Disturbance Decoupling for Descriptor Systems by State Feedback

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Abstract

We study the disturbance decoupling problem for linear time invariant descriptor systems. We give necessary and sufficient conditions for the existence of a solution to the disturbance decoupling problem with/without stability via a proportional and/or derivative feedback that also makes the resulting closed-loop system regular and/or of index at most one. All results are proved constructively based on condensed forms that can be computed using orthogonal matrix transformations, i.e., transformations that can be implemented in a numerically stable way.

Keywords: Descriptor system, state feedback, disturbance decoupling, stability, orthogonal matrix transformation.

AMS subject classification: 93B05, 93B40, 93B52, 65F35

1 Introduction

We consider linear and time-invariant continuous descriptor systems of the form

\[
\begin{align*}
E \dot{x}(t) &= Ax(t) + Bu(t) + Gq(t); \quad x(t_0) = x_0, \quad t \geq t_0 \\
y(t) &= Cx(t),
\end{align*}
\]

where \(E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, G \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{q \times n}\), and \(\dot{x} = dx/dt\). The term \(q(t), t \geq 0\) represents a disturbance, which may represent modelling or measuring errors, noise or higher order terms in linearization. We study the problem of constructing feedbacks that suppress this disturbance in the sense that \(q(t)\) does not affect the input-output behaviour of the system. In this paper, we only study square systems \((E, A\) are square\). This seems to be a restriction, since in general models that arise from automatic modelling or from heterogeneous systems are often rectangular, e.g., [16]. In [7], however, it was shown that for every rectangular system there exists an underlying square system that can be obtained via a numerically backwards stable procedure by removing redundancies and higher index uncontrollable and unobservable parts in the system. It is possible to formulate all the results in this paper also for rectangular systems, by including the transformations performed in [7] into the methods and the theorems. But this would make the paper, which is quite technical already, even
longer and more technical. For this reason we restrict ourselves to the square case. Similarly, we also assume without loss of generality that $B, G$ are full column rank, and $C$ is full row rank, i.e., rank $(B) = m$, rank $(G) = p$, rank $(C) = q$. If this is not the case then this can be easily achieved, via a numerically stable procedure, by removing the nullspaces and appropriate renaming of variables.

The theory for (1) is well established from the analytical, geometric and numerical point of view, see e.g. [11, 17, 22, 30]. Existence and uniqueness of (classical) solutions to (1) for sufficiently smooth input functions and consistent initial values is guaranteed if $(E, A)$ is regular, i.e., if $\det(\alpha E - \beta A) \neq 0$ for some $(\alpha, \beta) \in \mathbb{C}^2$. This means in particular that the system has to be square. The system (1) is said to have index at most one if the dimension of the largest nilpotent block in the Kronecker canonical form of $(E, A)$ is at most one [14, 4]. It is well-known that systems that are regular and of index at most one can be separated into purely dynamical and purely algebraic parts (fast and slow modes), and in theory the algebraic part can be eliminated to give a reduced-order standard system. The reduction process, however, may be ill-conditioned with respect to numerical computation. For this reason it is preferable to use descriptor system models rather than turning the problem into a standard system. Nonetheless most numerical simulation methods work well for systems of index at most one, see [4] and the usual class of piecewise continuous input functions can be used. Also classical techniques for important control applications like stabilization, pole assignment or linear quadratic control can be applied, see e.g., [22, 5, 6].

If the index is larger than one, however, then impulses arise in the response of the system if the control is not sufficiently smooth [5, 11]. This restricts the set of admissible input functions and also impulses can arise due to the presence of modelling, measurement, linearization and roundoff errors in the real system. Furthermore the use of numerical integration methods is restricted, see [4].

There are essentially two possibilities to deal with higher index systems in the context of control systems. Either an index reduction is performed, see e.g., [7, 18], to obtain an equivalent system of index at most one, or an appropriate feedback control is chosen to ensure that the closed-loop system is regular and of index at most one. Techniques for the construction of such feedbacks were developed in [5, 6] based on transformations to condensed forms via orthogonal matrix transformations, which can be implemented as numerically stable algorithms. This paper, which is strongly inspired by the work of [5, 6], extends these techniques to the solution of the disturbance decoupling problem.

The disturbance decoupling problem for descriptor systems (1) can be stated as follows: Find necessary and sufficient conditions under which there exists a proportional and derivative feedback of the form $u(t) = F x(t) - K \dot{x}(t)$, such that matrix pencil $(E + BK, A + BF)$ is regular and of index at most one and $C(s(E + BK) - (A + BF))^{-1}G = 0$, where $C(s(E + BK) - (A + BF))^{-1}G$ is the transfer-function matrix of the closed-loop system

$$
(E + BK) \dot{x}(t) = (A + BF)x(t) + Gq(t), \quad y(t) = Cx(t).
$$

For standard systems $(E = I)$ this problem is well studied see [23, 24, 25, 26, 27, 29, 30]. Our attention, however, will focus on the case that $E$ is singular. Let us briefly summarize some previous results. The disturbance decoupling problem for continuous-time descriptor systems was first formulated in [21] and the problem was solved under the assumption, among other conditions, that the output is independent of the input disturbance in the sense that there is a set of admissible initial conditions such that the response of the system is zero. But, since
the disturbance input is usually unknown, it is not clear how, and if at all, a given initial state \( x_0 \) can be qualified as an admissible initial condition. In [3] the problem was solved from the geometric point of view, using the concepts of sliding and coasting subspaces by means of a set of necessary and sufficient conditions for obtaining disturbance decoupling in implicit discrete systems. These results are not constructive and numerically stable methods cannot be based on this approach. Furthermore the index of the system is not considered. In [19, 20] again the discrete time disturbance decoupling problem is discussed and structurally equivalent characterizations are presented for the solvability of the disturbance decoupling problems for implicit discrete-time systems.

Recently, in [1] the standard disturbance decoupling problem for continuous-time descriptor systems was considered as formulated in the standard state-space system theory [30], i.e., given the system (1), find (if possible) a proportional state feedback such that, regardless of the initial value of \( x_0 \), the disturbance input has no influence on the output of the systems for \( t \geq 0 \), and yet the uniqueness of solutions for the closed-loop system is ensured. Also in [1] necessary and sufficient conditions were given for solvability of the disturbance decoupling problem under the assumptions \( \text{rank} \ E \ G = n \) and \( \text{rank} \ A \ B \ G = n \). But the obtained conditions are rather cumbersome and are only partly given in terms of the original data \((E, A, B, C, G)\). Moreover, combined derivative and proportional state feedback, the index and stability of the system and also numerical aspects of the algorithms so far have not been considered in the literature.

To demonstrate the great flexibility of our matrix pencil approach, we also discuss the extra requirement that the closed loop system is stable, i.e., that all the finite generalized eigenvalues of \( s(E + BK) - (A + BF) \) are in the open left half plane. Furthermore a similar approach yields also the solution for partly measurable disturbances, i.e., we also study the use of a proportional and derivative feedback of the form \( u(t) = Fx(t) - K \dot{x}(t) + Hq(t) \), such that the matrix pencil \((E + BK, A + BF)\) is regular, of index at most one, and

\[
C (s(E + BK) - (A + BF))^{-1} (G + BH) = 0.
\]

Again, we also include the stability of the closed loop system as extra requirement.

All our results are proved constructively, based on condensed forms under orthogonal matrix transformations which can be directly implemented as numerically stable algorithms.

The paper is organized as follows: In Section 2 we introduce some notation and give some preliminary results. In Sections 3 and 4 we solve the disturbance decoupling problems without and with stability, respectively. We discuss separately the case that the system is only regularized and that it also has index at most one. It should be noted that several more results on this topic could have been included but are omitted for lack of space. See the technical reports [8, 9].

## 2 Preliminaries

In this section we introduce the notation and give some preliminary results. We denote by \( \text{deg}(p) \) the degree of a polynomial \( p \) and by \( \text{rank}_2 [M(s)] \) the generic rank of a rational matrix valued function \( M(s) \) and by \( \mathbb{C}^+ \) the closed right half plane. Let the orthogonal complement of the space spanned by the columns of a matrix \( M \) be denoted by \( M^\perp \). A matrix with orthogonal columns spanning the right nullspace of a matrix \( M \) is denoted by \( S_\infty(M) \) and a matrix with orthogonal columns spanning the right nullspace of \( M^T \) by \( T_\infty(M) \). For convenience of
notation we identify a subspace and a matrix whose columns form an orthonormal basis of this subspace. These orthonormal bases will be available from the condensed forms that we determine.

In principle the complete analysis of descriptor systems could be based on the Kronecker canonical forms of the associated matrix pencils, [14, 13], but it is in general impossible to compute the Kronecker canonical form with a finite precision numerical algorithm, since small changes in the data can drastically change the canonical form. Instead one can obtain a condensed form under orthogonal equivalence transformations. This form, the generalized upper triangular (GUPTRI) form is well analyzed [12, 13], and numerically stable algorithms are available and have been implemented in LAPACK [2]. The generalized upper triangular form displays all the invariants, in particular the left and right Kronecker indices, but it is not the complete canonical form.

**Lemma 1** [12] Given a matrix pencil \((E, A)\), \(E, A \in \mathbb{R}^{n \times l}\) there exist orthogonal matrices \(P \in \mathbb{R}^{n \times n}\), \(Q \in \mathbb{R}^{l \times l}\) such that \((PEQ, PAQ)\) are in the following generalized upper triangular form:

\[
P(sE - A)Q = \begin{bmatrix}
  i_1 & i_2 & i_3 & i_4 \\
  n_1 & sE_{11} - A_{11} & sE_{12} - A_{12} & sE_{13} - A_{13} & sE_{14} - A_{14} \\
  n_2 & 0 & sE_{22} - A_{22} & sE_{23} - A_{23} & sE_{24} - A_{24} \\
  n_3 & 0 & 0 & sE_{33} - A_{33} & sE_{34} - A_{34} \\
  n_4 & 0 & 0 & 0 & sE_{44} - A_{44}
\end{bmatrix}.
\]  

(3)

Here \(n_2 = i_2, n_3 = i_3, sE_{11} - A_{11}\) and \(sE_{44} - A_{44}\) contains all left and right singular Kronecker blocks of \(sE - A\), respectively. Furthermore, \(sE_{22} - A_{22}\) and \(sE_{33} - A_{33}\) are regular and contain the regular finite and infinite structur of \(sE - A\), respectively.

Based on the form (3), we introduce the following spaces, which we will use to describe a geometric, coordinate-free, characterization of the solution to the disturbance decoupling problem.

**Definition 2** [12] Given a matrix pencil \((E, A)\), \(E, A \in \mathbb{R}^{n \times l}\) and orthogonal matrices \(P, Q\) such that \(P(sE - A)Q\) is of the form (3). Then

1. The minimal left reducing subspace \(V_{m-1}[E, A]\) of \((E, A)\) is the space spanned by the leading \(n_1\) columns of \(P^T\);

2. The minimal right reducing subspace \(V_{m-1}[E, A]\) of \((E, A)\) is the space spanned by the leading \(i_1\) columns of \(Q\);

3. The left reducing subspace corresponding to the finite spectrum of \((E, A)\), denoted by \(V_{l-1}[E, A]\), is the space spanned by the leading \(n_1 + n_2\) columns of \(P^T\);

4. The right reducing subspace corresponding to the finite spectrum of \((E, A)\), denoted by \(V_{l-1}[E, A]\), is the space spanned by the leading \(i_1 + i_2\) columns of \(Q\).

The problem of constructing feedbacks such that the closed loop system is regular, of index at most one and stable has already been studied in detail in the literature. We summarize some of the relevant results in the following Lemmas.
**Lemma 3** [5, 11, 28]. Given $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.

(i) There exists $F \in \mathbb{R}^{m \times n}$ such that $(E, A + BF)$ is regular if and only if

$$\text{rank}_g \left[ \begin{array}{cc} sE - A & B \\ C & 0 \end{array} \right] = n.$$  

(ii) There exists $F \in \mathbb{R}^{m \times n}$ such that $(E, A + BF)$ is regular and of index at most one if and only if

$$\text{rank} \left[ \begin{array}{ccc} E & A \mathcal{S}_\infty(E) & B \\ C \end{array} \right] = n.$$  

(iii) There exists $F, K \in \mathbb{R}^{m \times n}$ such that $(E + BK, A + BF)$ is regular and of index at most one if and only if

$$\text{rank} \left[ \begin{array}{ccc} E & A \mathcal{S}_\infty(T_\infty(B)E) & B \\ C \end{array} \right] = n.$$  

(iv) There exists a matrix $F \in \mathbb{R}^{m \times n}$ such that $(E, A + BF)$ is regular and stable if and only if

$$\text{rank} \left[ \begin{array}{cc} sE - A & B \\ C \end{array} \right] = n, \quad \forall s \in \mathbb{C}^+.$$  

(v) There exists a matrix $F \in \mathbb{R}^{m \times n}$ such that $(E, A + BF)$ is regular, stable and of index at most one if and only if conditions (5) and (7) hold.

(vi) There exist matrices $F, G \in \mathbb{R}^{m \times n}$ such that $(E + BG, A + BF)$ is regular and stable if and only if condition (7) holds.

(vii) There exist matrices $F, G \in \mathbb{R}^{m \times n}$ such that $(E + BG, A + BF)$ is regular, stable and of index at most one if and only if conditions (6) and (7) hold.

**Remark 1** Condition (5) is often called *controllability at infinity*, since if it holds, then the Jordan structure of the spectrum at infinity can be modified arbitrarily. This condition is not invariant under derivative feedback, see [5], hence condition (6) is needed when combined state and derivative is used. Condition (4) is sometimes called *regularizability* [7]. If a system satisfies both (5) and (7) then it is called *strongly stabilizable*, see ([22], p.14).

The spaces occurring in Lemma 3 can be easily computed via numerically stable procedures, like singular value decomposition or rank revealing QR-decompositions [15, 2]. Thus, they can be checked numerically within the limitations of numerical rank decisions and nullspace computations in finite arithmetic.

For the disturbance decoupling problem, we need the following lemma.

**Lemma 4** Consider a system of the form (1). If $(E, A)$ is regular, then $C(sE - A)^{-1}G = 0$ if and only if

$$\text{rank}_g \left[ \begin{array}{ccc} sE - A & G \\ C & 0 \end{array} \right] = n.$$  

**Proof.** The proof follows directly from the fact that for any $s \in \mathbb{C}$ with $\det(sE - A) \neq 0$ we have

$$\text{rank} \left( C(sE - A)^{-1}G \right) = \text{rank} \left[ \begin{array}{cc} sE - A & G \\ C & 0 \end{array} \right] - \text{rank} (sE - A).$$

We close this section with a technical lemma that we will use frequently in subsequent sections.
Lemma 5 Given matrices $E, A, B$ such that

$$sE - A := \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} t \\ -A_2 \end{bmatrix}, \quad B := \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

with $l_1 \leq t$ and $B_2$ of full row rank.

(i) If

$$\text{rank}_g \begin{bmatrix} sE_1 - A_1 & B_1 \\ -A_2 & B_2 \end{bmatrix} = i_1 + i_2,$$

then there exist a matrix $F \in \mathbb{R}^{t \times t}$ and a nonsingular matrix $Z \in \mathbb{R}^{t \times t}$, such that

$$(sE - A - BF)Z = \begin{bmatrix} i_1 & t - i_1 \\ l_2 & 0 \\ 0 & 0 \end{bmatrix}$$

with $(\Theta_1, \Phi_1)$ regular.

(ii) If (8) holds and furthermore

$$\text{rank} \begin{bmatrix} E_1 & A_1 S_\infty(E_1) & B_1 \\ 0 & A_2 S_\infty(E_1) & B_2 \end{bmatrix} = i_1 + i_2,$$

then there exist a matrix $F \in \mathbb{R}^{t \times t}$ and a nonsingular matrix $Z \in \mathbb{R}^{t \times t}$, such that $(sE - A - BF)Z$ has partitioning (9) with $(\Theta_1, \Phi_1)$ regular and of index at most one.

Proof. Let $W \in \mathbb{R}^{t \times t}$ and $Q \in \mathbb{R}^{r \times r}$ be orthogonal matrices such that

$$(sE - A)W = \begin{bmatrix} i_1 & t - i_1 \\ l_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} i_1 & r - i_2 \\ l_2 & B_{11} \\ B_{21} & B_{12} \end{bmatrix}$$

with $B_{21}$ nonsingular, (note that $B_2$ has full row rank).

(i) Condition (8) is equivalent to

$$\text{rank}_g \begin{bmatrix} sE_{11} - A_{11} & A_{12} & B_{11} & B_{12} \\ -A_{21} & -A_{22} & B_{21} & B_{22} \end{bmatrix} = i_1 + i_2.$$

Since $B_{21}$ is nonsingular, using Schur complements, this is equivalent to

$$\text{rank}_g \begin{bmatrix} sE_{11} - (A_{11} - B_{11} B_{21}^{-1} A_{21}) & A_{12} - B_{11} B_{21}^{-1} A_{22} & B_{12} \\ -A_{21} & -A_{22} & B_{21} \end{bmatrix} = i_1.$$

Then, applying Lemma 3(iii) immediately gives the existence of a regularizing feedback.

(ii) Analogously, (10) is equivalent to

$$\text{rank} \begin{bmatrix} E_{11} & A_{11} S_\infty(E_{11}) & A_{12} & B_{11} & B_{12} \\ 0 & A_{21} S_\infty(E_{11}) & A_{22} & B_{21} & B_{22} \end{bmatrix} = i_1 + i_2,$$
which is equivalent to
\[ \text{rank} \begin{bmatrix} E_{11} & (A_{11} - B_{11}B_{21}^{-1}A_{21})S_{\infty}(E_{11}) & (A_{12} - B_{11}B_{21}^{-1}A_{22}) & B_{12} \\ \end{bmatrix} = l_1. \]

Lemma 3(i) then gives the existence of a feedback that makes the system regular and of index at most one.

Actually the feedbacks in both cases can be constructed explicitly as follows. Let
\[ Z := W \begin{bmatrix} I & 0 \\ \hat{Z} & I \end{bmatrix}, \quad F := Q \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & 0 \end{bmatrix} W^T, \]
where \( F_{11} \) and \( F_{12} \) are constructed from
\[ B_{21} \begin{bmatrix} F_{11} & F_{12} \end{bmatrix} = - \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \]
and \( \hat{Z} \) and \( F_{21} \) are constructed via the algorithm given in the Appendix of [5].

**Remark 2** The construction of the feedback in the proof of Lemma 5 needs the solution of a linear system in \((E_{11}, A_{11})\). This system may be very ill-conditioned and hence a numerical solution of this system may create large errors. A different possibility to construct the desired feedback is the following.

Using a rank revealing QR-decomposition, construct an orthogonal matrix \( P \), such that
\[ P \begin{bmatrix} \hat{B}_{11} \\ \hat{B}_{21} \end{bmatrix} =: \begin{bmatrix} 0 \\ \hat{B}_{21} \end{bmatrix}, \]
with \( \hat{B}_{21} \) nonsingular. Set, with compatible partitioning,
\[ P \begin{bmatrix} sE_{11} - A_{11} & A_{12} \\ -A_{21} & A_{22} \end{bmatrix} =: \begin{bmatrix} s\hat{E}_{11} - \hat{A}_{11} & \hat{A}_{12} \\ s\hat{E}_{21} - \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad P \begin{bmatrix} B_{12} \\ 0 \end{bmatrix} =: \begin{bmatrix} \hat{B}_{12} \\ \hat{B}_{22} \end{bmatrix}. \]
Then apply the algorithm in the appendix of [5] to compute \( \hat{Z} \) and \( F_{21} \) such that \((\hat{E}_{11}, \hat{A}_{11} + \hat{A}_{12}X + \hat{B}_{12}F_{21})\) is regular and of index at most one.

After having introduced the preliminaries, in the next sections we now discuss the disturbance decoupling problem.

### 3 The Disturbance Decoupling Problem

In this section, we first establish a condensed form for matrix quintuples \((E, A, B, C, G)\) under orthogonal equivalence transformations, and then solve the disturbance decoupling problem without the stability requirement.

The general philosophy is to generate an equivalent representation of the original system, from which the system properties can be read off easily and that can be used to construct the desired feedbacks in the solution of the disturbance decoupling problem. The main feature of the new condensed form is that it is (in contrast to canonical forms that are used in previous work on this subject, like, e.g., [1]) based on orthogonal matrix transformations, which can be implemented as numerically stable algorithms, thus guaranteeing robust computation of the
desired quantities, if this is possible. The spaces that we will need for the solution are the
following (with the notation introduced in Section 2).

$$\Pi := T_\infty \left( \begin{bmatrix} B & G \\ 0 & 0 \end{bmatrix} \right), \quad \Psi := T_\infty (G), \quad \Lambda_r := V_{j-r}[\Pi^T \begin{bmatrix} E \\ 0 \end{bmatrix}, \Pi^T \begin{bmatrix} A \\ C \end{bmatrix}],$$

$$\Lambda_i := V_{j-i}[\Pi^T \begin{bmatrix} E \\ 0 \end{bmatrix}, \Pi^T \begin{bmatrix} A \\ C \end{bmatrix}], \quad \Lambda_t := \left[ \begin{bmatrix} I & 0 \\ 0 & A_t \end{bmatrix} \right]. \quad (12)$$

With the abbreviations $\Gamma_1 := \begin{bmatrix} 0 & \Psi^T E \\ 0 & 0 \end{bmatrix}$ and $\Gamma_2 := \begin{bmatrix} \Psi^T B & \Psi^T A \\ 0 & C \end{bmatrix}$ we further introduce the spaces

$$\Lambda_1 := \Lambda_r^T \begin{bmatrix} E \\ 0 \end{bmatrix} \Lambda_r, \quad \Lambda_2 := \Lambda_r^T \begin{bmatrix} A \\ C \end{bmatrix} \Lambda_r,$$

$$\Lambda_3 := \Lambda_r^T \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \Lambda_4 := \Pi_j V_{j-r}[\Gamma_1, \Gamma_2]^T \Pi_j V_{j-r}[\Gamma_1, \Gamma_2]. \quad (13)$$

These spaces can be easily obtained form the following condensed form under orthogonal transformations.

**Theorem 6** Given a system of the form (1) with system matrices $E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, G \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{p \times n}$. Then there exist orthogonal matrices $U, V \in \mathbb{R}^{n \times n}$ such that

$$UEV = \begin{bmatrix} n_1 & n_2 & n_3 \\ \tilde{n}_2 & E_{11} & E_{12} \\ \tilde{n}_3 & E_{21} & E_{22} \\ \tilde{n}_4 & E_{32} & E_{33} \\ \tilde{n}_5 & E_{42} & E_{43} \end{bmatrix}, \quad UAV = \begin{bmatrix} n_1 & n_2 & n_3 \\ \tilde{n}_2 & A_{11} & A_{12} \\ \tilde{n}_3 & A_{21} & A_{22} \\ \tilde{n}_4 & A_{31} & A_{32} \\ \tilde{n}_5 & A_{41} & A_{42} \end{bmatrix},$$

$$U = \begin{bmatrix} n_1 & n_2 & n_3 \\ \tilde{n}_2 & B_1 \\ \tilde{n}_3 & B_2 \\ \tilde{n}_4 & B_3 \\ \tilde{n}_5 & 0 \end{bmatrix}, \quad UG = \begin{bmatrix} n_1 & n_2 & n_3 \\ \tilde{n}_2 & G \\ \tilde{n}_3 & 0 \\ \tilde{n}_4 & 0 \\ \tilde{n}_5 & 0 \end{bmatrix}, \quad CV = \begin{bmatrix} n_1 & n_2 & n_3 \\ 0 & C_2 & C_3 \end{bmatrix}, \quad (14)$$

where $G_1, E_{21}, B_3$ and $E_{42}$ are of full row rank and furthermore

$$\text{rank}(\begin{bmatrix} sE_{42} - A_{42} & sE_{43} - A_{43} \\ 0 & sE_{53} - A_{53} \end{bmatrix}) = n_2 + n_3, \quad \forall s \in \mathbb{C}.$$ 

**Proof.** The proof is given constructively via Algorithm 1 in Appendix A. \[ \blacksquare \]

Using condensed form (14) we can determine directly the following important spaces and their dimensions.

**Lemma 7** Let $E, A, B, C, G$ be in the condensed form (14).
(i) We have

\[
\tau := \dim(V_{f-1}[^{\begin{bmatrix} 0 & \Psi T E \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \Psi T B & \Psi T A \\ 0 & C \end{bmatrix}}]) = \tilde{n}_2,
\]

\[
\mu := \dim(V_{f-\tau}[^{\begin{bmatrix} \Psi T B & \Psi T E \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \Psi T A \\ 0 & C \end{bmatrix}}]) = n_2 + n_3,
\]

\[
\eta := \dim(V_{f-\tau}[^{\begin{bmatrix} \Psi T B & \Psi T E \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \Psi T A \\ 0 & C \end{bmatrix}}]) = \tau = \tilde{n}_3. \quad (15)
\]

(ii) Let \( S := S_\infty \left( \begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} \right) \) then

\[
\begin{aligned}
\text{rank } \begin{bmatrix} E_{11} & A_{11}S & B_1 \\ E_{21} & A_{21}S & B_2 \\ 0 & A_{31}S & B_3 \end{bmatrix} &= \text{rank } \begin{bmatrix} A_1 & A_2S_\infty(A_1) & A_3 \end{bmatrix} \quad \text{with}
\text{rank } \begin{bmatrix} E_{11} \\ E_{21} \\ 0 \end{bmatrix} &= \text{rank}(A_1), \quad \text{rank } \begin{bmatrix} E_{32} & E_{33} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix} = \text{rank}(A_4). \quad (16)
\end{aligned}
\]

(iii) The matrix

\[
T_\infty^T \left( \begin{bmatrix} E_{11} & B_1 \\ E_{21} & B_2 \\ 0 & B_3 \end{bmatrix} \right) S_\infty^T \left( \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix} \right) \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{21} \\ 0 \end{bmatrix}
\]

has full row rank if and only if \( T_\infty^T \left( \begin{bmatrix} A_1 & A_3 \end{bmatrix} \right) A_2S_\infty(T_\infty^T(A_3)A_1) \) is of full row rank.

**Proof.** The proof is given in Appendix B. \( \square \)

Based on the condensed form (14) and Lemma 3 we obtain the following necessary conditions for the existence of feedbacks that make the system regular and of index at most one.

**Lemma 8** Given system (1) in condensed form (14). Then we have the following.

(i) If there exists a feedback \( F \in \mathbb{R}^{m \times n} \) such that \((E, A + BF)\) is regular and of index at most one, i.e., if condition (5) holds, then \( \tilde{n}_5 = n_3, E_{53} = 0, A_{53} \) is nonsingular, and furthermore,

\[
\text{rank } \begin{bmatrix} E_{32} \\ E_{42} \end{bmatrix} = \text{rank } \begin{bmatrix} E_{32} & E_{33} \\ E_{42} & E_{43} \end{bmatrix}. \quad (17)
\]

(ii) If there exist \( F, K \in \mathbb{R}^{m \times n} \) such that \((E + BK, A + BF)\) is regular and of index at most one, i.e., if condition (6) holds, then \( \tilde{n}_5 = n_3, E_{53} = 0, A_{53} \) is nonsingular and

\[
\begin{aligned}
\text{rank } \begin{bmatrix} E_{11} & E_{12} & E_{13} & B_1 \\ E_{21} & E_{22} & E_{23} & B_2 \\ 0 & E_{32} & E_{33} & B_3 \\ 0 & E_{42} & E_{43} & 0 \end{bmatrix} &= \text{rank } \begin{bmatrix} E_{11} & E_{12} & B_1 \\ E_{21} & E_{22} & B_2 \\ 0 & E_{32} & B_3 \\ 0 & E_{42} & 0 \end{bmatrix}. \quad (18)
\end{aligned}
\]
Proof. Let the system be given in the condensed form (14).

(i) If there exists \( F := \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix} \), partitioned conformly to (14), such that \((E, A + BF)\) is regular and of index at most one, then the last block row in (14), which cannot be modified by proportional feedback, must satisfy \( \text{rank}_2(sE_{53} - A_{53}) = \tilde{n}_5 \). But \( sE_{53} - A_{53} \) is of full column rank for any \( s \in \mathbb{C} \) and thus, \( n_3 = \tilde{n}_5 \) and \( \det(sE_{53} - A_{53}) = \det(-A_{53}) \). Therefore, the nonsingularity of \( A_{53} \) follows directly from the regularity of \((E, A + BF)\).

Moreover,

\[
\text{rank}(E) = \deg(\det(sE - A - BF))
\]

\[
= \deg(\det(sE_{11} - A_{11} - B_1F_1, sE_{12} - A_{12} - B_1F_2, sE_{21} - A_{21} - B_2F_1, sE_{22} - A_{22} - B_2F_2, -A_{31} - B_3F_1, -A_{32} - B_3F_2, sE_{31} - A_{31}, sE_{32} - A_{32}, 0, sE_{41} - A_{41}, sE_{42} - A_{42})) + \deg(\det(sE_{53} - A_{53}))
\]

\[
= \deg(\det(sE_{11} - A_{11} - B_1F_1, sE_{12} - A_{12} - B_1F_2, sE_{21} - A_{21} - B_2F_1, sE_{22} - A_{22} - B_2F_2, -A_{31} - B_3F_1, -A_{32} - B_3F_2, sE_{31} - A_{31}, sE_{32} - A_{32}, 0, sE_{41} - A_{41}, sE_{42} - A_{42})) \leq \text{rank}(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \\ 0 & E_{32} \\ 0 & E_{42} \end{bmatrix})
\]

But, we have \( \text{rank}(E) \geq \text{rank}(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \\ 0 & E_{32} \\ 0 & E_{42} \end{bmatrix}) + \text{rank}(E_{53}) \) and hence \( E_{53} = 0 \). We also have

\[
\text{rank}(E) = \text{rank}(\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ 0 & E_{32} & E_{33} \\ 0 & E_{42} & E_{43} \end{bmatrix}) = \text{rank}(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \\ 0 & E_{32} \\ 0 & E_{42} \end{bmatrix}),
\]

which implies (17).

(ii) If there exist \( F := \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix} \) and \( K := \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix} \) such that \((E + BK, A + BF)\) is regular and of index at most one, then, since \( E_{53}, A_{53} \) are not affected by these feedbacks, it follows from part (i) that \( \tilde{n}_5 = n_3, E_{53} = 0 \) and that \( A_{53} \) is nonsingular.

Similar as in part (i), we get that

\[
\text{rank}(E + BK) = \text{rank}(\begin{bmatrix} E_{11} + B_1K_1 & E_{12} + B_1K_2 \\ E_{21} + B_2K_1 & E_{22} + B_2K_2 \\ B_3K_1 & B_3K_2 \\ 0 & E_{42} \end{bmatrix}),
\]

However, since \( E_{53} = 0 \), we also have

\[
\text{rank}(E + BK) = \text{rank}(\begin{bmatrix} E_{11} + B_1K_1 & E_{12} + B_1K_2 & E_{13} + B_1K_3 \\ E_{21} + B_2K_1 & E_{22} + B_2K_2 & E_{23} + B_2K_3 \\ B_3K_1 & B_3K_2 & B_3K_3 \\ 0 & E_{42} & E_{43} \end{bmatrix})
\]
and hence
\[
\begin{align*}
\operatorname{rank}
\begin{bmatrix}
E_{11} + B_1 K_1 & E_{12} + B_1 K_2 & E_{13} + B_1 K_3 \\
E_{21} + B_2 K_1 & E_{22} + B_2 K_2 & E_{23} + B_2 K_3 \\
B_3 K_1 & E_{32} + B_3 K_2 & E_{33} + B_3 K_3 \\
0 & E_{42} & E_{43}
\end{bmatrix}
& = \operatorname{rank}
\begin{bmatrix}
E_{11} + B_1 K_1 & E_{12} + B_1 K_2 \\
E_{21} + B_2 K_1 & E_{22} + B_2 K_2 \\
B_3 K_1 & E_{32} + B_3 K_2 \\
0 & E_{42}
\end{bmatrix}
\end{align*}
\]
which implies (18). \( \Box \)

We will now apply these results to solve the disturbance decoupling problem.

We present the results in a coordinate free way, but note that all the quantities are available via numerically stable procedures from the results presented before.

**Theorem 9** Consider a system of the form (1). and let the spaces \( \Lambda_i \) be as in (13) and \( \tau, \eta \) and \( \mu \) as in (15).

(a) There exists a feedback matrix \( F \in \mathbb{R}^{m \times n} \) such that the pencil \( (E, A + BF) \) is regular, of index at most one, and \( C(sE - (A + B F))^{-1} G = 0 \) if and only if condition (5) and furthermore the following three conditions hold:

\[
\begin{align*}
\tau + \mu & \leq n - p; \\
\operatorname{rank}(\Lambda_1) + \operatorname{rank}(\Lambda_4) & = \operatorname{rank}(E); \\
\operatorname{rank}
\begin{bmatrix}
\Lambda_1 & \Lambda_2 S_\infty(\Lambda_1) & \Lambda_3
\end{bmatrix}
& = p + \tau + \eta.
\end{align*}
\]

(b) There exist feedback matrices \( F, K \in \mathbb{R}^{m \times n} \), such that the pencil \( (E + BK, A + BF) \) is regular of index at most one and \( C(sE - (A + BK) - (A + B F))^{-1} G = 0 \) if and only if condition (6) holds,

\[
\begin{align*}
\operatorname{rank}_2
\begin{bmatrix}
T_\infty^T(G)(sE - A) & T_\infty^T(G)B \\
C & 0
\end{bmatrix}
- \operatorname{rank}
\begin{bmatrix}
T_\infty^T(G)B \\
0
\end{bmatrix}
& \leq n - p.
\end{align*}
\]

and \( W_1 := T_\infty^T\left(\begin{bmatrix} \Lambda_1 & \Lambda_3 \end{bmatrix}\right)\Lambda_2 S_\infty(T_\infty^T(\Lambda_3)\Lambda_1) \) has full row rank.

**Proof.** By Theorem 6 there exist orthogonal matrices that transform the system to the form (14). Thus, for the proof we may assume, w.l.o.g., that the system is already in form (14).

(a) **Necessity:** Let \( F \in \mathbb{R}^{m \times n} \) be such that \( (E, A + BF) \) is regular, of index at most one and \( C(sE - (A + B F))^{-1} G = 0 \). Partition \( F =: \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix} \) compatibly with \( (E, A, B) \). Then (5) follows directly from Lemma 3 (ii). Furthermore, by Lemma 4 we have that

\[
\begin{align*}
\operatorname{rank}_2
\begin{bmatrix}
sE_{21} - A_{21} - B_2 F_1 \\
- A_{32} - B_3 F_1
\end{bmatrix}
& = n - p - n_2 - n_3. \quad \text{Hence, we obtain}
\end{align*}
\]

\[
\begin{align*}
n - p - n_2 - n_3 \geq \operatorname{rank}_2
\begin{bmatrix}
sE_{21} - A_{21} - B_2 F_1
\end{bmatrix}
\geq \operatorname{rank}(E_{21}) = \tilde{n}_2,
\end{align*}
\]

i.e., by (15), condition (19) holds.
To prove conditions (20)–(21), observe from Lemma 8, and since \((E, A + BF)\) is regular and of index at most one, we have

\[
\text{rank}(E) = \deg(\det(sE - A - BF)) = \deg(\det(sE_{53} - A_{53}))
\]

\[
+ \deg(\det\left[
\begin{array}{ccc}
E_{11} - A_{11} & B_1 F_1 & E_{12} - A_{12} - B_1 F_2 \\
E_{21} - A_{21} & B_2 F_1 & E_{22} - A_{22} - B_2 F_2 \\
0 & -A_{31} & E_{32} - A_{32} - B_3 F_3 \\
0 & 0 & E_{42} - A_{42}
\end{array}
\right]).
\]

\[
= \deg(\det\left[
\begin{array}{ccc}
E_{11} - A_{11} & B_1 F_1 & E_{12} - A_{12} - B_1 F_2 \\
E_{21} - A_{21} & B_2 F_1 & E_{22} - A_{22} - B_2 F_2 \\
0 & -A_{31} & E_{32} - A_{32} - B_3 F_3 \\
0 & 0 & E_{42} - A_{42}
\end{array}
\right]).
\]

Hence, from (23) it follows that

\[
\text{rank}(E) = \text{rank}\left[
\begin{array}{cc}
E_{11} & E_{12} \\
E_{21} & E_{22} \\
0 & E_{32} \\
0 & E_{42}
\end{array}
\right] = \deg(\det\left[
\begin{array}{ccc}
E_{11} - A_{11} & B_1 F_1 & E_{12} - A_{12} - B_1 F_2 \\
E_{21} - A_{21} & B_2 F_1 & E_{22} - A_{22} - B_2 F_2 \\
0 & -A_{31} & E_{32} - A_{32} - B_3 F_3 \\
0 & 0 & E_{42} - A_{42}
\end{array}
\right]).
\]

Using that \(E_{21}, E_{42}\) are of full row rank, we may assume, w.l.o.g., (by performing appropriate equivalence transformations) that

\[
E_{11} = \left[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & \Sigma_{11} & 0 \\
0 & 0 & 0
\end{array}
\right],
E_{21} = \left[
\begin{array}{ccc}
\Sigma_{21} & 0 & 0 \\
0 & 0 & 0
\end{array}
\right],
E_{32} = \left[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
\right],
E_{12} = \left[
\begin{array}{ccc}
\Sigma_{12} & 0 & 0 \\
0 & 0 & 0
\end{array}
\right],
E_{22} = 0,
\]

where \(\Sigma_{11} \in \mathbb{R}^{p_1 \times p_1}, \Sigma_{21} \in \mathbb{R}^{n_2 \times n_2}, \Sigma_{42} \in \mathbb{R}^{n_4 \times n_4}\) are nonsingular and \(\Theta_{32} \in \mathbb{R}^{(n_2 - n_4) \times t}\) is of full column rank. Partition accordingly, \(A_{42} = \left[
\begin{array}{ccc}
\Phi_{64} & \Phi_{65} & \Phi_{66}
\end{array}
\right],\)

\[
A_{11} + B_1 F_1 = \left[
\begin{array}{ccc}
\Phi_{11} & \Phi_{12} & \Phi_{13} \\
\Phi_{21} & \Phi_{22} & \Phi_{23} \\
\Phi_{31} & \Phi_{32} & \Phi_{33}
\end{array}
\right],
A_{12} + B_1 F_2 = \left[
\begin{array}{ccc}
\Phi_{14} & \Phi_{15} & \Phi_{16} \\
\Phi_{24} & \Phi_{25} & \Phi_{26} \\
\Phi_{34} & \Phi_{35} & \Phi_{36}
\end{array}
\right],
A_{21} + B_2 F_1 = \left[
\begin{array}{ccc}
\Phi_{41} & \Phi_{42} & \Phi_{43} \\
\Phi_{51} & \Phi_{52} & \Phi_{53}
\end{array}
\right],
A_{22} + B_2 F_2 = \left[
\begin{array}{ccc}
\Phi_{44} & \Phi_{45} & \Phi_{46} \\
\Phi_{54} & \Phi_{55} & \Phi_{56}
\end{array}
\right].
\]

Then (24) yields that

\[
\begin{bmatrix}
T^T(\vec{E}_{12})\Phi_{13} & T^T(\vec{E}_{12})\Phi_{14}\Phi_{43} & T^T(\vec{E}_{12})\Phi_{14}\Phi_{53} \\
T^T(\vec{E}_{12})\Phi_{43} & T^T(\vec{E}_{12})\Phi_{44}\Phi_{43} & T^T(\vec{E}_{12})\Phi_{44}\Phi_{53} \\
T^T(\vec{E}_{12})\Phi_{53} & T^T(\vec{E}_{12})\Phi_{54}\Phi_{43} & T^T(\vec{E}_{12})\Phi_{54}\Phi_{53}
\end{bmatrix}
\]

is nonsingular. Hence, we obtain that

\[
\text{rank}\left[
\begin{array}{cc}
T^T(\vec{E}_{12})\Phi_{13} & T^T(\vec{E}_{12})\Phi_{43} \\
T^T(\vec{E}_{12})\Phi_{53}
\end{array}
\right] = n_1 - p_1 - n_2.
\]

(25)
But, from Lemma 4 we have
\[
\text{rank}_2 \begin{bmatrix}
  s\Sigma_{21} - \Phi_{31} & -\Phi_{32} & -\Phi_{33} \\
  -\Phi_{41} & -\Phi_{42} & -\Phi_{43} \\
  -\Phi_{51} & -\Phi_{52} & -\Phi_{53}
\end{bmatrix} = n - p - n_2 - n_3 = n_1 - p
\]
and hence
\[
\text{rank} \begin{bmatrix}
  \Phi_{43} \\
  T^T_\infty(\Theta_{32})\Phi_{53}
\end{bmatrix} \leq \text{rank} \begin{bmatrix}
  \Phi_{43} \\
  \Phi_{53}
\end{bmatrix} \leq n_1 - p - \tilde{n}_2. \tag{26}
\]
Thus, by (25), we have
\[
\text{rank} \left( T^T_\infty(\tilde{E}_{12})\Phi_{13} \right) \geq p - p_1, \tag{27}
\]
where \( p - p_1 \) is the number of rows of \( \Phi_{13} \). This implies that \( \tilde{E}_{12} = 0 \) and hence (24) implies that
\[
\text{rank} (E) = \text{rank} \begin{bmatrix}
  E_{11} \\
  E_{21} \\
  0
\end{bmatrix} + \text{rank} \begin{bmatrix}
  E_{32} \\
  E_{42} \\
  0
\end{bmatrix}.
\]
and thus, Lemma 8 (i) yields that
\[
\text{rank}(E) = \text{rank} \begin{bmatrix}
  E_{11} \\
  E_{21} \\
  0
\end{bmatrix} + \text{rank} \begin{bmatrix}
  E_{32} & E_{33} \\
  E_{42} & E_{43} \\
  0 & E_{53}
\end{bmatrix}.
\]
Then, Lemma 7 (ii) gives (20). Furthermore, using (26) and (27) we have that
\[
\text{rank} (\Phi_{13}) = p - p_1, \quad \text{rank} \begin{bmatrix}
  \Phi_{43} \\
  \Phi_{53}
\end{bmatrix} = \text{rank} \begin{bmatrix}
  \Phi_{43} \\
  T^T_\infty(\Theta_{32})\Phi_{53}
\end{bmatrix} = n_1 - p - \tilde{n}_2
\]
and thus with \( S := S_\infty \left( \begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} \right) \), we obtain
\[
\text{rank} \begin{bmatrix}
  E_{11} & A_{11}S & B_1 \\
  E_{21} & A_{21}S & B_2 \\
  0 & A_{31}S & B_3
\end{bmatrix} = p + \tilde{n}_2 + \tilde{n}_3. \tag{28}
\]
Then (21) follows directly from Lemma 7 (ii).

**Sufficiency:** Using conditions (19) and (21) and Lemmas 3 (ii) and (5) (ii), there exist \( \tilde{F}_1 \in \mathbb{R}^{m \times n_1} \) and a nonsingular matrix \( Z \in \mathbb{R}^{n_1 \times n_1} \) such that
\[
\begin{bmatrix}
  sE_{11} - A_{11} - B_1\tilde{F}_1 \\
  sE_{21} - A_{21} - B_2\tilde{F}_1 \\
  -A_{31} - B_3\tilde{F}_1
\end{bmatrix} = \begin{bmatrix}
  p + \tilde{n}_2 & n_1 - p - \tilde{n}_2 \\
  \tilde{n}_2 & 0 \\
  \tilde{n}_3 & 0
\end{bmatrix}
\]
with \( \left( \begin{bmatrix} \Theta_{11} \\ \Theta_{21} \end{bmatrix}, \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} \right) \) regular and of index at most one.
By (5) and (20) and Lemma 8 it follows that $\tilde{n}_5 = n_3$, $E_{53} = 0$, $A_{53}$ is nonsingular, and furthermore

$$\text{rank}(E) = \text{rank} \begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} + \text{rank} \begin{bmatrix} E_{32} \\ E_{42} \end{bmatrix}. \quad (29)$$

Note that $B_3$ and $E_{42}$ are of full row rank, so by Lemma 3 (ii) there exist matrices $\tilde{F}_1 \in \mathbb{R}^{m \times (n_1 - p - n_2)}$ and $F_2 \in \mathbb{R}^{m \times n_2}$ such that

$$\begin{bmatrix} 0 & E_{32} \\ 0 & E_{42} \end{bmatrix}, \begin{bmatrix} B_3 & A_{32} + B_3 F_2 \\ 0 & A_{42} \end{bmatrix}$$

is regular and of index at most one. Taking $F_1 := \tilde{F}_1 + \begin{bmatrix} 0 & \tilde{F}_1 \end{bmatrix} Z^{-1}$, $F := \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}$ with $F_3$ arbitrary, it is easy to see that $(E, A + BF)$ is regular, of index at most one and $C(sE - A - BF)^{-1}G = 0$.

(b) **Necessity:** Let $F, K \in \mathbb{R}^{m \times n}$ be such that $(E + BK, A + BF)$ is regular of index at most one and $C(sE + BK - (A + BF))^{-1}G = 0$. Then condition (6) follows directly from Lemma 3 (iii). As in (a), by Lemma 4 we have that

$$\text{rank}_j \begin{bmatrix} s(E_{21} + B_2 K_1) - (A_{21} + B_2 F_1) \\ sB_3 K_1 - (A_{31} + B_3 F_1) \end{bmatrix} = n - p - n_2 - n_3,$$

which implies that

$$\text{rank}_j \begin{bmatrix} sE_{21} - A_{21} \\ -A_{31} \end{bmatrix} B_3 - \text{rank} \begin{bmatrix} B_3 \\ B_3 \end{bmatrix} = \text{rank}_j \begin{bmatrix} sE_{21} + B_2 K_1 - (A_{21} + B_2 F_1) \\ sB_3 K_1 - (A_{31} + B_3 F_1) \end{bmatrix} - \text{rank} \begin{bmatrix} B_3 \\ B_3 \end{bmatrix} \leq \text{rank}_j \begin{bmatrix} sE_{21} + B_2 K_1 - (A_{21} + B_2 F_1) \\ sB_3 K_1 - (A_{31} + B_3 F_1) \end{bmatrix} = n - p - n_2 - n_3.$$

Thus,

$$n - p \geq n_2 + n_3 + \text{rank}_j \begin{bmatrix} sE_{21} - A_{21} \\ -A_{31} \end{bmatrix} B_3 - \text{rank} \begin{bmatrix} B_3 \\ B_3 \end{bmatrix} \leq \text{rank}_j \begin{bmatrix} T^T_{\infty}(G)(sE - A) \\ C \end{bmatrix} - \text{rank} \begin{bmatrix} T^T_{\infty}(G)B \\ 0 \end{bmatrix},$$

i.e., (22) holds.

Since

$$\begin{bmatrix} E_{21} + B_2 K_1 \\ B_3 K_1 \\ B_3 \end{bmatrix}$$

is of full row rank, from (28) we have that

$$\begin{bmatrix} E_{11} + B_1 K_1 \\ E_{21} + B_2 K_1 \\ B_3 K_1 \end{bmatrix}$$

is of full row rank, where $\tilde{S} := S_{\infty}(\begin{bmatrix} E_{11} + B_1 K_1 \\ E_{21} + B_2 K_1 \\ B_3 K_1 \end{bmatrix})$. Equivalently, we obtain that

$$\begin{bmatrix} E_{11} + A_{11} \tilde{S}_{\infty} \\ E_{21} \\ 0 \end{bmatrix}$$

is of full row rank. Thus, using the relation between $S_{\infty}(T^T_{\infty}(\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}))$ and $\tilde{S}_{\infty}$, we
have that
\[
T^T_\infty \left( \begin{array}{cc} E_{11} & B_1 \\ E_{21} & B_2 \\ 0 & B_3 \end{array} \right) \left( \begin{array}{c} A_{11} \\ A_{21} \\ A_{31} \end{array} \right) S_\infty \left(T^T_\infty \left( \begin{array}{cc} B_1 \\ B_2 \\ 0 \end{array} \right) \right) \left( \begin{array}{c} E_{11} \\ E_{21} \\ 0 \end{array} \right)
\]
is of full row rank and using Lemma 7 (iii) also \( W_1 \).

**Sufficiency:** Since \( E_{21}, B_3 \) are of full row rank, using Lemma 1, we can determine orthogonal matrices \( P_1 \in \mathbb{R}^{p \times p}, P_2 \in \mathbb{R}^{(n_2+n_3) \times (n_2+n_3)} \) and \( W \in \mathbb{R}^{m \times m} \), such that

\[
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
\begin{bmatrix}
sE_{11} - A_{11} & sE_{12} - A_{12} & sE_{13} - A_{13} \\
sE_{21} - A_{21} & sE_{22} - A_{22} & sE_{23} - A_{23} \\
-A_{31} & sE_{32} - A_{32} & sE_{33} - A_{33}
\end{bmatrix}
\begin{bmatrix}
l_1 \\
l_2 \\
n_1 + n_2 - n_3
\end{bmatrix}
= \begin{bmatrix}
l_1 \\
l_2 \\
n_1 + n_2 - n_3
\end{bmatrix},
\]

with \( \tilde{B}_{22}, \Theta_{31} \) of full row rank and \( \tilde{B}_{41} \) nonsingular. Now determine \( K_{11}, K_{21} \) from

\[
\begin{bmatrix}
\tilde{B}_{21} & \tilde{B}_{22} \\
\tilde{B}_{41} & 0
\end{bmatrix}
\begin{bmatrix}
K_{11} \\
K_{21}
\end{bmatrix}
= - \begin{bmatrix}
\Theta_{21} \\
\Theta_{41}
\end{bmatrix}.
\]

The last two conditions imply that \( p + l_2 \leq n_1 \) and

\[
\text{rank } \begin{bmatrix}
\Theta_{11} + \tilde{B}_{11} K_{11} & \Phi_{11} \tilde{S} & \tilde{B}_{11} & 0 \\
0 & \Phi_{21} \tilde{S} & \tilde{B}_{21} & \tilde{B}_{22} \\
\Theta_{31} & \Phi_{31} \tilde{S} & 0 & 0 \\
0 & \Phi_{41} \tilde{S} & \tilde{B}_{41} & 0
\end{bmatrix} = p + \tilde{n}_2 + \tilde{n}_3,
\]

where \( \tilde{S} = S_\infty \left( \begin{array}{c}
\Theta_{11} + \tilde{B}_{11} K_{11} \\
\Theta_{31}
\end{array} \right) \). By Lemma 5 there exist \( F_{11}, F_{21} \) and a nonsingular matrix \( Z \) satisfying

\[
\begin{bmatrix}
s(\Theta_{11} + \tilde{B}_{11} K_{11}) - \Phi_{11} - \tilde{B}_{11} F_{11} \\
-\Phi_{21} - \tilde{B}_{21} F_{11} - \tilde{B}_{22} F_{21} \\
0 \\
-\Phi_{41} - \tilde{B}_{41} F_{11}
\end{bmatrix}
= \begin{bmatrix}
l_1 \\
l_2 \\
p + l_2 \\
\tilde{n}_2 + \tilde{n}_3 - l_2
\end{bmatrix},
\]

with \( \begin{bmatrix}
\Theta_{11} \\
0
\end{bmatrix}, \begin{bmatrix}
\Phi_{11} \\
\Phi_{21}
\end{bmatrix} \) regular and of index at most one. Using condition (6) and Lemma 8
By (18), and since we have from Lemma 8 that \( \tilde{n}_5 = n_3 \). Hence, we can determine \( \tilde{K}_{11}, \tilde{K}_{21}, K_{12}, K_{22} \), such that

\[
\begin{bmatrix}
\tilde{B}_{21} & \tilde{B}_{22}
\end{bmatrix}
\begin{bmatrix}
\tilde{K}_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
= -
\begin{bmatrix}
0 & \Theta_{22}
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
\tilde{B}_{41} \tilde{K}_{11} & \Theta_{42} + \tilde{B}_{41} K_{12} \\
0 & E_{42}
\end{bmatrix}
\]

is nonsingular. Let

\[
K_1 := \begin{bmatrix}
K_{11} \\
K_{21}
\end{bmatrix}
+ \begin{bmatrix}
0 & \tilde{K}_{11} \\
0 & \tilde{K}_{21}
\end{bmatrix} Z^{-1}
K_2 := \begin{bmatrix}
K_{12} \\
K_{22}
\end{bmatrix}
K_3 := \begin{bmatrix}
K_{13} \\
K_{23}
\end{bmatrix},
F_1 := \begin{bmatrix}
F_{11} \\
F_{21}
\end{bmatrix},
\]

where \( K_{13}, K_{23} \) will be determined later, and set \( K := \begin{bmatrix}
K_1 & K_2 & K_3
\end{bmatrix} \) and \( F := \begin{bmatrix}
F_1 & 0 & 0
\end{bmatrix} \). Then we have

\[
\begin{bmatrix}
P_1 & \quad P_2 & \quad I
\end{bmatrix}
\begin{bmatrix}
\quad & (E + BK) & \quad - (A + BF)
\end{bmatrix}
\begin{bmatrix}
\quad & Z & \quad
\end{bmatrix}
\begin{bmatrix}
\quad & \quad & \quad
\end{bmatrix}
\]

By (18), and since

\[
\begin{bmatrix}
\tilde{B}_{41} \tilde{K}_{11} & \Theta_{42} + \tilde{B}_{41} K_{12} \\
0 & E_{42}
\end{bmatrix}
\]

is nonsingular, we have that

\[
\text{rank}
\begin{bmatrix}
\tilde{\Theta}_{11} & \Theta_{12} & \Theta_{13} & \tilde{B}_{11} \\
\Theta_{31} & \Theta_{32} & \Theta_{33} & 0 \\
0 & \Theta_{42} & \Theta_{43} & \tilde{B}_{41} \\
0 & E_{42} & E_{43} & 0
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
\tilde{\Theta}_{11} & \Theta_{12} & \tilde{B}_{11} \\
\Theta_{31} & \Theta_{32} & 0 \\
0 & \Theta_{42} & \tilde{B}_{41} \\
0 & E_{42} & 0
\end{bmatrix}.
\]

Therefore, there exists \( K_{13} \) such that

\[
\text{rank}
\begin{bmatrix}
\tilde{\Theta}_{11} & \Theta_{12} + \tilde{B}_{11} K_{12} & \Theta_{13} + \tilde{B}_{11} K_{13} \\
\Theta_{31} & \Theta_{32} & \Theta_{33} \\
0 & \Theta_{42} + \tilde{B}_{41} K_{12} & \Theta_{43} + \tilde{B}_{41} K_{13} \\
0 & E_{42} & E_{43}
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
\tilde{\Theta}_{11} & \Theta_{12} + \tilde{B}_{11} K_{12} \\
\Theta_{31} & \Theta_{32} \\
0 & \Theta_{42} + \tilde{B}_{41} K_{12} \\
0 & E_{42}
\end{bmatrix}.
\]

Finally, we determine \( K_{23} \) from

\[
\tilde{B}_{22} K_{23} = -(\Theta_{23} + \tilde{B}_{21} K_{13}).
\]

We have from Lemma 8 that \( E_{43} = 0 \) and that \( A_{53} \) is nonsingular. Thus, for \( F, K \) determined by the described procedure, \( (E + BK, A + BF) \) is regular and of index at most one. It is also easy to see that \( C(s(E + BK) - (A + BF))^{-1} G = 0 \).
Remark 3 If the index one condition is not required, then in (a) necessary and sufficient conditions are given by (4), (19) and
\[
\operatorname{rank}_2 \begin{bmatrix} sE - A & B \\ C & 0 \end{bmatrix} = \operatorname{rank}_2 \begin{bmatrix} sE - A & B & G \\ C & 0 & 0 \end{bmatrix},
\]
and in (b) by (4), (22) and (30). The proof of the necessary and sufficient conditions and the construction of feedbacks that regularize the system without achieving index at most one are discussed in detail in [8].

Remark 4 The previous results can also be modified to the case when only derivative feedback is used. Since the results are essentially dual results to the ones for state feedback by exchanging the roles of $E$ and $A$, we omit these results. Details are given in the reports [8, 9].

4 The Disturbance Decoupling Problem with Stability

In this section we study the case where the extra requirement that the closed loop system is stable is added. Similar to Section 3, first we prove a condensed form for matrix quintuples $(E, A, B, C, G$ under orthogonal equivalence transformations and determine different left and right reducing subspaces that are needed for the solution of the disturbance decoupling problem with stability.

Theorem 10 Given a system of the form (1), there exist orthogonal matrices $U, V \in \mathbb{R}^{n \times n}$ such that
\[
U(sE - A)V = \begin{bmatrix} \tilde{n}_1 & \tilde{n}_2 & \tilde{n}_3 & \tilde{n}_4 \\ \tilde{\nu}_1 & sE_{11} - A_{11} & sE_{12} - A_{12} & sE_{13} - A_{13} & sE_{14} - A_{14} \\ \tilde{\nu}_2 & -A_{21} & sE_{22} - A_{22} & sE_{23} - A_{23} & sE_{24} - A_{24} \\ \tilde{\nu}_3 & -A_{31} & sE_{32} - A_{32} & sE_{33} - A_{33} & sE_{34} - A_{34} \\ \tilde{\nu}_4 & 0 & sE_{42} - A_{42} & sE_{43} - A_{43} & sE_{44} - A_{44} \\ \tilde{\nu}_5 & 0 & 0 & sE_{53} - A_{53} & sE_{54} - A_{54} \\ \tilde{\nu}_6 & 0 & 0 & 0 & sE_{64} - A_{64} \end{bmatrix},
\]
\[
UB = \begin{bmatrix} \tilde{n}_1 & \tilde{n}_2 & \tilde{n}_3 & \tilde{n}_4 \\ \tilde{\nu}_1 & B_1 & 0 & 0 \\ \tilde{\nu}_2 & 0 & B_3 & 0 \\ \tilde{\nu}_3 & B_3 & 0 & 0 \\ \tilde{\nu}_4 & 0 & 0 & 0 \\ \tilde{\nu}_5 & 0 & 0 & 0 \\ \tilde{\nu}_6 & 0 & 0 & 0 \end{bmatrix}, \quad UG = \begin{bmatrix} \tilde{n}_1 & \tilde{n}_2 & \tilde{n}_3 & \tilde{n}_4 \\ \tilde{\nu}_1 & G_1 & 0 & 0 \\ \tilde{\nu}_2 & 0 & G_2 & 0 \\ \tilde{\nu}_3 & 0 & 0 & G_3 \\ \tilde{\nu}_4 & 0 & 0 & 0 \\ \tilde{\nu}_5 & 0 & 0 & 0 \\ \tilde{\nu}_6 & 0 & 0 & 0 \end{bmatrix}, \quad CV = \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \\ 0 & 0 & C_3 & C_4 \end{bmatrix},
\]

where $E_{11}, G_2, B_3$ and $E_{53}$ are of full row rank, $E_{42}$ is square and nonsingular, (i.e., $n_2 = \tilde{n}_4$,) and furthermore for all $s \in C$
\[
\operatorname{rank}(sE_{64} - A_{64}) = n_4, \quad \operatorname{rank}\begin{bmatrix} sE_{53} - A_{53} \\ C_3 \end{bmatrix} = n_3,
\]
\[
\operatorname{rank}\begin{bmatrix} sE_{11} - A_{11} & B_1 & G_1 \\ -A_{21} & 0 & G_2 \\ -A_{31} & B_3 & G_3 \end{bmatrix} = \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3.
\]
Proof. The proof is constructive and can be obtained by an algorithm similar to Algorithm 1. A detailed description of the method can be found in [9].

Similar to the proof of Lemma 7, using the condensed form (31) we can characterize the following spaces

\[
\begin{align*}
\Delta_r & := V_{m-r}[\Pi^T \begin{bmatrix} E & 0 \\ 0 & C \end{bmatrix}], \\
\Delta_l & := [\Pi^{-} \Pi]^T \begin{bmatrix} I & 0 \\ 0 & \Delta_l \end{bmatrix}, \\
\Delta_1 & := \Delta_l^T \begin{bmatrix} E \\ 0 \end{bmatrix} \Delta_r, \\
\Delta_2 & := \Delta_l^T \begin{bmatrix} A \\ C \end{bmatrix} \Delta_r, \\
\Delta_3 & := \Delta_l^T \begin{bmatrix} B \\ 0 \end{bmatrix}, \\
\Delta_4 & := \Delta_l^T \begin{bmatrix} G \\ 0 \end{bmatrix},
\end{align*}
\]

where \( \Pi \) is as in (12). Introduce furthermore the following indices which are determined by the condensed form of Theorem 10.

\[
\begin{align*}
\nu & := \text{rank} \begin{bmatrix} B & G \end{bmatrix} + \text{rank}(\Delta_l), \\
\xi & := \text{rank}(\Delta_r), \\
\chi & := \text{dim}(V_{m-r} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B & G \\ C & 0 & 0 \end{bmatrix}) + \text{dim}(V_{l-1} \begin{bmatrix} E & B & G \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A & 0 & 0 \\ C & 0 & 0 \end{bmatrix}) \\
& - \text{dim}(V_{l-1} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & 0 & 0 \\ C & 0 & 0 \end{bmatrix}).
\end{align*}
\]

Form the form (31) we then immediately obtain that

\[\xi = n_1, \ \chi = \bar{n}_1 + \bar{n}_2, \ \nu = \bar{n}_1 + \bar{n}_2 + \bar{n}_3.\]

**Theorem 11** Given a system of the form (1), spaces \( \Delta_i \) as in (32) and indices \( \chi, \xi \) and \( \nu \) as in (33).

(a) There exist feedback matrices \( F \in \mathbb{R}^{m \times n} \) and \( H \in \mathbb{R}^{m \times d} \) such that \( (E, A + BF) \) is regular, of index at most one, stable and \( C (sE - (A + BF))^{-1} (G + BH) = 0 \) if and only if conditions (5) and (7) and the following three conditions hold

\[
\begin{align*}
\text{rank} \begin{bmatrix} s\Delta_1 - \Delta_2 & \Delta_3 \end{bmatrix} & = \nu, \ \forall s \in \mathbb{C}^{+}, \\
\text{rank} \begin{bmatrix} \Delta_1 & \Delta_2 S_\infty(\Delta_1) & \Delta_3 \end{bmatrix} & = \nu, \\
\chi & \leq \xi.
\end{align*}
\]

(b) There exist feedback matrices \( F, K \in \mathbb{R}^{m \times n} \) and \( H \in \mathbb{R}^{m \times d} \) such that \( (E + BK, A + BF) \) is regular of index at most one, stable and \( C (sE - (A + BF))^{-1} (G + BH) = 0 \) if and only if conditions (6), (7), (34),

\[
\text{rank}_s \begin{bmatrix} T^T_{\infty}(B)(sE - A) & T^T_{\infty}(B)G \\ C & 0 \end{bmatrix} \leq n
\]

hold and furthermore \( W_2 := T^T_{\infty}(\begin{bmatrix} \Delta_1 & \Delta_3 \end{bmatrix})\Delta_2 S_\infty(T^T_{\infty}(\Delta_3)\Delta_1) \) is of full row rank.
Proof. Let \( U,V \in \mathbb{R}^{n \times n} \) be orthogonal matrices such that \( U(sE - A)V, UB, CV, UG \) are in the form (31). Then condition (34) translates to
\[
\text{rank} \begin{bmatrix} sE_{11} - A_{11} & B_1 \\ -A_{21} & 0 \\ -A_{31} & B_3 \end{bmatrix} = \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3, \quad \forall s \in \mathbb{C}^+ \]
and (36) means that \( \tilde{n}_1 + \tilde{n}_2 \leq n_1 \). Condition (35) translates to
\[
\text{rank} \begin{bmatrix} E_{11} & A_{11}S_{\infty}(E_{11}) & B_1 \\ 0 & A_{21}S_{\infty}(E_{11}) & 0 \\ 0 & A_{31}S_{\infty}(E_{11}) & B_3 \end{bmatrix} = \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 \]
and \( W_2 \) has full rank if and only if
\[
T^T_\infty \begin{bmatrix} E_{11} & B_1 \\ 0 & 0 \\ 0 & B_3 \end{bmatrix} = A_{11} \begin{bmatrix} 0 \\ A_{21} \\ A_{31} \end{bmatrix} S_{\infty}(T^T_\infty \begin{bmatrix} B_1 \\ 0 \\ B_3 \end{bmatrix} = E_{11}) \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
has full row rank.

(a) Necessity: Let \( F \in \mathbb{R}^{m \times n} \) and \( H \in \mathbb{R}^{m \times d} \) be such that \((E, A + BF)\) is regular of index at most one and stable and \( C(sE - (A + BF))^{-1}(G + BH) = 0 \). Conditions (5) and (7) follow directly by Lemma 3 (v). If we partition \( FV = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 \end{bmatrix} \), then by Lemma 4 we have that
\[
\sum_{i=1}^{4} n_i = n = \text{rank}_2 \begin{bmatrix} sE - (A + BF) & G + BH \\ C & 0 \end{bmatrix} = n_2 + n_3 + n_4 + \text{rank}_2 \begin{bmatrix} sE_{11} - (A_{11} + B_1F_1) & G_1 + B_1H \\ -A_{21} & G_2 \\ -(A_{31} + B_3F_1) & G_3 + B_3H \end{bmatrix}.
\]
Hence
\[
\text{rank}_2 \begin{bmatrix} sE_{11} - (A_{11} + B_1F_1) & G_1 + B_1K \\ -A_{21} & G_2 \\ -(A_{31} + B_3F_1) & G_3 + B_3K \end{bmatrix} = n_1 \quad \text{(38)}
\]
which implies condition (36), since
\[
\tilde{n}_1 + \tilde{n}_2 = \text{rank}_2 \begin{bmatrix} sE_{11} - (A_{11} + B_1F_1) & G_1 + B_1H \\ -A_{21} & G_2 \end{bmatrix} \leq n_1.
\]
To show (34) let \( P_1 \) be an orthogonal matrix such that
\[
P_1^T \begin{bmatrix} G_1 + B_1H \\ G_2 \\ G_3 + B_3H \end{bmatrix} = t_1 \begin{bmatrix} \tilde{G}_1 \\ 0 \end{bmatrix} \quad \text{(39)}
\]
with $\tilde{G}_1$ of full row rank. Set

$$
\begin{align*}
t_1 & \begin{bmatrix} s\tilde{E}_{11} - \tilde{A}_{11} \\ s\tilde{E}_{21} - \tilde{A}_{21} \end{bmatrix} := P_1^T \begin{bmatrix} sE_{11} - (A_{11} + B_1 F_1) \\ -A_{21} \\ -(A_{31} + B_3 F_1) \end{bmatrix} \\
t_2 & \end{align*}
$$

and compute the generalized upper triangular form of $(\tilde{E}_{21}, \tilde{A}_{21})$

$$
\hat{P}_1^T (s\tilde{E}_{21} - \tilde{A}_{21}) Q_1 = t_2 \begin{bmatrix} r_1 \\ t_3 \\ r_2 \end{bmatrix} \begin{bmatrix} s\Theta_{21} - \Phi_{21} \\ s\Theta_{22} - \Phi_{22} \\ 0 \end{bmatrix}
$$

with $\Theta_{21}$ of full row rank and $s\Theta_{32} - \Phi_{32}$ of full column rank for all $s \in C$. Set

$$
\begin{align*}
\begin{bmatrix} s\Theta_{11} - \Phi_{11} \\ s\Theta_{12} - \Phi_{12} \end{bmatrix} := (s\tilde{E}_{11} - \tilde{A}_{11}) Q_1, \\
t_1 & \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix} := \begin{bmatrix} I \\ \hat{P}_1^T \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \\ B_3 \end{bmatrix}.
\end{align*}
$$

Since

$$
\begin{align*}
\text{rank} \begin{bmatrix} sE_{11} - A_{11} & B_1 \\ -A_{21} & 0 & G_1 \\ -A_{31} & B_3 & G_3 \end{bmatrix} = \text{rank} \begin{bmatrix} s\Theta_{11} - \Phi_{11} & s\Theta_{12} - \Phi_{12} & \Psi_1 & \tilde{G}_1 \\ s\Theta_{21} - \Phi_{21} & s\Theta_{22} - \Phi_{22} & \Psi_2 & 0 \\ 0 & s\Theta_{32} - \Phi_{32} & \Psi_3 & 0 \end{bmatrix} = \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3 = t_1 + t_2 + t_3
\end{align*}
$$

for all $s \in C$, it follows that rank $\begin{bmatrix} s\Theta_{32} - \Phi_{32} & \Psi_3 \end{bmatrix} = t_3$ for all $s \in C$. By (38) we also have that $t_1 + t_2 + r_2 = t_1 + \text{rank}_s (s\Theta_{21} - \Phi_{21}) + r_2 = \tilde{n}_1 = r_1 + r_2$, or equivalently we have that

$$
t_1 + t_2 = r_1 \quad \text{and} \quad \begin{bmatrix} s\Theta_{11} - \Phi_{11} \\ s\Theta_{21} - \Phi_{21} \end{bmatrix} \text{ is square.}
$$

We know that $(E, A + BF)$ is regular and stable, so we have that $\begin{bmatrix} \Theta_{11} \\ \Theta_{21} \end{bmatrix}$, $\begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix}$ is regular and stable. Therefore, we have for all $s \in C^+$ that

$$
\begin{align*}
\text{rank} \begin{bmatrix} sE_{11} - A_{11} & B_1 \\ -A_{21} & 0 \\ -A_{31} & B_3 \end{bmatrix} = \text{rank} \begin{bmatrix} sE_{11} - (A_{11} + B_1 F_1) & B_1 \\ -A_{21} & 0 \\ -(A_{31} + B_3 F_1) & B_3 \end{bmatrix} = t_1 + t_2 + \text{rank} \begin{bmatrix} s\Theta_{32} - \Phi_{32} & \Psi_3 \end{bmatrix} = t_1 + t_2 + t_3 = \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3
\end{align*}
$$

which gives condition (34).

We have that $E_{11}, G_2$ and $B_3$ are of full row rank, so that $\begin{bmatrix} \Theta_{32} & \Psi_3 \end{bmatrix}$ is also of full row rank. Since $(E, A + BF)$ is regular and of index at most one, we have that $\begin{bmatrix} \Theta_{11} \\ \Theta_{21} \end{bmatrix}$ and $\begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix}$
is also regular and of index at most one. Then using the standard characterization of pencils that are regular and of index at most one, e.g., [5] we obtain that \( \begin{bmatrix} \Theta_{11} & \Phi_{11} \hat{S} \\ \Theta_{21} & \Phi_{21} \hat{S} \end{bmatrix} = t_1 + t_2 \) with \( \hat{S} = S_\infty( \begin{bmatrix} \Theta_{11} \\ \Theta_{21} \end{bmatrix} ) \). Therefore, we have

\[
\bar{n}_1 + \bar{n}_2 + \bar{n}_3 = t_1 + t_2 + t_3 \geq \text{rank} \begin{bmatrix} E_{11} & A_{11} S_\infty(E_{11}) & B_1 \\ 0 & A_{21} S_\infty(E_{11}) & 0 \\ 0 & A_{31} S_\infty(E_{11}) & B_3 \end{bmatrix} = \text{rank} \begin{bmatrix} E_{11} & (A_{11} + B_1 F_1) S_\infty(E_{11}) & B_1 \\ 0 & A_{21} S_\infty(E_{11}) & 0 \\ 0 & (A_{31} + B_3 F_1) S_\infty(E_{11}) & B_3 \end{bmatrix} = \text{rank} \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Phi_{11} \hat{S} & \Psi_1 \\ \Theta_{21} & \Theta_{22} & \Phi_{21} \hat{S} & \Psi_2 \\ 0 & 0 & \Psi_3 \end{bmatrix} = t_1 + t_2 + t_3, \]

where

\[
\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \\ 0 & \Phi_{32} \end{bmatrix}, \quad \hat{S} = S_\infty( \begin{bmatrix} \Theta_{11} \\ \Theta_{21} \end{bmatrix} ).
\]

Hence, (35) follows.

**Sufficiency:** Since conditions (7) and (5) hold, similar to Lemma 8, we have

\[
\text{rank}(E) = \text{rank}(E_{11}) + \text{rank} \begin{bmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix}, \quad \bar{n}_6 = n_4, \ E_{64} = 0, \ \det(A_{64}) \neq 0. \quad (44)
\]

Hence, by condition (36), we have that \( n_2 + n_3 \leq \bar{n}_3 + n_2 + \bar{n}_5 \) and therefore, as in Theorem 2.4 of [10], we can compute a matrix \( X \in \mathbb{R}^{n_2 \times n_2} \) such that

\[
\text{rank} \begin{bmatrix} E_{32} + X E_{22} & E_{33} + X E_{23} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix} = \text{rank} \begin{bmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix}.
\]

As a consequence, we obtain

\[
\text{rank}(E) = \text{rank}(E_{11}) + \text{rank} \begin{bmatrix} E_{32} + X E_{22} & E_{33} + X E_{23} \\ E_{42} & E_{43} \\ 0 & E_{53} \end{bmatrix}. \quad (45)
\]

Since conditions (34), (36) and (35) hold, by a slight modification of Lemma 5, there exist \( \tilde{F}_1 \in \mathbb{R}^{m \times n_1} \) and a nonsingular matrix \( Z \) such that

\[
\begin{bmatrix} s E_{11} - (A_{11} + B_1 \tilde{F}_1) \\ -A_{21} \\ -(A_{31} + X A_{21} + B_3 \tilde{F}_1) \end{bmatrix} Z = \begin{bmatrix} \bar{n}_1 \\ \bar{n}_2 \\ \bar{n}_5 \end{bmatrix} \begin{bmatrix} s \Theta_{11} - \Phi_{11} \\ -\Phi_{12} \\ s \Theta_{21} - \Phi_{21} \\ -\Phi_{22} \\ 0 \end{bmatrix} = n_1 - (\bar{n}_1 + \bar{n}_2) \].
with \( \begin{bmatrix} \Theta_{11} \\ \Theta_{21} \end{bmatrix}, \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} \) regular, of index at most one and stable. By condition \((7)\), we obtain for all \( s \in C^+ \)

\[
\begin{bmatrix}
  s \bar{E}_{32} - \bar{A}_{32} & s \bar{E}_{33} - \bar{A}_{33} & s \bar{E}_{34} - \bar{A}_{34} & B_3 \\
  s E_{42} - A_{42} & s E_{43} - A_{43} & s E_{44} - A_{44} & 0 \\
  0 & s E_{53} - A_{53} & s E_{54} - A_{54} & 0 \\
  0 & 0 & s E_{64} - A_{64} & 0
\end{bmatrix} = n - (\bar{n}_1 + \bar{n}_2),
\]

where

\[
\begin{align*}
\bar{E}_{32} &= E_{32} + X E_{22}, & \bar{A}_{32} &= A_{32} + X A_{42}, & \bar{E}_{33} &= E_{33} + X E_{23}, \\
\bar{A}_{33} &= A_{33} + X A_{23}, & \bar{E}_{34} &= E_{34} + X E_{24}, & \bar{A}_{34} &= A_{34} + X A_{24}.
\end{align*}
\]

We also have that \( B_3, E_{53} \) are of full row rank, \( E_{42} \) and \( A_{64} \) are nonsingular and \( E_{64} = 0 \). Moreover, condition \((45)\) obviously says that

\[
\begin{bmatrix}
  E_{32} + X E_{12} & E_{33} + X E_{23} \\
  E_{42} & E_{43} \\
  0 & E_{53}
\end{bmatrix} = \begin{bmatrix}
  E_{32} + X E_{12} & E_{33} + X E_{23} & E_{34} + X E_{24} \\
  E_{42} & E_{43} & E_{44} \\
  0 & E_{53} & E_{54}
\end{bmatrix}.
\]

Thus, by Lemma 3(iv), there exists a matrix \( \begin{bmatrix} n_1 - (\bar{n}_1 + \bar{n}_2) & n_2 & n_3 & n_4 \end{bmatrix} \) such that

\[
\begin{bmatrix}
  0 & \bar{E}_{32} & \bar{E}_{33} & \bar{E}_{34} \\
  0 & E_{42} & E_{43} & E_{44} \\
  0 & 0 & E_{53} & E_{54} \\
  0 & 0 & 0 & E_{64}
\end{bmatrix} = \begin{bmatrix} B_3 \bar{F}_1 & \bar{A}_{32} + B_3 F_2 & \bar{A}_{33} + B_3 F_3 & \bar{A}_{34} + B_3 F_4 \\
  0 & A_{42} & A_{43} & A_{44} \\
  0 & 0 & A_{53} & A_{54} \\
  0 & 0 & 0 & A_{64}
\end{bmatrix}
\]

is regular, of index at most one, and stable.

Let

\[
F_1 := \bar{F}_1 + \begin{bmatrix} 0 \end{bmatrix} Z^{-1}, \quad F := \begin{bmatrix} F_1 & F_2 & F_3 & F_4 \end{bmatrix} V^T,
\]

we have that \((E, A + BF)\) is regular, of index at most one and stable. Furthermore, if we compute \(H\) from \((G_3 + X G_2) + B_3 H = 0\), then we also have disturbance decoupling.

(b) **Necessity:** Let \( F, K \in R^{m \times n}\) and \( H \in R^{m \times d}\) be such that \((E + BK, A + BF)\) is regular, of index at most one, stable and the disturbances are decoupled. Then conditions \((6)-(7)\) and \((37)\) follow directly Lemma 3(vii), Lemma 4 and the inequality

\[
\begin{bmatrix} T_{\infty}(B)(s E - A) \\ C \end{bmatrix} \leq \begin{bmatrix} T_{\infty}(B)G \end{bmatrix} \begin{bmatrix} s E - (A + BF) \\ G + BH \end{bmatrix} = n.
\]

Partition

\[
FV =: \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \end{bmatrix}, \quad KV =: \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \end{bmatrix}.
\]
To prove (34), note that $E_{11}$, $G_2$ and $B_3$ are of full row rank, so there exists an orthogonal matrix $\hat{P}_1$ such that in

$$
\begin{bmatrix}
\hat{t}_1 \\
\hat{t}_2 \\
\hat{t}_3
\end{bmatrix}
\begin{bmatrix}
sE_{11} - \hat{A}_{11} \\
-\hat{A}_{21} \\
-\hat{A}_{31}
\end{bmatrix}
= \hat{P}_1^T
\begin{bmatrix}
s(E_{11} + B_1K_1) - A_{11} \\
-A_{21} \\
s(B_3K_1) - A_{31}
\end{bmatrix},
$$

$$
\begin{bmatrix}
\hat{t}_1 \\
\hat{t}_2 \\
\hat{t}_3
\end{bmatrix}
\begin{bmatrix}
\hat{B}_1 \\
0 \\
\hat{B}_3
\end{bmatrix}
= \hat{P}_1^T
\begin{bmatrix}
B_1 \\
0 \\
B_3
\end{bmatrix},
\begin{bmatrix}
\hat{t}_1 \\
\hat{t}_2 \\
\hat{t}_3
\end{bmatrix}
\begin{bmatrix}
\hat{G}_1 \\
\hat{G}_2 \\
\hat{G}_3
\end{bmatrix}
= \hat{P}_1^T
\begin{bmatrix}
G_1 \\
G_2 \\
G_3
\end{bmatrix},
$$

$\hat{E}_{11}$, $\hat{G}_2$ and $\hat{B}_3$ have full row rank.

If we set $\hat{P} := \begin{bmatrix} \hat{P}_1 & I \end{bmatrix} \in \mathbb{R}^{n \times n}$, then we obtain that $\hat{P}^T U (s(E + BK) - A)V, \hat{P}^T UB, \hat{P}^T UG, CV$ are in the condensed form (31). Since there exist $F \in \mathbb{R}^{n \times n}$ and $K \in \mathbb{R}^{m \times p}$ such that $(E + BK, A + BF)$ is regular, stable and the disturbances are decoupled, it follows from part (a) that

$$
\text{rank}
\begin{bmatrix}
sE_{11} - A_{11} & B_1 \\
-A_{21} & 0 \\
-\hat{A}_{31} & B_3
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
s\hat{E}_{11} - \hat{A}_{11} & \hat{B}_1 \\
-\hat{A}_{21} & 0 \\
-\hat{A}_{31} & \hat{B}_3
\end{bmatrix}
= \hat{t}_1 + \hat{t}_2 + \hat{t}_3 = \hat{n}_1 + \hat{n}_2 + \hat{n}_3
$$

for all $s \in \mathbb{C}^+$, which is condition (34). Moreover, by part (a) we also have

$$
\text{rank}
\begin{bmatrix}
E_{11} + B_1K_1 & A_{11}S & B_1 \\
0 & A_{21}S & 0 \\
B_3K_1 & A_{31}S & B_3
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
\hat{E}_{11} & \hat{A}_{11}S_{\infty} & \hat{B}_1 \\
0 & \hat{A}_{21}S_{\infty} & 0 \\
0 & \hat{A}_{31}S_{\infty} & \hat{B}_3
\end{bmatrix}
= \hat{t}_1 + \hat{t}_2 + \hat{t}_3 = \hat{n}_1 + \hat{n}_2 + \hat{n}_3
$$

with $\hat{S} = S_{\infty} \left( \begin{bmatrix} E_{11} + B_1K_1 \\
0 \\
B_3K_1 \end{bmatrix} \right)$. Thus we have that $T_{\infty}^T \left( \begin{bmatrix} E_{11} & 0 & 0 \\
0 & A_{21} & 0 \\
0 & 0 & A_{31} \end{bmatrix} \right) \hat{S}$ is of full row rank and hence also $W_2$.

**Sufficiency:** Since (6) and (7) hold, it follows by Lemma 3 (vii) that there exist $F_0, K_0$ such that $(E + BK_0, A + BF_0)$ is regular and of index at most one. Hence

$$
\text{det}(U(s(E + BK_0) - (A + BF_0)V) \neq 0,
\text{deg} \left( \text{det}(U(s(E + BK_0) - (A + BF_0)V) \right) = \text{rank}(U(E + BK_0)).
$$

Similar to Lemma 8, a direct calculation yields that

$$
\text{rank}
\begin{bmatrix}
E_{22} & E_{23} \\
E_{42} & E_{43} \\
0 & E_{53}
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
E_{22} & E_{23} & E_{24} \\
E_{42} & E_{43} & E_{44} \\
0 & E_{53} & E_{54}
\end{bmatrix},
\hat{n}_6 = n_4, \text{ } E_{64} = 0, \text{ } A_{64} \text{ is nonsingular.}
$$

(46)
Note $E_{11}, G_2$ and $B_3$ are of full row rank, there exists an orthogonal matrix $P_1 \in \mathbb{R}^{n \times n}$, such that

$$
P_1^T \begin{bmatrix}
s E_{11} - A_{11} & s E_{12} - A_{12} & s E_{13} - A_{13} & s E_{14} - A_{14} & B_1 & G_1 \\
- A_{21} & s E_{22} - A_{22} & s E_{23} - A_{23} & s E_{24} - A_{24} & 0 & G_2 \\
- A_{31} & s E_{32} - A_{32} & s E_{33} - A_{33} & s E_{34} - A_{34} & B_3 & G_3 \\
\end{bmatrix}
$$

with $\tilde{E}_{11}$, and $\tilde{B}_3$ full row rank. Then (37) implies that $t_1 + \tilde{n}_2 \leq n_1$. Furthermore, by (46) we have $n_2 + n_3 \leq t_3 + n_2 + \tilde{n}_2$. Note that $E_{42}$ is nonsingular and $E_{53}$ is full row rank, thus, as in the construction given Theorem 2.4 of [10], there exists a matrix $X \in \mathbb{R}^{t \times n_2}$ such that

$$
\text{rank} \begin{bmatrix}
X E_{22} & X E_{23} \\
E_{42} & E_{43} \\
0 & E_{53}
\end{bmatrix} = \text{rank} \begin{bmatrix}
E_{22} & E_{23} \\
E_{42} & E_{43} \\
0 & E_{53}
\end{bmatrix}.
$$

With this $X$, by (46) and (48) we have

$$
\text{rank} \begin{bmatrix}
\tilde{E}_{11} & \tilde{E}_{12} & \tilde{E}_{13} & \tilde{E}_{14} \\
0 & E_{22} & E_{23} & E_{24} \\
0 & X E_{22} & X E_{23} & X E_{24} \\
0 & E_{42} & E_{43} & E_{44} \\
0 & 0 & E_{53} & E_{54}
\end{bmatrix} = \text{rank}(\tilde{E}_{11}) + \text{rank} \begin{bmatrix}
X E_{22} & X E_{23} \\
E_{42} & E_{43} \\
0 & E_{53}
\end{bmatrix}.
$$

Determine $K := \begin{bmatrix} K_1 & K_2 & K_3 & K_4 \end{bmatrix}$ from

$$
\begin{bmatrix}
\tilde{E}_{31} & \tilde{E}_{32} & \tilde{E}_{33} & \tilde{E}_{34}
\end{bmatrix} + \begin{bmatrix}
\tilde{E}_{11} & \tilde{E}_{12} & \tilde{E}_{13} & \tilde{E}_{14}
\end{bmatrix} = 0,
$$

then, since $W_2$ has full rank if and only $\text{rank}(\tilde{A}_{11} S_{1 \infty}(\tilde{E}_{11})) = \tilde{n}_2$, and since we have already shown that (46) and (49) hold, similar to the sufficiency in part (a), we obtain the desired feedback matrices $F$ and $H$. See [9] for details. \(\square\)

**Remark 5** If the index one condition is not required, then in (a) necessary and sufficient conditions are given by (7), (34), (36) and in (b) by (7), (34) and (37). The proof of these conditions and the construction of feedbacks that regularize the system without achieving index at most one are discussed in detail in [9].

**Remark 6** We can extend these results to systems that include a feedthrough term, which arises frequently in $H_\infty$ and LQG control, see [26], i.e., systems of the form

$$
\begin{align*}
E \ddot{x}(t) &= Ax(t) + Bu(t) + Gq(t); \ x(t_0) = x_0, \ t \geq t_0 \\
y(t) &= Cx(t) + Du(t),
\end{align*}
$$

where $E, A, B, G, C$ are as in (1) and $D \in \mathbb{R}^{p \times m}$. 24
Using the singular value decomposition of $D$, e.g., [15]

$$P^T DW = \begin{bmatrix} r_d & m - r_d \\ q - r_d & \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with $P, W$ orthogonal and $D_1$ of size $r_d \times r_d$ and nonsingular, we can set

$$BW =: \begin{bmatrix} r_d & m - r_d \\ B_1 & B_2 \end{bmatrix}, \quad P^T C =: \begin{bmatrix} r_d & 0 \\ q - r_d & \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad W^T u =: \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad P^T y =: \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and furthermore for $F \in \mathbb{R}^{m \times n}$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} := W^T F - \begin{bmatrix} D_1^{-1} C_1 \\ 0 \end{bmatrix}.$$  

Then for the solution of the disturbance decoupling problem it suffices to perform the analysis of the augmented system

$$\begin{align*}
\dot{\hat{x}} &= \hat{A} \hat{x} + \hat{B} \hat{u} + \hat{G} q \\
\dot{y} &= \hat{C} \hat{x},
\end{align*}$$

where

$$\begin{align*}
\hat{E} &:= \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{A} := \begin{bmatrix} A & B_1 \\ C_1 & D_1 \end{bmatrix} \in \mathbb{R}^\hat{n} \times \hat{n}, \\
\hat{B} &:= \begin{bmatrix} B_2 \\ 0 \end{bmatrix} \in \mathbb{R}^\hat{n} \times \hat{m}, \quad \hat{C} := \begin{bmatrix} C_2 & 0 \end{bmatrix} \in \mathbb{R}^\hat{p} \times \hat{n}, \quad \hat{G} := \begin{bmatrix} G \\ 0 \end{bmatrix} \in \mathbb{R}^\hat{p} \times \hat{d}
\end{align*}$$

and $\hat{n} := n + r_d, \hat{m} := m - r_d, \hat{p} := p - r_d$. For this system the results from the previous section apply and hence the case of systems with feedthrough can be reduced to the previous results. See [9] for more details.

5 Conclusions

In this paper we have studied the disturbance decoupling problem with/without stability for descriptor systems. We have given necessary and sufficient conditions for solving this problem and at the same time ensuring that the resulting closed-loop system is regular and has index at most one. The proofs are constructive, based on condensed forms that can be computed via orthogonal matrix transformations and can be implemented as numerically stable procedures.

References


**Appendix A Constructive Proof of Theorem 6.**

In the following algorithm we need row compressions, column compressions or simultaneous row and column compressions of matrices. Such compressions can be obtained in the usual way via QR-factorizations, rank revealing QR-factorizations, URV-decompositions or singular value decompositions, see [15, 2].

We also need the computation of generalized upper triangular forms which can be obtained via the LAPACK routine DGGBAK from LAPACK [2]:

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Algorithm 1

Input: Matrices $E$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $G \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{q \times n}$.


Step 1: Perform a row compression and a column compression such that

$$UB =: \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}, \quad UG =: \begin{bmatrix} G_1 \\ 0 \\ 0 \end{bmatrix}, \quad CV =: \begin{bmatrix} 0 \\ C_3 \end{bmatrix}$$

with $G_1, B_2$ of full row rank and $C_3$ of full column rank. Set

$$U(sE - A)V =: \begin{bmatrix} sE_{11} - A_{11} & sE_{13} - A_{13} \\ sE_{21} - A_{21} & sE_{23} - A_{23} \\ sE_{31} - A_{31} & sE_{33} - A_{33} \end{bmatrix}.$$

Step 2: Compute the generalized upper triangular form of $(E_{31}, A_{31})$,

$$U_2(sE_{31} - A_{31})V_2 =: \begin{bmatrix} sE_{31} - A_{31} & sE_{32} - A_{32} \\ 0 & sE_{42} - A_{42} \end{bmatrix},$$

with $E_{31}$ of full row rank and $sE_{42} - A_{42}$ of full column rank for any $s \in C$. Set

$$V_2 =: \begin{bmatrix} sE_{11} - A_{11} & sE_{12} - A_{12} \\ sE_{21} - A_{21} & sE_{22} - A_{22} \end{bmatrix}, \quad U_2(sE_{33} - A_{33}) =: \begin{bmatrix} sE_{33} - A_{33} \\ sE_{43} - A_{43} \end{bmatrix}.$$

Step 3: Compute the generalized upper triangular form of $(E_{42}, A_{43})$,

$$U_3\begin{bmatrix} sE_{42} - A_{42} & sE_{43} - A_{43} \end{bmatrix}V_3 =: \begin{bmatrix} sE_{42} - A_{42} & sE_{43} - A_{43} \\ 0 & sE_{53} - A_{53} \end{bmatrix},$$

with $E_{42}$ of full row rank and $sE_{53} - A_{53}$ of full column rank for any $s \in C$. Set

$$V_3 =: \begin{bmatrix} sE_{12} - A_{12} & sE_{13} - A_{13} \\ sE_{22} - A_{22} & sE_{23} - A_{23} \\ sE_{32} - A_{32} & sE_{33} - A_{33} \end{bmatrix}$$

and $\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} =: \begin{bmatrix} 0 \\ C_3 \end{bmatrix}V_3$.

Step 4: Perform a row compression:

$$U_4\begin{bmatrix} sE_{21} - A_{21} \\ sE_{31} - A_{31} \end{bmatrix} =: \begin{bmatrix} sE_{21} - A_{21} \\ -A_{31} \end{bmatrix},$$

with $E_{21}$ of full row rank. Set

$$U_4\begin{bmatrix} sE_{22} - A_{22} & sE_{23} - A_{23} \\ sE_{32} - A_{32} & sE_{33} - A_{33} \end{bmatrix} := U_4\begin{bmatrix} sE_{22} - A_{22} & sE_{23} - A_{23} \\ sE_{32} - A_{32} & sE_{33} - A_{33} \end{bmatrix}, \quad \begin{bmatrix} B_2 \\ B_3 \end{bmatrix} := U_4\begin{bmatrix} B_2 \\ 0 \end{bmatrix}$$

and

$$U := \begin{bmatrix} I \\ U_4 \\ I \end{bmatrix}, \quad U_3 \begin{bmatrix} I \\ U_3 \end{bmatrix}U_1, \quad V := V_4\begin{bmatrix} V_2 \\ I \end{bmatrix}V_3.$$
Appendix B  Proof of Lemma 7.

Let \( \bar{U} \) be an orthogonal matrix such that \( \bar{U} \begin{bmatrix} B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} l_2 \\ \bar{n}_2 + \bar{n}_3 \end{bmatrix} - l_2 \begin{bmatrix} \bar{B}_2 \\ 0 \end{bmatrix} \) with \( \bar{B}_2 \) of full row rank. Set

\[
\tilde{U} = \begin{bmatrix} sE_{21} - A_{21} & sE_{22} - A_{22} & sE_{23} - A_{23} \\ -A_{31} & sE_{32} - A_{32} & sE_{33} - A_{33} \end{bmatrix} = \begin{bmatrix} l_2 \\ \bar{n}_2 + \bar{n}_3 \end{bmatrix} - l_2 \begin{bmatrix} \tilde{E}_{21} - A_{21} \\ \tilde{E}_{31} - A_{31} \end{bmatrix}
\]

and partition \( \hat{U} = \begin{bmatrix} l_2 \\ \bar{n}_2 + \bar{n}_3 \end{bmatrix} - l_2 \begin{bmatrix} \hat{U}_2 \\ \hat{U}_3 \end{bmatrix} \). Then

\[
\begin{bmatrix} \tilde{E}_{21} - A_{21} \\ \tilde{E}_{31} - A_{31} \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \begin{bmatrix} n_3 \\ 0 \end{bmatrix} + \tilde{U}_2 \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}
\]

and \( \tilde{U}_2 \) is of full column rank and \( G_1, E_{21} \) and \( B_3 \) are of full row rank for all \( s \in C \), we have that \( \bar{E}_{31} \) is also of full row rank, and

\[
\Pi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{n_4 + \bar{n}_5 + q} & 0 \\ 0 & 0 & I_{n_4 + \bar{n}_5 + q} \end{bmatrix} \in \mathbb{R}^{(n + q) \times (n + q - l_2)}, \quad \Pi - \Pi \end{bmatrix} = \begin{bmatrix} W_{\pi}^T & 0 \\ 0 & I_{n_4 + \bar{n}_5 + q} \end{bmatrix} \in \mathbb{R}^{(n + q) \times (n + q)},
\]

\[
A_r = \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} W_r \in \mathbb{R}^{n \times n_1}, \quad A_l = \begin{bmatrix} I_{n_2 + \bar{n}_5 - l_2} \\ 0 \end{bmatrix} W_{l} \in \mathbb{R}^{(n + q - l_2) \times (n_2 + \bar{n}_5 - l_2)},
\]

\[
A_l = \begin{bmatrix} W_{\pi}^T \\ I_{n_4 + \bar{n}_5 + q} \\ I_{n_4 + \bar{n}_5 + q} \end{bmatrix} = \begin{bmatrix} I_{p + l_2} \\ 0 \end{bmatrix} W_{l} \in \mathbb{R}^{(n + q + \bar{n}_5 + q)},
\]

where \( W_{\pi}, W_r, W_{l} \) are orthogonal. If we set \( W_{l} = \begin{bmatrix} I_{p + l_2} \\ -W_{l} \end{bmatrix} W_{\pi} \), then we obtain

\[
A_1 = W_{l} \begin{bmatrix} E_{11} \\ E_{21} \\ 0 \end{bmatrix} W_r, \quad A_2 = W_{l} \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix} W_r, \quad A_3 = W_{l} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}.
\]

Note that

\[
T_{\infty}(W_{l}) = \begin{bmatrix} E_{11}W_r B_1 \\ B_2 \\ 0 \end{bmatrix} W_{l} = \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix} S_{\infty}(W_{l}) = \begin{bmatrix} E_{11} \\ E_{21} \\ 0 \end{bmatrix} S_{\infty}(W_{l}),
\]

\[
T_{\infty}(W_{l}) = \begin{bmatrix} E_{11} \Phi_{32} \\ E_{21} \Phi_{32} \\ 0 \end{bmatrix} W_{l} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} W_{l} = \begin{bmatrix} \Phi_{32} \\ \Phi_{32} \end{bmatrix} W_{l}.
\]

Since \( B_3 \) is of full row rank, there exists an orthogonal matrix \( \tilde{V} \), such that \( \begin{bmatrix} B_3 \\ A_{31} \end{bmatrix} \tilde{V} = \begin{bmatrix} 0 \\ \Phi_{32} \end{bmatrix} \) with \( \Phi_{32} \) nonsingular. Set (with conformal partitioning) \( \begin{bmatrix} B_2 \\ A_{21} \end{bmatrix} \tilde{V} = \begin{bmatrix} \Phi_{21} \\ \Phi_{22} \end{bmatrix} \),
\[
\begin{bmatrix}
0 & E_{21} \\
\Theta_{21} & \Theta_{22}
\end{bmatrix}
\text{and}
\hat{V} :=
\begin{bmatrix}
\hat{V} \\
I
\end{bmatrix}
\in \mathbb{R}^{(n+m) \times (n+m)}. \text{ Then, we obtain that}
\[
\begin{bmatrix}
-\Psi B & \Psi(sE - A) \\
0 & C
\end{bmatrix}
\hat{V} =
\begin{bmatrix}
s\Theta_{21} - \Phi_{21} & s\Theta_{22} - \Phi_{22} & sE_{22} - A_{22} & sE_{23} - A_{23} \\
0 & -\Phi_{32} & sE_{32} - A_{32} & sE_{33} - A_{33} \\
0 & 0 & sE_{42} - A_{42} & sE_{43} - A_{43} \\
0 & 0 & 0 & sE_{53} - A_{53}
\end{bmatrix}.
\]

Since \(E_{21}, B_2\) are of full row rank and \(\Phi_{32}\) is nonsingular, it follows that \(\Theta_{21}\) is also of full row rank. Note that
\[
\begin{bmatrix}
-\Phi_{32} & sE_{42} - A_{42} & sE_{43} - A_{43} \\
0 & 0 & sE_{53} - A_{53} \\
0 & C_2 & C_3
\end{bmatrix}
\text{is of full column rank for any } s \in \mathbb{C}.
\]

Hence,
\[
\text{rank}(A_4) = \text{rank}
\begin{bmatrix}
0 & E_{32} & E_{33} \\
0 & E_{42} & E_{43} \\
0 & 0 & E_{53}
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
E_{32} & E_{33} \\
E_{42} & E_{43} \\
0 & E_{53}
\end{bmatrix}.
\]

Since
\[
\begin{bmatrix}
\Psi(sB) & \Psi(sE - A) \\
0 & -C
\end{bmatrix}
= \begin{bmatrix}
sB_2 & sE_{21} - A_{21} & sE_{22} - A_{22} & sE_{23} - A_{23} \\
0 & sB_3 & -A_{31} & sE_{32} - A_{32} & sE_{33} - A_{33} \\
0 & 0 & sE_{42} - A_{42} & sE_{43} - A_{43} \\
0 & 0 & 0 & sE_{53} - A_{53} \\
0 & 0 & -C_2 & -C_3
\end{bmatrix},
\]

\[
\begin{bmatrix}
B_2 & E_{21} \\
B_3 & 0
\end{bmatrix}
\text{are of full row rank and}
\begin{bmatrix}
sE_{42} - A_{42} & sE_{43} - A_{43} \\
0 & sE_{53} - A_{53} \\
-C_2 & -C_3
\end{bmatrix}
\text{is of full column rank for all } s \in \mathbb{C}. \quad \square