Consequently, since none of $\mathcal{H}', \mathcal{Z}'$, and $Q_i', i = 1, \ldots, 3$ is an orbit of system (1.3), there are no orbits connecting with $O$ in the direction $\theta = \theta_2$. The proof is completed.

Summarizing results in Lemmas 4 and 5, we obtain Theorem 2 immediately and omit the statement of its proof.

REFERENCES