LEXICOGRAPHIC METHOD FOR MATRIX GAMES WITH PAYOFFS OF TRIANGULAR FUZZY NUMBERS

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The purpose of the paper is to study how to solve a type of matrix games with payoffs of triangular fuzzy numbers. In this paper, the value of a matrix game with payoffs of triangular fuzzy numbers has been considered as a variable of the triangular fuzzy number. First, based on two auxiliary linear programming models of a classical matrix game and the operations of triangular fuzzy numbers, fuzzy optimization problems are established for two players. Then, based on the order relation of triangular fuzzy numbers the fuzzy optimization problems for players are decomposed into three-objective linear programming models. Finally, using the lexicographic method maximin and minimax strategies for players and the fuzzy value of the matrix game with payoffs of triangular fuzzy numbers can be obtained through solving two corresponding auxiliary linear programming problems, which are easily computed using the existing Simplex method for the linear programming problem. It has been shown that the models proposed in this paper extend the classical matrix game models. A numerical example is provided to illustrate the methodology.

Keywords: Group decision making; fuzzy matrix game; fuzzy number; fuzzy multi-objective optimization; linear programming.

1. Introduction

Fuzzy Game Theory is an active research field in Operational Research and Systems Engineering. It provides an efficient framework in which solves the real-life conflict problems with fuzzy information and has achieved a success. Recently, an increasing number of papers and books [1–3, 5–7, 9–17, 20–21, 24] have been published on this topic in which several types of fuzzy games have been investigated and many typical real-life conflict problems such as the moving-objective search, missiles vs bomb carriers have been discussed [9]. Campos [5] solved a matrix game with imprecise payoffs by deriving two auxiliary fuzzy linear programming models from the game and obtaining a fuzzy value and optimal strategies from their solutions. Dubois and Prade in Chapter 9 of [13] gave a brief overview and discussion on the fuzzy game with crisp sets of strategies

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and fuzzy payoffs due to the lack of precision on the knowledge of the associated payoffs. Bector et al. [1] and Nishizaki and Sakawa [16] make good overviews on update research of this topic. The common features of the above methods are that all fuzzy data are first transformed into crisp ones and classical game theory is then used to solve the corresponding problems.

In this paper, we focus on reporting a further study of the solution method and computation procedure for the fuzzy matrix game with payoffs of triangular fuzzy numbers in which its value is considered as a triangular fuzzy number. Using the operations of triangular fuzzy numbers and the order relation between triangular fuzzy numbers, new auxiliary multi-objective programming models and corresponding lexicographic method for the fuzzy matrix game with payoffs of triangular fuzzy numbers are established. This study focuses on the solution method and procedure of the fuzzy matrix game and is significantly different from other methods discussed by Campos [5], Campos and Gonzalez [6], Campos et al. [7] and Bector et al. [2]. The main differences are that in our new models the value of the fuzzy matrix game is considered as a fuzzy variable and both the players do not need to use subjective ranking criteria or functions or choose some subjective parameters any more if the order relation between triangular fuzzy numbers is determined a prior. Normally, it is very difficult for players to choose subjective ranking criteria or functions or parameters. Furthermore, this study is also different from the previous models proposed by Li [9–12] in that in the models proposed in this paper the solution method of the fuzzy matrix game with payoffs of triangular fuzzy numbers is based on two auxiliary linear programming models transformed from corresponding three-objective programming models. Therefore, the auxiliary models and the solution method proposed in this paper are superior to the other ones in the following aspects. On the one hand, the auxiliary models proposed in this paper do not use subjective ranking criteria or functions except for determining the order relation between triangular fuzzy numbers, i.e., Definition 1. This decreases burden of players and raises efficiency of decision-making. On the other hand, the linear programming models transformed from corresponding three-objective programming models for the fuzzy matrix game with payoffs of triangular fuzzy numbers may be solved directly using the existing Simplex method for the linear programming problem. This can decrease computation and it is easy to obtain the minimax and maximin strategies for players and the fuzzy value of the fuzzy matrix game with payoffs of triangular fuzzy numbers.

This paper is organized as follows. In Section 2 a pair of the existing auxiliary linear programming problems derived from the crisp matrix game is examined and the main operations of triangular fuzzy numbers are also discussed. In Section 3 the new multi-objective linear programming models and the corresponding linear programming models for the fuzzy matrix game with payoffs of triangular fuzzy numbers and its lexicographic method are proposed. An illustrative numerical example is given in Section 4. This paper ends with short concluding remarks in Section 5.
2. Linear Programming Models of Matrix Games and Triangular Fuzzy Numbers

Assume that \( S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \) and \( S_2 = \{\beta_1, \beta_2, \ldots, \beta_n\} \) are sets of pure strategies for two players I and II, respectively. A payoff matrix for the player I is given by

\[
A = (a_{ij})_{mn}
\]

The vectors \( y = (y_1, y_2, \ldots, y_m)^T \) and \( z = (z_1, z_2, \ldots, z_n)^T \) are mixed strategies for players I and II, respectively, where \( y_i \ (i = 1, 2, \ldots, m) \) and \( z_j \ (j = 1, 2, \ldots, n) \) are probabilities which the players I and II choose their pure strategies \( \alpha_i \in S_1 \ (i = 1, 2, \ldots, m) \) and \( \beta_j \in S_2 \ (j = 1, 2, \ldots, n) \), respectively. Sets of all mixed strategies for players I and II are denoted by \( Y \) and \( Z \), respectively, i.e.

\[
Y = \{ y = (y_1, y_2, \ldots, y_m)^T \mid \sum_{i=1}^{m} y_i = 1, y_i \geq 0 \ (i = 1, 2, \ldots, m) \}
\]

and

\[
Z = \{ z = (z_1, z_2, \ldots, z_n)^T \mid \sum_{j=1}^{n} z_j = 1, z_j \geq 0 \ (j = 1, 2, \ldots, n) \}
\]

By a (crisp) two-person zero-sum matrix game \( G \) we mean the triplet \( G = \{Y, Z, A\} \). In the following, we shall often call a two-person zero-sum matrix game simply as a matrix game \( G \) or \( G = \{Y, Z, A\} \).

Then, an optimal strategy \( y^* \in Y \) for the player I is the solution of the following linear programming problem

\[
\begin{align*}
\max \{ \nu \} \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} y_j \geq \nu \quad (j = 1, 2, \ldots, n) \\
& \sum_{i=1}^{m} y_i = 1 \\
& y_i \geq 0 \quad (i = 1, 2, \ldots, m)
\end{align*}
\]

Similarly, an optimal strategy \( z^* \in Z \) for the player II is the solution of the following linear programming problem

\[
\begin{align*}
\min \{ \omega \} \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} z_j \leq \omega \quad (i = 1, 2, \ldots, m) \\
& \sum_{j=1}^{n} z_j = 1 \\
& z_j \geq 0 \quad (j = 1, 2, \ldots, n)
\end{align*}
\]

It is easily seen that Eq. (2) is a dual programming problem of Eq. (1). So the maximum of \( \nu \) will be equal to the minimum of \( \omega \). Their common value is the value of the matrix game [9, 18]. In a fuzzy matrix game, however, this cannot be guaranteed [5, 9, 11] and in general \( \nu \) and \( \omega \) are fuzzy numbers.
A triangular fuzzy number [13] with the following membership function is denoted by $\tilde{a} = (a, a, \overline{a})$

$$
\mu_\epsilon(x) =
\begin{cases}
0 & \text{if } x < a - \epsilon \\
\frac{x - a + \epsilon}{\epsilon} & \text{if } x \in [a - \epsilon, a) \\
1 & \text{if } x = a \\
\frac{\overline{a} + x - a}{\epsilon} & \text{if } x \in (a, \overline{a}) \\
0 & \text{if } x > \overline{a}
\end{cases}
$$

where $a$ is the mode of the triangular fuzzy number $\tilde{a}$, and $\overline{a} \geq 0$ and $\underline{a} \geq 0$ are the right and left spreads, respectively.

Note that if $\overline{a} = \underline{a} = 0$ then $\tilde{a}$ is reduced to a real number $a$. The set of all triangular fuzzy numbers is denoted by $\text{TFN}(R)$.

For triangular fuzzy numbers and according to the extension principle [9, 13, 25], the following conclusions may be derived.

**Lemma 1.** Let $\tilde{a} = (a, a, \overline{a}) \in \text{TFN}(R)$ and $\tilde{b} = (b, b, \overline{b}) \in \text{TFN}(R)$ be triangular fuzzy numbers, and $\beta \neq 0$ be any real number. Then

1. $\tilde{a} + \tilde{b} = (a + b, a + b, \overline{a} + \overline{b})$
2. $\tilde{a} - \tilde{b} = (a - b, a - b, \overline{a} + \overline{b})$
3. $\beta \tilde{a} = \begin{cases} (\beta a, \beta a, \beta \overline{a}) & \text{if } \beta > 0 \\
(\beta a, -\beta a, -\beta \overline{a}) & \text{if } \beta < 0 \end{cases}$

The proof of Lemma 1 may be found in Dubois and Prade [13].

The ranking of fuzzy numbers is a difficult problem. The definition of $\tilde{a} \leq \tilde{b}$ is given for general fuzzy numbers in Ramik and Rimanek [19]. In this paper, the order relations “$\leq$” and “$\geq$” are used only for triangular fuzzy numbers, not for general fuzzy numbers. So the meaning of $\tilde{a} \leq \tilde{b}$ and $\tilde{a} \geq \tilde{b}$ is given for the special case of triangular fuzzy numbers in the following Definition 1.

**Definition 1.** Let $\tilde{a} = (a, a, \overline{a}) \in \text{TFN}(R)$ and $\tilde{b} = (b, b, \overline{b}) \in \text{TFN}(R)$ be triangular fuzzy numbers. Then, $\tilde{a} \leq \tilde{b}$ if and only if $a - a \leq b - b$, $a \leq b$, and $a + \overline{a} \leq b + \overline{b}$. Similarly, $\tilde{a} \geq \tilde{b}$ if and only if $a - a \geq b - b$, $a \geq b$, and $a + \overline{a} \geq b + \overline{b}$.

The validity of Definition 1 may be discussed in a similar way to that in [19].

“$\leq$” and “$\geq$” are fuzzy versions of the order relations “$\leq$” and “$\geq$” in three-dimension vector space, respectively, and have the linguistic interpretation “essentially less than or equal to” and “essentially greater than or equal to”.

From Lemma 1 and Definition 1, a triangular fuzzy number $\tilde{a} = (a, a, \overline{a}) \in \text{TFN}(R)$ may be regarded as a three-dimension vector and the order relations “$\leq$” and “$\geq$” given in Definition 1 are similar to the order relations in the vector space. So the definition of maximizing and minimizing triangular fuzzy numbers can be given as follows.
Definition 2. Let \( \tilde{a} = (a, a, a) \in \text{TFN}(R) \) be triangular fuzzy number. A maximization problem of triangular fuzzy numbers is described as follows:

\[
\begin{align*}
\max & \{ \tilde{a} \} \\
\text{s.t.} & \quad \tilde{a} \in \Omega_1 \cap \text{TFN}(R)
\end{align*}
\]

which is equivalent to the following three-objective mathematical programming problem

\[
\begin{align*}
\max & \{ a - \tilde{a}, a, a + \tilde{a} \} \\
\text{s.t.} & \quad \tilde{a} \geq 0 \\
& \quad \frac{1}{\tilde{a}} \geq 0 \\
& \quad (a - \tilde{a}, a, a + \tilde{a}) \in \Omega_1
\end{align*}
\]

where \( \Omega_1 \) is a set of constraint conditions which the variable \( \tilde{a} \) should satisfy according to requirements in the real situation.

Definition 3. Let \( \tilde{a} = (a, a, a) \in \text{TFN}(R) \) be triangular fuzzy number. A minimization problem of triangular fuzzy numbers is described as follows:

\[
\begin{align*}
\min & \{ \tilde{a} \} \\
\text{s.t.} & \quad \tilde{a} \in \Omega_2 \cap \text{TFN}(R)
\end{align*}
\]

which is equivalent to the following three-objective mathematical programming problem

\[
\begin{align*}
\min & \{ a - \tilde{a}, a, a + \tilde{a} \} \\
\text{s.t.} & \quad \tilde{a} \geq 0 \\
& \quad \frac{1}{\tilde{a}} \geq 0 \\
& \quad (a - \tilde{a}, a, a + \tilde{a}) \in \Omega_2
\end{align*}
\]

where \( \Omega_2 \) is a set of constraint conditions which the variable \( \tilde{a} \) should satisfy according to requirements in the real situation.

Definitions 2 and 3 can be used to transform the corresponding fuzzy optimization problems of the fuzzy matrix game with payoffs of triangular fuzzy numbers into three-objective linear programming problems, which may be solved using the existing multi-objective programming methods [8, 9, 23].

3. Lexicographic Method for Fuzzy Matrix Games

Let’s consider the matrix game with fuzzy payoffs, where crisp sets of pure strategies \( S_i \) and \( S_j \) and crisp sets of mixed strategies \( Y \) and \( Z \) for two players I and II are defined as above Section 2. Without loss of generality, assume that the payoff matrix of the player I is given by

\[
\tilde{A} = (\tilde{a}_{ij})_{n \times m}
\]
where each $\tilde{a}_{ij} = (a_{ij}, \bar{a}_{ij}, \overline{a}_{ij}) \in \text{TFN}(R)$ \quad (i = 1, 2, \ldots, m ; \ j = 1, 2, \ldots, n) \quad$ is a triangular fuzzy number defined as above. Then a two-person zero-sum matrix game with payoffs of triangular fuzzy numbers is the triplet
\[
FG = \{Y, Z, \hat{A}\}
\]

In the following, we shall often call a two-person zero-sum matrix game with payoffs of triangular fuzzy numbers simply as a fuzzy matrix game $FG$ or $FG = \{Y, Z, \hat{A}\}$.

According to Lemma 1, the fuzzy expectation payoff (or value) of the player I can be computed as follows
\[
\hat{E}(\hat{A}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{a}_{ij}y_{i}z_{j} = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}y_{i}z_{j}, \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{a}_{ij}y_{i}z_{j}, \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{a}_{ij}y_{i}z_{j} \right)
\]
which is a triangular fuzzy number.

As the fuzzy matrix game $FG$ is zero-sum, according to Lemma 1, the fuzzy expectation payoff of the player II is
\[
\hat{E}(-\hat{A}) = \sum_{i=1}^{m} \sum_{j=1}^{n} (-\tilde{a}_{ij})y_{i}z_{j} = \left( -\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}y_{i}z_{j}, -\sum_{i=1}^{m} \sum_{j=1}^{n} \bar{a}_{ij}y_{i}z_{j}, -\sum_{i=1}^{m} \sum_{j=1}^{n} \overline{a}_{ij}y_{i}z_{j} \right)
\]
which is still a triangular fuzzy number. Thus, in general, player I’s gain-floor and player II’s loss-ceiling should be triangular fuzzy numbers, denoted by $\tilde{v} = (v, \bar{v}, \overline{v})$ and $\hat{w} = (\omega, \bar{w}, \overline{w})\), respectively.

Since the fuzzy expectation payoffs of the players and the player I’s gain-floor and player II’s loss-ceiling are triangular fuzzy numbers, so according to Definitions 2 and 3 the concept of the solution of the fuzzy matrix game $FG$ may be given using that of the Pareto optimal solution or the vector optimal solution as follows. The notion of a reasonable solution of a fuzzy matrix game has been introduced for the first time by Bector et al [2], which is the generalization of the matrix game [18].

**Definition 4.** Let $\tilde{v} = (v, \bar{v}, \overline{v}) \in \text{TFN}(R)$ and $\hat{w} = (\omega, \bar{w}, \overline{w}) \in \text{TFN}(R)$ be triangular fuzzy numbers. Assume that there exist $y' \in Y$ and $z' \in Z$. Then $(y', z', \tilde{v}, \hat{w})$ is called a reasonable solution of the fuzzy matrix game $FG$ for any $z \in Z$ and $y \in Y$ satisfying both the following conditions
\[
(1) \quad (y')^{T}\hat{A}z \geq \tilde{v}
\]
and
\[
(2) \quad y^{T}\hat{A}z' \leq \hat{w}
\]

If $(y', z', \tilde{v}, \hat{w})$ is a reasonable solution of the fuzzy matrix game $FG$ then $\tilde{v}$ and $\hat{w}$ are called reasonable values for players I and II, respectively. Similarly, $y'$ and $z'$ are called reasonable strategies for players I and II, respectively.

Let $V$ and $W$ be the sets of all reasonable values $\tilde{v}$ and $\hat{w}$ (triangular fuzzy numbers) for players I and II, respectively.
It is worth noticing that Definition 4 only gives the notion of the reasonable solution, not the notion of an optimal solution. In other words, the reasonable solution is not the optimal solution or the solution of the fuzzy matrix game. So the concept of the solution of the game is given in the following Definition 5.

**Definition 5.** Assume that there exist $\bar{v} \in V$ and $\bar{\omega} \in W$. If there do not exist any $\bar{v} \in V$ ($\bar{v} \neq \bar{u}$) and $\bar{\omega} \in W$ ($\bar{\omega} \neq \bar{\omega}'$) such that

1. $\bar{\nu} \subseteq \bar{u}$ and
2. $\bar{\omega} \subseteq \bar{\omega}'$

Then $(\bar{v}', \bar{z}', \bar{v}, \bar{\omega}')$ is called a solution of the fuzzy matrix game $FG$. Where, $\bar{y}'$ is called a maximin strategy for player I and $\bar{z}'$ is called a minimax strategy for player II, respectively; $\bar{v}'$ and $\bar{\omega}'$ are called player I’s gain-floor and player II’s loss-ceiling, respectively.

Let

$$\bar{v}' = \bar{v}' \cap \bar{\omega}'$$

with the membership function

$$\mu_{\bar{v}'}(x) = \min\{\mu_{\bar{v}}(x), \mu_{\bar{\omega}}(x)\}$$

Then $\bar{v}'$ is called a fuzzy value of the fuzzy matrix game $FG$ as Fig. 1.

![Fig. 1. A fuzzy value $\bar{v}'$.](image)

Notice that $\bar{v}'$ must not be always a normal triangular fuzzy number.

According to Definitions 1, 4 and 5, the maximin strategy $\bar{y}' \in Y$ for player I and minimax strategy $\bar{z}' \in Z$ for player II can be generated by solving the following fuzzy mathematical programming problems

$$\max \{\bar{v}\}$$

$$s.t.:$$

$$y^T A z \geq \bar{v} \text{ for all } z \in Z$$

$$y \in Y$$

$$\bar{v} \in \text{TFN}(R)$$

(3)
and

\[
\min \{ \mathbf{\omega} \} \quad \begin{cases} 
\mathbf{y}^T \mathbf{A} \mathbf{z} \preceq \mathbf{\omega} & \text{for all } \mathbf{y} \in \mathbf{Y} \\
\mathbf{z} \in \mathbf{Z} \\
\mathbf{\omega} \in \text{TFN}(R) 
\end{cases} \quad (4)
\]

respectively.

It makes sense to consider only the extreme points of sets \( \mathbf{Y} \) and \( \mathbf{Z} \) in the constraints of Eqs. (3) and (4) since “≤” and “≥” preserve the ranking when triangular fuzzy numbers are multiplied by positive scalars according to Lemma 1 and Definition 1. Therefore, Eqs. (3) and (4) will be converted into the following fuzzy mathematical programming problems

\[
\begin{align*}
\max \{ \mathbf{\omega} \} & \\
\sum_{i=1}^{m} \tilde{a}_i y_i \preceq \mathbf{\tilde{\omega}} & \quad (j = 1, 2, \ldots, n) \\
\text{s.t.} & \\
\sum_{i=1}^{m} y_i = 1 \\
y_i \geq 0 & \quad (i = 1, 2, \ldots, m) \\
\mathbf{\tilde{\omega}} \in \text{TFN}(R) 
\end{align*} \quad (5)
\]

and

\[
\begin{align*}
\min \{ \mathbf{\omega} \} & \\
\sum_{j=1}^{n} \tilde{a}_j z_j \preceq \mathbf{\tilde{\omega}} & \quad (i = 1, 2, \ldots, m) \\
\text{s.t.} & \\
\sum_{j=1}^{n} z_j = 1 \\
z_j \geq 0 & \quad (j = 1, 2, \ldots, n) \\
\mathbf{\tilde{\omega}} \in \text{TFN}(R) 
\end{align*} \quad (6)
\]

respectively.

Using operations of triangular fuzzy numbers (i.e., Lemma 1 and Definition 1), in general the following Theorem 1 can be obtained.

**Theorem 1.** Assume that \( \mathbf{y}^* \) and \( \mathbf{\tilde{\omega}}^* \) be an optimal solution of Eq. (5), and \( \mathbf{z}^* \) and \( \mathbf{\tilde{\omega}}^* \) be an optimal solution of Eq. (6). Then, \( \mathbf{\tilde{\omega}}^* \) and \( \mathbf{\tilde{\omega}}^* \) are triangular fuzzy numbers and \( \mathbf{\tilde{\omega}}^* \preceq \mathbf{\tilde{\omega}}^* \).

**Proof.** Lemma 1 implies that \( \mathbf{\tilde{\omega}}^* \) and \( \mathbf{\tilde{\omega}}^* \) are triangular fuzzy numbers. Furthermore, since \( \mathbf{y}^* \) and \( \mathbf{\tilde{\omega}}^* \) is an optimal solution of Eq. (5), and \( \mathbf{z}^* \) and \( \mathbf{\tilde{\omega}}^* \) is an optimal solution of Eq. (6), these imply that \( \mathbf{y}^* \) and \( \mathbf{\tilde{\omega}}^* \) is a feasible solution of Eq. (5), and \( \mathbf{z}^* \) and \( \mathbf{\tilde{\omega}}^* \) is a feasible solution of Eq. (6). Hence, it follows from the above Eqs. (5) and (6) that...
\[ \vec{v}^* = \sum_{j=1}^{n} \vec{v}^* z_j^* \leq \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} y_i^* \right) z_j^* \]
\[ = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} \bar{a}_{ij} z_j^* \right) y_i^* \]
\[ \leq \sum_{i=1}^{m} \bar{\vec{v}}^* y_i^* = \vec{\omega}^* \]

i.e.

\[ \vec{v}^* \leq \vec{\omega}^* \]

Theorem 1 means that player I’s gain-floor “essentially cannot exceed” player II’s loss-ceiling in the sense of Definition 1.

Equations (5) and (6) are general fuzzy mathematical programming problems in which may involve in different solutions [8, 22, 23]. But in this study fuzzy optimization is made in the sense of Definition 2 or 3. In the following, we will focus on studying the solution method and procedure of Eqs. (5) and (6).

Using Lemma 1 and Definition 2, Eqs. (5) and (6) can be converted into the following three-objective optimization problems

\[
\begin{align*}
\text{max} & \quad \{ f^i = v - \nu, \quad f = v, \quad f^* = v + \bar{v} \} \\
\text{s.t.} & \quad \sum_{j=1}^{n} \left( a_{ij} - \bar{a}_{ij} \right) y_j \geq v - \nu \quad (j = 1, 2, \ldots, n) \\
& \quad \sum_{j=1}^{n} a_{ij} y_j \geq v \quad (j = 1, 2, \ldots, n) \\
& \quad \sum_{i=1}^{m} \left( \bar{a}_{ij} + a_{ij} \right) y_j \geq \nu + \bar{v} \quad (j = 1, 2, \ldots, n) \\
& \quad \sum_{i=1}^{m} y_i = 1 \\
& \quad y_i \geq 0 \quad (i = 1, 2, \ldots, m) \\
& \quad \nu \geq 0, \quad \bar{v} \geq 0
\end{align*}
\]

and

\[
\begin{align*}
\text{min} & \quad \{ g^i = \omega - \omega, \quad g = \omega, \quad g^* = \omega + \bar{\omega} \} \\
\text{s.t.} & \quad \sum_{j=1}^{n} \left( \bar{a}_{ij} - a_{ij} \right) z_j \leq \omega - \omega \quad (i = 1, 2, \ldots, m) \\
& \quad \sum_{i=1}^{m} \bar{a}_{ij} z_j \leq \omega \quad (i = 1, 2, \ldots, m) \\
& \quad \sum_{i=1}^{m} \left( a_{ij} + \bar{a}_{ij} \right) z_j \leq \omega + \bar{\omega} \quad (i = 1, 2, \ldots, m) \\
& \quad \sum_{i=1}^{m} z_i = 1 \\
& \quad z_j \geq 0 \quad (j = 1, 2, \ldots, n) \\
& \quad \omega \geq 0, \quad \bar{\omega} \geq 0
\end{align*}
\]
respectively.

In Eq. (7), adding constraints of \( \varpi \geq 0 \) and \( \bar{\varpi} \geq 0 \) can make its solution \( \omega, \bar{\omega} \) and \( \bar{\omega} \) constitute a triangular fuzzy number \( \tilde{\omega} = (\omega, \bar{\omega}, \bar{\omega}) \). In fact, \( \omega, \bar{\omega} \) and \( \bar{\omega} \) constitute a triangular fuzzy number \( \tilde{\omega} = (\omega, \bar{\omega}, \bar{\omega}) \) if and only if \( \omega + \bar{\omega} \geq \omega \geq \omega - \bar{\omega} \), while \( \omega + \bar{\omega} \geq \omega \). In a similar consideration, adding constraints of \( \omega \geq 0 \) and \( \bar{\omega} \geq 0 \) in Eq. (8) can make its solution \( \omega, \bar{\omega} \) and \( \bar{\omega} \) constitute a triangular fuzzy number \( \tilde{\omega} = (\omega, \bar{\omega}, \bar{\omega}) \).

Equations (7) and (8) are three-objective optimization problems. There are few standard ways of defining a solution. Normally, the concept of Pareto optimal solutions/efficient solutions is commonly-used [8, 22, 23]. There exist several solution methods for them [8, 22, 23] such as utility theory, goal programming, fuzzy programming and interactive approaches. However, in this study we focus on establishing and proposing the lexicographic method.

In Eq. (7), the priority of the objective functions \( f' = \omega - \varpi \), \( f = \omega \) and \( f^* = \omega + \bar{\omega} \) should be different. In fact, the fuzzy objective function value \( \tilde{f} = (f', f, f^*) \), where \( f = \omega \) is the center of the triangular fuzzy number \( \tilde{f} \), and \( f' = \omega - \varpi \) and \( f^* = \omega + \bar{\omega} \) are lower and upper limits of the triangular fuzzy number \( \tilde{f} \), respectively. The priority of the objective function \( f = \omega \) should be higher than that of both the objective functions \( f' = \omega - \varpi \) and \( f^* = \omega + \bar{\omega} \), and the priority of \( f' = \omega - \varpi \) and \( f^* = \omega + \bar{\omega} \) may be the same because the priority of the center of the triangular fuzzy number is much higher than that of its lower and upper limits according to the fuzzy sets [13, 25]. Hence, Eq. (7) may be solved using the lexicographic method. Its first priority is given to the objective function \( f = \omega \). Its second priority is given to the objective functions \( f' = \omega - \varpi \) and \( f^* = \omega + \bar{\omega} \). Thus, solving Eq. (7) becomes solving the following Eqs. (9) and (10) successively.

Firstly, according to Eq. (7), the following linear programming problem is easily constructed and solved

\[
\text{max} \{ f = \omega \} \quad \text{s.t.} \quad \sum_{j=1}^{n} (a_j - a_y) y_j \geq \omega - \varpi \quad (j = 1, 2, \ldots, n) \\
\sum_{j=1}^{n} a_j y_j \geq \omega \quad (j = 1, 2, \ldots, n) \\
\sum_{j=1}^{n} (a_j + a_y) y_j \geq \omega + \bar{\omega} \quad (j = 1, 2, \ldots, n) \\
\sum_{i=1}^{m} y_i = 1 \\
y_i \geq 0 \quad (i = 1, 2, \ldots, m) \\
\varpi \geq 0, \bar{\omega} \geq 0
\]

Where \( y = (y_1, y_2, \ldots, y_m)^T \), \( \omega \), \( \varpi \) and \( \bar{\omega} \) are decision variables. Using the existing Simplex method for the linear programming problem, the optimal solution of Eq. (9) is obtained, denoted by \( y^*, \omega^*, \varpi^* \) and \( \bar{\omega}^* \).
Secondly, the following linear programming problem is constructed and solved

\[ \max \{ f^i = u' - u, \quad f^u = u' + \bar{u} \} \]

\[
\begin{align*}
& \sum_{i=1}^{n} (a_i - a_j) y_i \geq u' \quad (j = 1, 2, \cdots, n) \\
& \sum_{i=1}^{n} a_i y_i \geq u' \quad (j = 1, 2, \cdots, n) \\
& -\bar{u} + \sum_{i=1}^{n} (a_i + \bar{a}_j) y_i \geq u' \quad (j = 1, 2, \cdots, n) \\
& v' - v \geq u' - u' \\
& v' + \bar{u} \geq u' + \bar{u}' \\
& \sum_{i=1}^{n} y_i = 1 \\
& y_i \geq 0 \quad (i = 1, 2, \cdots, m) \\
& v \geq 0, \bar{v} \geq 0
\end{align*}
\]

s.t.

\[ (10) \]

i.e.

\[
\max \{ f^i = -u, \quad f^u = \bar{u} \}
\]

\[
\begin{align*}
& \sum_{i=1}^{n} (a_i - a_j) y_i \geq u' \quad (j = 1, 2, \cdots, n) \\
& \sum_{i=1}^{n} a_i y_i \geq u' \quad (j = 1, 2, \cdots, n) \\
& -\bar{u} + \sum_{i=1}^{n} (a_i + \bar{a}_j) y_i \geq u' \quad (j = 1, 2, \cdots, n) \\
& v \leq u' \\
& \bar{v} \geq \bar{u}' \\
& \sum_{i=1}^{n} y_i = 1 \\
& y_i \geq 0 \quad (i = 1, 2, \cdots, m) \\
& v \geq 0, \bar{v} \geq 0
\end{align*}
\]

s.t.

\[ (11) \]

Where \( y = (y_1, y_2, \cdots, y_m) \), \( u \) and \( \bar{u} \) are decision variables.

In Eq. (10) (or Eq. (11)), adding the constraints \( u' - v \geq u' - u' \) and \( v' + \bar{u} \geq u' + \bar{u}' \) (i.e., \( u \leq u' \) and \( \bar{v} \geq \bar{u}' \)) aim to improve \( u \) and \( \bar{v} \), respectively. Minimizing \( u \) aims to avoid \( u \) be an arbitrary large positive number while maximizing \( \bar{v} \) aims to avoid \( \bar{v} \) close the minimum zero. Thus \( \bar{u} = (u, \bar{v}, \bar{v}) \) is a real triangular fuzzy number. These are the real reason why the second linear programming problem (10) (or (11)) was introduced after the first linear programming problem (9).

In Eq. (11), the objective functions \( f^i = -u \) and \( f^u = \bar{u} \) may be regarded as equal importance, i.e., their weights are the same. Therefore the bi-objective linear programming problem (11) can be aggregated into the following linear programming problem.
Using the existing Simplex method of the linear programming problem, the optimal solution of Eq. (12) is obtained, denoted by \( y^*, \tilde{v}^* \) and \( \overset{\vee}{v} \). It is not difficult to prove that \((y^*, \tilde{v}^*)\) is a Pareto optimal solution of Eq. (7), where \( \tilde{v} = (\tilde{v}^*, \overset{\vee}{v}, \overset{\wedge}{v}) \) is a triangular fuzzy number. Thus, the maximin strategy \( y^* \) and the optimal gain-floor \( \overset{\vee}{v} = (\overset{\vee}{v}^*, \overset{\wedge}{v}, \overset{\wedge}{v}) \) for the player I can be obtained.

In a similar way, in Eq. (8), the priority of the objective functions \( g' = \omega - \omega \), \( g = \omega \) and \( g^* = \omega + \bar{\omega} \) should be different. That is, the priority of the objective function \( g = \omega \) may be regarded to be higher than that of both the objective functions \( g' = \omega - \omega \) and \( g^* = \omega + \bar{\omega} \), and the priority of \( g' = \omega - \omega \) and \( g^* = \omega + \bar{\omega} \) may be assumed to be the same in that \( g = \omega \) may be regarded as the center of the fuzzy objective function value \( \tilde{\omega} = (\omega, \omega, \bar{\omega}) \), and \( g' = \omega - \omega \) and \( g^* = \omega + \bar{\omega} \) are regarded as lower and upper limits of the triangular fuzzy number \( \tilde{\omega} = (\omega, \omega, \bar{\omega}) \), respectively, and the priority of the center of the triangular fuzzy number is much higher than that of its lower and upper limits. Therefore its first priority is given to the objective function \( g = \omega \), its second priority is given to the objective functions \( g' = \omega - \omega \) and \( g^* = \omega + \bar{\omega} \). Thus, solving Eq. (8) becomes solving the following Eqs. (13) and (14) successively. Namely, the first linear programming problem is constructed as follows:

\[
\min \{ g = \omega \} \quad \text{s.t.} \quad \sum_{j=1}^{m} (a_i - a_{ij}) z_j \leq \omega - \omega \quad (i=1,2,\ldots,m) \\
\sum_{j=1}^{m} a_i z_j \leq \omega \quad (i=1,2,\ldots,m) \\
\sum_{j=1}^{m} (a_i + \bar{a}_{ij}) z_j \leq \omega + \bar{\omega} \quad (i=1,2,\ldots,m) \\
\sum_{j=1}^{n} z_j = 1 \\
z_j \geq 0 \quad (j=1,2,\ldots,n) \\
\omega \geq 0, \bar{\omega} \geq 0
\]
where $z = (z_1, z_2, \ldots, z_n)^T$, $\omega$, $\omega'$, and $\bar{\omega}$ are decision variables. Let the optimal solution of Eq. (13) be denoted by $z^0$, $\omega^0$, $\omega'^0$, and $\bar{\omega}'^0$.

The second linear programming problem is constructed as follows

$$\min \{ g' = \omega' - \omega, \ g'' = \omega' + \bar{\omega} \}$$

$$\begin{align*}
\omega + \sum_{j=1}^{n} (a_{j} - \bar{a}_{j})z_j & \leq \omega' \quad (i = 1, 2, \cdots, m) \\
\sum_{j=1}^{n} a_{j}z_j & \leq \omega' \quad (i = 1, 2, \cdots, m) \\
-\bar{\omega} + \sum_{j=1}^{n} (a_{j} + \bar{a}_{j})z_j & \leq \omega' \quad (i = 1, 2, \cdots, m)
\end{align*}$$

(14)

$$\begin{align*}
\omega' - \omega & \leq \omega' - \omega^0 \\
\omega' + \bar{\omega} & \leq \omega' + \bar{\omega}' \\
\sum_{j=1}^{n} z_j & = 1 \\
z_j & \geq 0 \quad (j = 1, 2, \cdots, n) \\
\omega & \geq 0, \bar{\omega} & \geq 0
\end{align*}$$

i.e.,

$$\begin{align*}
\min \{ g' = -\omega, \ g'' = \bar{\omega} \} \\
\omega + \sum_{j=1}^{n} (a_{j} - \bar{a}_{j})z_j & \leq \omega' \quad (i = 1, 2, \cdots, m) \\
\sum_{j=1}^{n} a_{j}z_j & \leq \omega' \quad (i = 1, 2, \cdots, m) \\
-\bar{\omega} + \sum_{j=1}^{n} (a_{j} + \bar{a}_{j})z_j & \leq \omega' \quad (i = 1, 2, \cdots, m)
\end{align*}$$

(15)

$$\begin{align*}
\omega & \geq \omega^0 \\
\bar{\omega} & \leq \bar{\omega}' \\
\sum_{j=1}^{n} z_j & = 1 \\
z_j & \geq 0 \quad (j = 1, 2, \cdots, n) \\
\omega & \geq 0, \bar{\omega} & \geq 0
\end{align*}$$

where $z = (z_1, z_2, \ldots, z_n)^T$, $\omega$, $\omega'$, and $\bar{\omega}$ are decision variables.

In a similar consideration in Eq. (10) (or Eq. (11)), adding the constraints $\omega' - \omega \leq \omega' - \omega^0$ and $\omega' + \bar{\omega} \leq \omega' + \bar{\omega}'$ (or $\omega \geq \omega^0$ and $\bar{\omega} \leq \bar{\omega}'$) in Eq. (14) (or Eq. (15)) aim to improve $\omega$ and $\bar{\omega}$, respectively. Maximizing $\omega$ aims to avoid $\omega$ close the minimum zero while minimizing $\bar{\omega}$ aims to avoid $\bar{\omega}$ be an arbitrary large positive number. Thus $\hat{\omega} = (\omega, \omega, \bar{\omega})$ is a real triangular fuzzy number.

In a similar consideration in Eq. (11), the objective functions $\bar{g}' = -\bar{\omega}$ and $\bar{g}' = \bar{\omega}$ in Eq. (15) may be regarded as equal importance, i.e., their weights are the same. Therefore the bi-objective linear programming problem (15) can be aggregated into the following linear programming problem.
\begin{aligned}
\min \left\{ \frac{\overline{a} - \omega}{2} \right\} \\
\omega + \sum_{j=1}^{n} (a_{ij} - w_{ij}) z_{j} \leq \omega^* \\
\sum_{j=1}^{n} w_{ij} z_{j} \leq \omega^* \\
-\overline{a} + \sum_{j=1}^{n} (a_{ij} + \overline{a}_{ij}) z_{j} \leq \omega^* \\
s.t., \quad \omega \geq \omega^* \\
\overline{a} \leq \overline{w} \\
\sum_{j=1}^{n} z_{j} = 1 \\
z_{j} \geq 0 \quad (j = 1, 2, \ldots, n) \\
\omega \geq 0, \overline{\omega} \geq 0
\end{aligned}

Let the optimal solution of Eq. (16) be denoted by \( \mathbf{z}^*, \overline{ \omega}^* \) and \( \overline{ \omega}^* \). It is easily proved that \( (\mathbf{z}^*, \overline{ \omega}^*) \) is a Pareto optimal solution of Eq. (8), where \( \overline{ \omega} = (\omega^*, \overline{ \omega}^*, \overline{ \omega}^*) \) is a triangular fuzzy number. Thus, the minimax strategy \( \mathbf{z}^* \) and the optimal loss-ceiling \( \overline{ \omega} = (\overline{ \omega}^*, \overline{ \omega}^*, \overline{ \omega}^*) \) for player II can be obtained.

Hence, \( \mathbf{y}^*, \mathbf{z}^* \) and \( \overline{ \nu} = \overline{ \nu} \wedge \overline{ \omega} \) are the solution and fuzzy value of the fuzzy matrix game \( FG \) with payoffs of triangular fuzzy numbers, respectively.

4. A Numerical Example and Comparison Analysis

Let's consider a simple numerical example which is taken from Campos [5]. Suppose the fuzzy payoff matrix for player I is given by Campos [5]

\[
\tilde{A} = \begin{pmatrix}
180,5,10 & 156,6,2 \\
90,10,10 & 180,5,10
\end{pmatrix}
\]

where each element of the above payoff matrix \( \tilde{A} \) is a triangular fuzzy number defined as above.

According to Eq. (9), the following linear programming problem is obtained

\[
\begin{aligned}
\max \{ f = \mathbf{y} \} \\
175y_1 + 80y_2 \geq \nu - \mathbf{y} \\
150y_1 + 175y_2 \geq \nu - \mathbf{y} \\
180y_1 + 90y_2 \geq \nu \\
156y_1 + 180y_2 \geq \nu \\
190y_1 + 100y_2 \geq \nu + \overline{\mathbf{y}} \\
158y_1 + 190y_2 \geq \nu + \overline{\mathbf{y}} \\
y_1 + y_2 = 1 \\
y_1, y_2, \mathbf{y}, \overline{\mathbf{y}} \geq 0
\end{aligned}
\]

(17)
Solving Eq. (17) using the simplex method for the linear programming problem, the optimal solution can be obtained as follows

\[ y^0 = (0.789, 0.211)^T \]

and

\[ \nu^* = 161.06, \quad \nu^0 = 99.652, \quad \nu^0 = 2.013 \]

According to Eq. (12), the following linear programming problem can be obtained

\[
\max \{ f = \frac{\nu - y}{2} \}
\]

\[
\begin{align*}
\nu + 175y_1 + 80y_2 & \geq 161.06 \\
\nu + 150y_1 + 175y_2 & \geq 161.06 \\
180y_1 + 90y_2 & \geq 161.06 \\
156y_1 + 180y_2 & \geq 161.06 \\
-\nu + 190y_1 + 100y_2 & \geq 161.06 \\
-\nu + 158y_1 + 190y_2 & \geq 161.06 \\
y_1 + y_2 & = 1 \\
y_1, y_2, \nu & \geq 0
\end{align*}
\]

Solving Eq. (18) using the simplex method for the linear programming problem, the optimal solution can be obtained as follows

\[ y^* = (0.789, 0.211)^T \]

and

\[ \nu^* = 6.105, \quad \nu^* = 3.692 \]

Therefore, the maximin strategy and the optimal gain-floor for the player I are given by

\[ y^* = (0.789, 0.211)^T \]

and

\[ \nu^* = (161.06, 6.105, 3.692) \]

respectively.

In a similar way, according to Eq. (13), the following linear programming problem can be obtained

\[
\min \{ g = \omega \}
\]

\[
\begin{align*}
175z_1 + 150z_2 & \leq \omega - \omega \\
80z_1 + 175z_2 & \leq \omega - \omega \\
180z_1 + 156z_2 & \leq \omega \\
90z_1 + 180z_2 & \leq \omega \\
190z_1 + 158z_2 & \leq \omega + \omega \\
100z_1 + 190z_2 & \leq \omega + \omega \\
z_1 + z_2 & = 1 \\
z_1, z_2, \omega, \omega & \geq 0
\end{align*}
\]
Solving Eq. (19), the optimal solution can be obtained as follows
\[ z^* = (0.211, 0.789)^T \]
and
\[ \bar{\omega}^* = 161.06, \quad \bar{\omega}^0 = 2.152, \quad \bar{\omega}^\circ = 178.56 \]

According to Eq. (16), the following linear programming problem can be obtained
\[
\min \{ \bar{\pi} = \frac{\bar{\omega} - \omega}{2} \}
\begin{align*}
\bar{\omega} + 175z_1 + 150z_2 & \leq 161.06 \\
\bar{\omega} + 80z_1 + 175z_2 & \leq 161.06 \\
180z_1 + 156z_2 & \leq 161.06 \\
90z_1 + 180z_2 & \leq 161.06 \\
-\bar{\omega} + 190z_1 + 158z_2 & \leq 161.06 \\
-\bar{\omega} + 100z_1 + 190z_2 & \leq 161.06 \\
z_1 + z_2 & = 1 \\
z_1, z_2, \omega, \bar{\omega} & \geq 0
\end{align*}
\]
(20)

Solving Eq. (20), the optimal solution can be obtained as follows
\[ z^* = (0.211, 0.789)^T \]
and
\[ \bar{\omega}^* = 5.785, \quad \bar{\omega}^\circ = 9.95 \]

Thus the minimax strategy and the optimal loss for the player II are given by
\[ z^* = (0.211, 0.789)^T \]
and
\[ \bar{\omega}^* = (161.06, 5.785, 9.95) \]
respectively.

Hence, \((y^*, z^*)^T\) is a solution of the fuzzy matrix game \(FG = (Y, Z, \bar{A})\) with payoffs of triangular fuzzy numbers. Furthermore, the fuzzy value of the fuzzy matrix game \(FG = (Y, Z, \bar{A})\) with payoffs of triangular fuzzy numbers can be obtained as follows
\[ \bar{\nu}^* = \bar{\nu} \wedge \bar{\omega}^* = (161.06, 5.785, 3.692) \]
which means that the fuzzy value of the fuzzy matrix game \(FG = (Y, Z, \bar{A})\) with payoffs of triangular fuzzy numbers is “around 161.06”. In other words, player I’s minimum reward is 155.275 while his maximum reward is 164.752. He could win any intermediate value \(t\) between 155.275 and 164.752 with possibility of \(\mu_\tau (t)\)

\[
\mu_\tau (t) = \begin{cases} 
0 & \text{if } t < 155.275 \\
\frac{t - 155.275}{5.785} & \text{if } t \in [155.275, 161.06] \\
\frac{164.752 - t}{3.692} & \text{if } t \in (161.06, 164.752] \\
0 & \text{if } t > 164.752 
\end{cases}
\]
as Fig. 2.

![Fig. 2, The fuzzy value $\tilde{v}$](image)

Campos [5] solved the above fuzzy matrix game $FG = (Y,Z,\tilde{A})$ with payoffs of triangular fuzzy numbers by deriving two auxiliary fuzzy linear programming models from the matrix game $FG = (Y,Z,\tilde{A})$ according to four different kinds of ranking method for fuzzy numbers, and obtained its four fuzzy values and maximin and minimax strategies, respectively. Maximin and minimax strategies for both players provided by Campos [5] are almost the same as that generated in this paper. However, the ranking method for fuzzy numbers needs to be determined a prior when the method proposed by Campos [5] is employed to solve the fuzzy matrix game $FG = (Y,Z,\tilde{A})$ with payoffs of triangular fuzzy numbers. Obviously, it is difficult for players to determine what kind of ranking method should be chosen. Moreover, fuzzy values generated using the method proposed by Campos [5] are dependent on some additional parameters which are not easy to be chosen for each player.

5. Conclusions

We have discussed the fuzzy zero-sum two-person game with payoffs of triangular fuzzy numbers in which the game value is considered as a triangular fuzzy variable. Based on both the linear programming models of a classical matrix game and the operations of triangular fuzzy numbers, new auxiliary fuzzy multi-objective linear programming models are constructed for each player. Furthermore, based on the order relation between triangular fuzzy numbers, each auxiliary fuzzy multi-objective linear programming problem is decomposed into two linear programming models and corresponding effective solution method which allows us to use the existing Simplex method software for the linear programming problem is proposed. These models are proposed to extend the models used to solve a classical game. In fact, if each payoff $\bar{a}_i$ is a real number, i.e., $\bar{a}_i = \underline{a}_i = 0$, then $\tilde{u} = 0 = \tilde{v}$ and $\underline{w} = \bar{w} = 0$. Hence, Eqs. (7) and (8) are reduced to Eqs. (1) and (2), respectively.

Obviously, the models and method proposed in this paper are significantly different from other existing models in that subjective criteria or functions for ranking fuzzy
numbers as required in other methods are not needed in the proposed approach. This avoids the choice of a specific ranking criterion or function, which is difficult to do since different ranking criteria or functions generate different solutions.

In the above, we do not say anything about the existence of solutions for Eq. (3) or (4) in which fuzzy optimization problem is made in the sense of Definition 2 or 3. In other words, we do not make any conclusion about the existence of the solution of the fuzzy matrix game $FG$ with payoffs of triangular fuzzy numbers in the sense discussed in this paper. Therefore these problems need a further investigation in the future.

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7. References