Edge bounds in nonhamiltonian $k$-connected graphs

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Abstract

Let $G$ be a $k$-connected graph of order $n$ with $|E(G)| > (n-k) + k^2$. Then for ($k = 1$, $n \geq 3$), ($k = 2$, $n \geq 10$), and ($k = 3$, $n \geq 16$), $G$ is hamiltonian. The bounds are tight and for ($k = 1$, $n \geq 12$), and ($k = 3$, $n \geq 18$) the extremal graphs are unique. A general bound will also be given for the number of edges in a nonhamiltonian $k$-connected graph, but the bound is not tight.

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1. Introduction

We begin with some standard definitions and notation. For the purposes of this paper, $G$ will represent a simple, undirected graph of order $n$. The complement of $G$ will be denoted by $\overline{G}$ and the edge and vertex sets of $G$ by $E(G)$ and $V(G)$, respectively. For $I \subset V(G)$, $G - I$ represents the subgraph of $G$ induced on $V(G) - I$. We say $G$ is $k$-connected ($k \geq 1$) if $G$ is connected and deleting any $k - 1$ vertices (and incident edges) results in a connected graph; we say $G$ has connectivity $k$ if it is $k$-connected but is not $(k + 1)$-connected. The degree of a vertex $x$ will be denoted $d_G(x)$, or simply $d(x)$ if the context is clear. The neighborhood of a vertex $x$ consists of all vertices adjacent to $x$ and will be denoted $N(x)$.

The independence number of $G$ is the size of the largest independent (mutually nonadjacent) set of vertices and will be denoted by $\alpha(G)$, while $\sigma_k(G) = \min\left\{\sum_{i=1}^{k} d(x_i) : \{x_1, x_2, \ldots, x_k\} \text{ is an independent set of } k \text{ vertices in } G\right\}$. Finally, $G$ is hamiltonian if it contains a cycle that uses each vertex exactly once.

We now give some preliminary results. The following well-known theorem is due to Ore, and provided the motivation for much subsequent work in hamiltonian theory.

**Theorem 1 (Ore [8]).** If $G$ is a graph of order $n$ with $\sigma_2(G) \geq n$, then $G$ is hamiltonian.

Many generalizations of Ore’s Theorem followed, leading to various sufficient conditions under which a graph would be hamiltonian. Two excellent hamiltonian survey articles written by Ron Gould summarize much of the work that has been done to date (see [5,6]).

The purpose of this article is to provide an upper bound on the number of edges in a $k$-connected nonhamiltonian graph. The bounds will be sharp for $k \leq 3$ and $k = (n - 1)/2$. The next two theorems give sufficient conditions on $\sigma_k(G)$ for which $G$ will be hamiltonian, and they will be essential for our purposes.

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Theorem 2 (Bauer et al. [1]). If $G$ is a 2-connected graph of order $n$ and connectivity $c$ such that $\sigma_3(G) \geq n + c$, then $G$ is hamiltonian.

Theorem 3 (Harkat-Benhamdine et al. [7]). Let $G$ be a 3-connected graph of order $n$ and independence number $\alpha$. If $\sigma_4(G) \geq n + 2\alpha - 2$, then $G$ is hamiltonian.

2. Main results

We begin with the following simple lemma, which will be used quite often throughout the paper.

Lemma 4. Let $G$ be a nonhamiltonian, $k$-connected graph of order $n$. Then $k \leq (n - 1)/2$ and $|E(G)| \geq \left(\frac{k+1}{2}\right) + (k+1)(n-k-1) - \sigma_{k+1}(G)$.

Proof. By a result of Chvátal and Erdős [4], $k$-connected nonhamiltonian graphs must contain a set $I = \{x_1, x_2, \ldots, x_{k+1}\}$ of $k+1$ independent vertices. Clearly, if the $n-(k+1)$ vertices in $G - I$ are deleted, then the resulting graph is disconnected. Since $G$ is $k$-connected, it follows that $n-(k+1) \geq k-1$, so $k < n/2$.

Now, suppose $I$ is chosen so that $\sum_{i=1}^{k+1} d(x_i) = \sigma_{k+1}(G)$. Let $G_I$ represent the edges in $G$ that are incident with at least one vertex of $I$. Then $G_I$ contains $\left(\frac{k+1}{2}\right)$ edges with both endpoints in $I$ and $\sum_{i=1}^{k+1} (n-1-k-d_G(x_i))$ edges with exactly one endpoint in $I$. It follows that $|E(G)| \geq |G_I| = \left(\frac{k+1}{2}\right) + (k+1)(n-1-k) - \sigma_{k+1}(G)$.

The following result is a simple consequence of the previous lemma and was certainly known by Ore.

Theorem 5. Let $G$ be a graph of order $n \geq 3$. If $|E(G)| > \left(\frac{n-1}{2}\right) + 1$, then $G$ is hamiltonian. This result is best possible and the extremal graph is unique.

Proof. Assume $G$ is not hamiltonian. Using the converse of Theorem 1, $\sigma_2(G) \leq n-1$. Since $|E(G)| > \left(\frac{n-1}{2}\right) + 1$, $G$ is connected. Then, using $k = 1$ in Lemma 4, $|E(G)| \geq 1 + 2(n-2) - (n-1) = n - 2$. It follows that $E(G) \leq \left(\frac{n}{2}\right) - (n-2) = \left(\frac{n-1}{2}\right) + 1$.

To see that the bound is tight, note that the graph formed by joining one vertex of $K_{n-1}$ to one other vertex contains $\left(\frac{n-1}{2}\right) + 1$ edges and is nonhamiltonian. Equality on the bound for $|E(G)|$ occurs only if there exist nonadjacent vertices $x$ and $y$ such that each of the $n-2$ edges in $G$ is incident with $x$ or $y$ and $d_G(x) + d_G(y) = n-1$. Now, if $d(x) \geq 2$ and $d(y) \geq 2$, then the graph is hamiltonian; therefore, assume $d(x) = 1$ and $d(y) = n-2$. Since all other vertices are mutually adjacent, the extremal graph must be the one just described.

A hamiltonian-connected graph is one in which each pair of vertices is joined by a hamiltonian path. Ore proved in [9] that if $G$ is a graph of order $n$ with $\sigma_2(G) \geq n + 1$, then $G$ is hamiltonian-connected. Using a proof similar to the one given for Theorem 5, he proved the following related result (although his description of the extremal graph was not given explicitly).

Corollary 6. Let $G$ be a graph of order $n \geq 7$. If $|E(G)| > \left(\frac{n-1}{2}\right) + 2$, then $G$ is hamiltonian-connected. The graph formed by joining two vertices of $K_{n-1}$ to one other vertex is the unique graph of order $n$ with $\left(\frac{n-1}{2}\right) + 2$ edges that is not hamiltonian-connected.

Essential to our work in the 3-connected case will be the concept of closure, due to Bondy and Chvátal [2]. Define the closure of $G$, denoted $cl(G)$, to be the graph obtained from $G$ by recursively joining two nonadjacent vertices with degree sum at least $n$. They proved that $cl(G)$ is well-defined (i.e., independent of the order in which the edges are added) and that $G$ is hamiltonian if and only if $cl(G)$ is hamiltonian. Their result yields the following immediate corollary.

Corollary 7. Suppose \( cl(G) = G \) for a nonhamiltonian graph \( G \) of order \( n \). Then \( d(x) + d(y) \leq n - 1 \) for any pair \( \{x, y\} \) of nonadjacent vertices.

The following theorem relates the number of edges in a closed nonhamiltonian graph with the independence number of the graph.

Theorem 8. Suppose \( G = cl(G) \) for a nonhamiltonian graph \( G \) of order \( n \), and \( m \leq \alpha(G) \). Then

\[
|E(\overline{G})| \geq \begin{cases} 
\frac{m}{2}(n-m) & \text{for } n \text{ odd,} \\
\frac{m}{2}(n-m) + \frac{m}{2} - 1 & \text{for } n \text{ even.}
\end{cases}
\]

Proof. Let \( I = \{x_1, x_2, \ldots, x_m\} \) be a set of independent vertices. Using the same argument as in the proof of Lemma 4, but replacing \( k + 1 \) with \( m \), we obtain

\[
|E(\overline{G})| \geq |\overline{G}_I| = \left(\frac{m}{2}\right) + m(n-m) - \sum_{i=1}^{m} d(x_i).
\]

From Corollary 7, \( d(x_i) + d(x_j) \leq n - 1 \) for \( i \neq j \). When \( n \) is odd, it follows that \( \sum_{i=1}^{m} d(x_i) \) is maximized when \( d(x_i) = (n-1)/2 \) for \( 1 \leq i \leq m \), and when \( n \) is even, \( \sum_{i=1}^{m} d(x_i) \) is maximized when \( d(x_1) = n/2 \) and \( d(x_i) = n/2 - 1 \) for \( 2 \leq i \leq m \). Therefore, for odd \( n \),

\[
|E(\overline{G})| \geq \left(\frac{m}{2}\right) + m(n-m) - m \left(\frac{n-1}{2}\right) = \frac{m}{2}(n-m),
\]

and for even \( n \),

\[
|E(\overline{G})| \geq \left(\frac{m}{2}\right) + m(n-m) - \left(\frac{n}{2} + \frac{(m-1)(n-2)}{2}\right) = \frac{m}{2}(n-m) + \frac{m}{2} - 1. \quad \square
\]

The above theorem leads to the following general bound on the maximum number of edges in a \( k \)-connected nonhamiltonian graph.

Theorem 9. Let \( G \) be a \( k \)-connected graph of order \( n \). If \( |E(G)| > \left(\frac{n}{2}\right) - (k+1)(n-k-1)/2 \), then \( G \) is hamiltonian.

Proof. Assume that \( G \) is nonhamiltonian and let \( H = cl(G) \); then \( H \) is \( k \)-connected and nonhamiltonian. By the Chvátal and Erdős result \([4]\), \( \alpha(H) \geq k + 1 \). Using Theorem 8, \( |E(H)| \geq (k+1)(n-k-1)/2 \), and it follows that \( |E(G)| \leq |E(H)| \leq \left(\frac{n}{2}\right) - (k+1)(n-k-1)/2 \). \( \square \)

We now present the main result of the paper.

Theorem 10. Let \( G \) be a 3-connected graph of order \( n \geq 16 \). If \( |E(G)| > \left(\frac{n-3}{2}\right) + 9 \), then \( G \) is hamiltonian. This result is the best possible, and for \( n \geq 18 \) the extremal graph is unique.

Proof. Assume \( G \) is nonhamiltonian. By the comments preceding Corollary 7, we may assume that \( G = cl(G) \), in which case \( d(x) + d(y) \leq n - 1 \) for any two nonadjacent vertices \( x \) and \( y \). It suffices to prove that \( |E(\overline{G})| \geq \left(\frac{n}{2}\right) - \left(\left(\frac{n-3}{2}\right) + 9\right) = 3n - 15. \)

Note first that if \( \sigma_4(G) \leq n + 5 \), by Lemma 4,

\[
|E(\overline{G})| \geq |\overline{G}_I| \geq 6 + 4(n-4) - (n+5) = 3n - 15,
\]

as desired. We now assume \( \sigma_4(G) \geq n + 6 \) and consider two cases, depending on the relationship between \( n \) and \( \alpha. \)
Case 1: Assume \( n > (8x + 10)/3 \).

Let \( I = \{x_1, x_2, x_3, x_4\} \) be a set of independent vertices satisfying \( \sum_{i=1}^{4} d(x_i) = \sigma_4(G) \), and assume without loss of generality that \( d(x_1) \geq \sigma_4(G)/4 \geq n + \frac{x}{7} \). We will consider two subcases, depending on the degree of \( x_1 \).

Subcase 1a: Suppose \( d(x_1) \geq n - 6 \). Choose vertex \( v \in V(G) - I - N(x_1) \). Such a vertex must exist, because if \( V(G) = I \cup N(x_1) \), then \( d(x_1) = n - 4 \); since \( d(x_1) + d(x_i) \leq n - 1 \) for \( 2 \leq i \leq 4 \), it follows that \( d(x_i) \leq 3 \) for \( 2 \leq i \leq 4 \), which contradicts \( \sigma_4(G) \geq n + 6 \). Notice then that \( d_G(v) = n - 1 - d_G(v) \geq d_G(x_1) \geq n - 6 \). Therefore, \( G \) contains at least \( n - 6 - |I| = n - 10 \) edges with neither endpoint in \( I \). Counting again as we did in the proof of Lemma 4, but including the extra edges in \( G \) incident with \( v \) and substituting for \( \sigma_4(G) \) using the bound in Theorem 3, we obtain

\[
|E(G)| \geq \left(4 \over 2\right) + 4(n - 4) + (n - 10) - \sigma_4(G) \\
\geq 5n - 20 - (n + 2x - 3) \\
\geq (3n - 15) + \frac{8x + 10}{3} - (2x + 2) \\
= (3n - 15) + \frac{2x - 4}{3} > 3n - 15, \text{ as desired.}
\]

Subcase 1b: Suppose next that \( d(x_1) \leq n - 7 \). Then there exist distinct vertices \( v_1, v_2, v_3 \in V(G) - I - N(x_1) \), and \( G \) contains at least \( (d_G(v_1) - 4) + (d_G(v_2) - 5) + (d_G(v_3) - 6) \) edges with neither endpoint in \( I \). Since for each \( i, 1 \leq i \leq 3 \), \( d_G(v_i) \geq d_G(x_1) \geq (n + 6)/4 \), we obtain at least \( 3(n + 6)/4 - 15 \) edges in \( G \) of this type. Counting as we did in the previous case, we obtain

\[
|E(G)| \geq \left(4 \over 2\right) + 4(n - 4) + \frac{3(n + 6)}{4} - 15 - (n + 2x - 3) \\
= (3n - 15) + \frac{3}{4}n - 2x - 5 \\
> (3n - 15) + \frac{3}{4} \left(8x + 10 \over 3\right) - 2x - 5 \over 2 \geq 3n - 15, \text{ as desired.}
\]

Case 2: Assume \( n \leq (8x + 10)/3 \).

In this case, \( x \geq (3n - 10)/8 \). By Theorem 8, \( |E(G)| \geq \frac{x}{2}(n - 2) \). For fixed \( n \) and \( (3n - 10)/8 \leq x \leq n - (3n - 10)/8 \), this function is minimized at \( x = (3n - 10)/8 \). Therefore, in this range,

\[
|E(G)| \geq \frac{x}{2}(n - 2) \geq \frac{1}{2} \left(3n - 10 \over 8\right) \left(5n + 10 \over 8\right) \\
= {15n^2 - 20n - 100 \over 128} \geq 3n - 15 \text{ for } n \geq 22.
\]

Technically speaking, in this last case we still need to consider the possibility that \( x > n - (3n - 10)/8 = (5n + 10)/8 \). In this case, however, \( x \) is so large compared to \( n \) that the \( \left(5n + 10 \over 8\right) \) edges in \( G \) induced on an independent (in \( G \)) set of \( x \) vertices will exceed \( 3n - 15 \) for all \( n \).

All cases have been considered and the proof on the edge bound is complete for \( n \geq 22 \). Note that equality will occur only when \( \sigma_4(G) = n + 5 \) and all edges in \( G \) are incident with at least one vertex of the independent set \( I = \{x_1, x_2, x_3, x_4\} \); the graph formed by joining each vertex in \( \{x_1, x_2, x_3\} \) with the same three vertices of \( K_{n-3} \) is a 3-connected nonhamiltonian graph of order \( n \) with these properties. To see that this is the only such graph for \( n \geq 18 \) and to prove the bound on \( |E(G)| \) is satisfied when \( 16 \leq n \leq 21 \), please refer to the technical details in Section 4. \( \square \)

Note that \( n \geq 16 \) is required for the bound to hold, because the graph \( H \) formed by joining 8 independent vertices to each vertex of \( K_7 \) is easily seen to be nonhamiltonian of order \( n = 15 \), but \( |E(H)| = 77 > \left(12 \over 2\right) + 9 \).

We now state and prove the analogous result for 2-connected graphs.

**Theorem 11.** Let \( G \) be a 2-connected graph of order \( n \geq 10 \). If \( |E(G)| > \left(n - \frac{2}{2}\right) + 4 \), then \( G \) is hamiltonian. This result is best possible and for \( n \geq 12 \), the extremal graph is unique.
Therefore, \( c/afii9846c \) uniquely determined to be the one described above. Furthermore, for \( k \geq 3 \), then \( dG(x) \) has degree at least 2.

**Proof.** Assume \( G \) is not hamiltonian and, as before, that \( cl(G) = G \). Consider first the case where \( G \) has connectivity \( c=2 \). In this case, \( x(G) \geq 3 \) and by Theorem 2, there exist independent vertices \( x, y, \) and \( z \) such that \( d(x) + d(y) + d(z) = \sigma_3(G) \leq n + c - 1 = n + 1 \). Counting again as in the proof of Lemma 4, we see that \( |E(G)| \geq 3 + 3(n - 3) - (n + 1) = 2n - 7 \). It follows that \( |E(G)| \leq \binom{n}{2} - (2n - 7) = \binom{n-2}{2} + 4 \), as desired.

Now assume that \( G \) has connectivity \( c > 2 \), i.e. \( G \) is 3-connected. It again suffices to prove that \( |E(G)| \geq 2n - 7 \).

**Case 1:** Assume \( \sigma_4(G) \leq 2n - 3 \). Using Lemma 4 yet again, we obtain \( |E(G)| \geq 6 + 4(n - 4) = 2n - 7 \).

**Case 2:** Assume \( \sigma_4(G) \geq 2n - 2 \). Since we have assumed \( G \) is nonhamiltonian, from Theorem 3, \( \sigma_4(G) < n + 2x - 2 \). Therefore, \( 2n - 2 < n + 2x - 2 \), whence \( x > n/2 \). Since \( G \) contains at least \( x \) independent vertices, \( |E(G)| \geq \binom{x}{2} \geq 2n - 7 \), for \( n \geq 10 \). (One can check that the only instance in which equality is actually attained is when \( n = 11 \) and \( x = 5 \).)

To see that the bound is tight, note that the graph formed by joining two vertices of \( K_{n-2} \) to two other independent vertices contains \( \binom{n-2}{2} + 4 \) edges and is nonhamiltonian. We next show why for \( n \geq 12 \) this graph is the only one meeting these criteria.

When \( G \) has connectivity \( c = 2 \), equality in the bound on \( |E(G)| \) will hold if and only if all edges in \( G \) are incident with a vertex of the independent set \( \{x, y, z\} \) and \( dG(x) + dG(y) + dG(z) = n + 1 \). So, \( G \) can be formed by adding \( n + 1 \) edges between the graph \( K_{n-3} \) and three independent vertices \( x, y, \) and \( z \). Since \( G \) is 2-connected, each vertex has degree at least 2. Furthermore, for \( n \geq 7 \), it is easily seen that if two or more vertices in \( \{x, y, z\} \) have degree at least 3, then \( G \) is hamiltonian. Therefore, for \( n \geq 7 \), we may assume that \( dG(x) = dG(y) = 3 \) and \( dG(z) = n - 3 \); in this case, it is apparent that if \( N(x) \neq N(y) \), then \( G \) will be hamiltonian. Therefore, \( N(x) = N(y) \), and the extremal graph is uniquely determined to be the one described above.

For connectivity \( c \geq 3 \), we maintain that equality cannot occur in the edge bounds for \( |E(G)| \) in cases 1 and 2 for \( n \geq 12 \). For if it did, then in the proof of Lemma 4, the set \( I = \{x_1, x_2, x_3, x_4\} \) of independent vertices in \( G \) will have degree sum \( 2n - 3 \); furthermore, in order that \( |E(G)| = |G\ell| \), the subgraph \( G - I \) must be a clique on \( n - 4 \) vertices. Now, since \( G \) is 3-connected and \( I \) is an independent set, \( 3 \leq d(x_i) \leq n - 4 \) for \( 1 \leq i \leq 4 \). Putting all these facts together, we will see that \( G \) must be hamiltonian. Note
d\( (x_1) + d(x_2) + d(x_3) + (n - 4) \geq \sum_{i=1}^{4} d(x_i) = 2n - 3 \)
from which it follows that \( d(x_1) + d(x_2) + d(x_3) \geq n + 1 \). By labeling the vertices appropriately, we may assume \( d(x_3) \geq \binom{3+1}{2} \geq 5 \) and \( 3 \leq d(x_1) \leq d(x_2) \leq 5 \leq d(x_3) \leq d(x_4) \). From this it is clear that the vertices of \( I \) each have a sufficient number of neighbors in the clique \( G - I \) in order to create a hamiltonian cycle in \( G \). This is a contradiction, and it follows that for \( n \geq 12 \), the unique nonhamiltonian graph of order \( n \) containing \( \binom{n-2}{2} + 4 \) edges is the 2-connected one described previously, which completes the proof. \( \square \)

Note that the graph formed by joining 6 independent vertices to each vertex of \( K_3 \) and the graph formed by joining two vertices of \( K_2 \) to two other vertices are both 2-connected nonhamiltonian graphs of order 11 containing \( \binom{9}{2} + 4 \) edges.

### 3. Conclusion

We have given bounds on the maximum number of edges in a \( k \)-connected, nonhamiltonian graph of order \( n \); for sufficiently large \( n \), the bounds are tight for \( 1 \leq k \leq 3 \). For these \( k \)-values, the extremal graphs contain \( \binom{n-k}{2} + k^2 \) edges and are formed by joining \( k \) vertices of \( K_{n-k} \) to each of \( k \)-independent vertices. However, it is apparent that when \( k \) is relatively large compared to \( n \), the extremal graphs will not be of the type described. Specifically, let \( p = \lfloor (n - 1)/2 \rfloor \) and note that for \( k \leq p \) the graph \( H \) formed by joining \( n - p \) independent vertices to each vertex of \( K_p \) will be a \( k \)-connected nonhamiltonian graph of order \( n \). For \( k = p - 1 \) in particular (in fact, for \( (n + 1)/6 < k < p \)), this graph \( H \) will have more edges than the graphs described above. However, when \( k = p = (n - 1)/2 \) (the maximum possible value for \( k \) by Lemma 4), the two graphs described above are identical; moreover, they are extremal in that they meet the general edge bound given for nonhamiltonian connected graphs in Theorem 9.
Obviously, it is desirable to obtain tight bounds on the size of a \( k \)-connected nonhamiltonian graph for \( 3 < k < (n - 1)/2 \). Using our approach to do so would require a bound on \( \sigma_{k+1}(G) \) of the form \( n + f(k) \) for some “small” function \( f \).

4. Technical details

We now complete the proof of Theorem 10. In the first subsection we prove the theorem for small values of \( n \) and in the second subsection we prove the uniqueness of the extremal graphs.

4.1. Proof of theorem 10 for small values of \( n \)

We first assume \( G \) is a 3-connected nonhamiltonian graph of order \( 16 \leq n \leq 21 \) with \( z(G) \geq (3n - 10)/8 \). We will show that \( |E(G)| \geq 3n - 15 \), with strict inequality for \( n \geq 18 \).

Let \( I = \{ x_1, x_2, \ldots, x_d \} \) be a set of independent vertices in \( G \) and observe that \( d(x_i) \leq n - z \) for \( 1 \leq i \leq z \). Recall that we let \( \overline{G} \) represent the edges in \( G \) that are incident with at least one vertex of \( I \).

We will have several cases and subcases, depending on \( n \) and \( z \). In each instance, we will isolate the specific \( (n, z) \) pairs that cannot be handled by general inequalities, and we will address them later in the section.

Case 1: \( n \) is odd. As demonstrated in Theorem 8, \( |E(G)| \geq |\overline{G}| \geq (n/2)(n - z) \), with equalities precisely when \( E(G) = \overline{G} \) and \( d(x_i) = (n - 1)/2 \) for each \( x_i \in I \). In this situation, \( (n - 1)/2 \leq n - z \), which implies that \( z \leq (n + 1)/2 \).

Subcase 1a. If \( z = (n + 1)/2 \) or \( z = (n - 1)/2 \), then
\[
|E(G)| \geq \frac{n}{2}(n - z) = \frac{1}{8}(n^2 - 1) \geq 3n - 15
\]
for \( n \geq 17 \), with strict inequality in the latter case when \( n \geq 18 \).

Subcase 1b. Here we consider \( z \) values for which \( (3n - 10)/8 \leq z < (n - 1)/2 \). When \( n = 17 \), \( 6 \leq z \leq 7 \), and we only need to verify that \( |E(G)| \geq 36 \) for \( z = 6 = (n - 5)/2 \) and \( z = 7 = (n - 3)/2 \). Since \((6/2)(17 - 6) = 33\), our task will be completed if we demonstrate that, when \( z = 6 \), \( \overline{G} \) must contain at least four edges not in \( \overline{G}_1 \). Likewise, since \((7/2)(17 - 7) = 35\), we will need to show that when \( z = 7 \), \( \overline{G} \) must contain at least two edges not in \( \overline{G}_1 \).

Similarly, when \( n = 19 \), we have \( 6 \leq z \leq 8 \). However, when \( z = 7 \) or \( 8 \), \( |E(G)| \geq (z/2)(n - z) \geq 3n - 15 = 42 \), with the latter inequality becoming tight only when \( z = 7 \). Thus, we will need to verify that when \( z = 6 = (n - 7)/2 \), \( |E(G)| > 42 \); observe that \((6/2)(19 - 6) = 39\), so our task will be completed if we verify that when \( z = 6 \), \( \overline{G} \) must contain at least four edges not in \( \overline{G}_1 \). If \( |E(G)| = 3n - 15 \) when \( z = 7 \), then \( E(G) = \overline{G}_1 \); this case, where \( G - I = K_{n-z} \) and \( d(x_i) = (n - 1)/2 \) for \( x_i \in I \), will be addressed later.

When \( n = 21 \), \( (z/2)(n - z) > 3n - 15 \) for all values of \( z \) for which \( (3n - 10)/8 \leq z < (n - 1)/2 \).

Case 2: \( n \) is even. Again by Theorem 8, \( |E(G)| \geq |\overline{G}| \geq (z/2)(n - z) + z/2 - 1 \), with equality holding precisely when \( d(x_i) = n/2 \) for some \( x_i \in I \), \( d(x_j) = (n - 2)/2 \) for \( j \neq i \), and \( E(G) = \overline{G}_1 \). Now, it must be that \( n/2 \leq n - z \), so \( z \leq n/2 \).

Subcase 2a. If \( z = n/2 \), then
\[
|E(G)| \geq \frac{n}{2}(n - z) + \frac{z}{2} - 1 = \frac{1}{8}n^2 + \frac{1}{4}n - 1 \geq 3n - 15
\]
for \( n \geq 16 \).

Subcase 2b. Here we consider \( z \) values for which \( (3n - 10)/8 \leq z < n/2 \). When \( n = 16 \), \( 5 \leq z \leq 7 \). Furthermore, for \( z = 7 \),
\[
|E(G)| \geq (z/2)(n - z) + z/2 - 1 = (7/2)(17 - 7) + 7/2 - 1 = 36.5
\]
\[
> 3n - 15 = 33.
\]
Thus, we only need to verify that \( |E(G)| > 33 \) for \( z = 5 = (n - 6)/2 \) and \( z = 6 = (n - 4)/2 \). As in the case with \( n \) odd, since \((5/2)(16 - 5) + (5/2) - 1 = 29\), our task will be completed if we demonstrate that, when \( z = 5 \), \( \overline{G} \) must contain at least five edges not in \( \overline{G}_1 \). Likewise, since \((6/2)(16 - 6) + (6/2) - 1 = 32\), we need to show that when \( z = 6 \), \( \overline{G} \) must contain at least two edges not in \( \overline{G}_1 \).
Similarly, when \( n = 18, 6 \leq x \leq 8 \). However, when \( x = 7 \) or \( x = 8 \), \((x/2)(n-x)+x/2-1 > 3n-15 = 39\). Thus, we will need to verify that when \( x = 6 = (n-7)/2 \), \(|E(\overline{G})| > 3\); observe that \((\frac{n}{2}) (18 - 6) + \frac{6}{2} - 1 = 38\), so we will be done once we demonstrate that when \( x = 6 \), \( \overline{G} \) must contain at least two edges not in \( \overline{G}_1 \).

When \( n = 20, (x/2)(n-x)+x/2-1 \), which exceeds \( 3n-15 \) for all values of \( x \) for which \((3n-10)/8 \leq x < n/2\).

Special cases of \((n, x)\): We now examine \(|E(\overline{G}) - \overline{G}_I|\) for the specific aforementioned \((n, x)\) pairs in Cases 1 and 2. For the pairs \((16, 5), (17, 6)\), and \((19, 6)\), notice that \( x \leq (n-5)/2 \). If \( 0 < |E(\overline{G}) - \overline{G}_I| \leq 4 \), then

\[
\min \{d(y) + d(z) : y, z \in G - I \} \geq 2(n - x) - (|E(\overline{G}) - \overline{G}_I| + 1) \\
\geq 2n - 2 \left( \frac{n - 5}{2} \right) - 5 = n.
\]

Therefore, if there exist independent vertices \( y \) and \( z \) in \( G - I \), we contradict our assumption that the degree sum of nonadjacent vertices is at most \( n - 1 \). Thus, for \( x \leq (n-5)/2 \), either \(|E(\overline{G}) - \overline{G}_I| > 4 \) (and we have, in fact, \(|E(\overline{G})| > 3n-15\)), or \(|E(\overline{G}) - \overline{G}_I| = 0\), and \( G - I = K_{n-2} \).

For the remaining \((n, x)\) pairs, we wish to verify that there is not exactly one edge in \( E(\overline{G}) - \overline{G}_I \); i.e., that our bound is satisfied for the pairs \((16, 6), (17, 7)\), and \((18, 6)\) if \(|E(\overline{G}) - \overline{G}_I| \geq 2 \). In each of these cases, \( x \leq (n-3)/2 \).

If \(|E(\overline{G}) - \overline{G}_I| = 1 \), then

\[
\min \{d(y) + d(z) : y, z \in G - I \} \geq 2(n - x) - (|E(\overline{G}) - \overline{G}_I| + 1) \\
\geq 2 \left( n - \left( \frac{n - 3}{2} \right) \right) - 2 \\
= n + 1 \geq n.
\]

As before, this shows that if \( G - I \) contains a pair \( \{y, z\} \) of independent vertices, we reach a contradiction. Therefore, for these three \((n, x)\) pairs, when \( x \leq (n-3)/2 \), either \(|E(\overline{G}) - \overline{G}_I| \geq 2 \) and \(|E(\overline{G})| > 3n-15\), or \(|E(\overline{G}) - \overline{G}_I| = 0 \) and \( G - I = K_{n-2} \).

We finally address the situation in which \( G - I = K_{n-2} \), with \( x \leq (n-3)/2 \). Suppose first that \( n \) is odd and let \( G' = G - x_1 \), a graph of order \( n - 1 \). In \( G' \), it is still the case that \( d(x_i) = (n-1)/2 \) for \( 2 \leq i \leq x \), while \( d(y) \geq n - x - 1 \geq (n-1)/2 \) for \( y \in G' - I \). Thus, for any pair of vertices, \( \{v, w\} \), in \( G' \), \( d(v) + d(w) \geq 2((n-1)/2) = n - 1 \). So \( G' \) is hamiltonian by Ore’s Theorem.

Let \( C = (v_1, v_2, \ldots, v_{n-1}, v_n = v_1) \) be a hamiltonian cycle in \( G' \). If \( C \) has two adjacent vertices, \( v_i \) and \( v_{i+1} \), in \( N(x_i) \), then \( (v_1, v_2, \ldots, v_{i}, x_i, v_{i+1}, \ldots, v_{n-1}, v_n = v_1) \) is a hamiltonian cycle in \( G \), contradicting our assumption that \( G \) was nonhamiltonian.

Suppose that \( C \) does not contain any pair \( v_i, v_{i+1} \in N(x_i) \). Note first that since \( d(x_1) = (n-1)/2 \) and \(|V(G')| = n-1 \), the hamiltonian cycle in \( G' \) must alternate neighbors and nonneighbors of \( x_1 \); that is, \( C = (a_1, v_1, a_2, v_2, a_3, v_3, \ldots, a_{n-1}, v_n) \), \( v_{n-1}/2, a_1) \), where the \( a_i \) are neighbors of \( x_1 \) and the \( v_i \) are not. Observe that there must be at least two vertices in \( K_{n-2} \) that are not neighbors of \( x_1 \) since \((n-x) - |N(x_1)| \geq (n - (n-3)/2) - (n-1)/2 = 2 \). Choose two such vertices, \( v_i, v_j \in V(K_{n-2}) - N(x_1) \). Then \( (a_1, v_1, \ldots, a_i, v_i, v_j, a_j, v_{i+1}, \ldots, a_j, x_1, a_{i+1}, \ldots, a_{n-1}/2, v_{n-1}/2, a_1) \) is a hamiltonian cycle in \( G \), again contradicting our assumption that \( G \) was nonhamiltonian. Thus, when \( n \) is odd and \( x \leq (n-3)/2 \), if \( G - I = K_{n-2} \), then \( G \) must be hamiltonian.

Suppose next that \( n \) is even and \( G - I = K_{n-2} \). Let \( x_1 \) be the vertex of degree \( n/2 \) and recall that \( d(x_i) = (n-2)/2 \) for \( 2 \leq i \leq x \). Again, let \( G' = G - x_1 \). Now \( G' \) is a graph of order \( n' = n - 1 \) (so that \( n' \) is odd) and \( G' - \{x_2, x_3, \ldots, x_x\} = K_{n'-1} \). Furthermore, \( d_G'(x_i) = (n'-1)/2 \) for \( i = 2, 3, \ldots, x \), so by the preceding work we conclude that the graph \( G' \) is hamiltonian. Consider any hamiltonian cycle, \( C = (v_1, v_2, \ldots, v_{n-1}, v_1) \), in \( G' \). Since \( d(x_1) = n/2 \) and \( n' = n - 1 \), there must be two adjacent vertices, \( v_i \) and \( v_{i+1} \), in \( N(x_1) \), and as before, \( (v_1, v_2, \ldots, v_i, x_1, v_{i+1}, \ldots, v_{n-1}, v_1) \) is a hamiltonian cycle in \( G \). Thus, we have shown that in all relevant cases where \( G - I = K_{n-2}, \) with \( x \leq (n-3)/2 \), \( G \) must be hamiltonian.

4.2. Uniqueness of extremal graphs

We now prove that the extremal graphs are unique for \( n \geq 18 \). Recall that we may assume that \( G = cl(G) \) is a nonhamiltonian, 3-connected graph of order \( n \geq 18 \) with \( \sigma_4(G) = n + 5 \). Let \( I = \{x_1, x_2, x_3, x_4\} \) be a set of independent
vertices such that $\sum_{i=1}^{4} d(x_i) = \sigma_4(G)$. We may further assume that all edges in $\overline{G}$ have at least one endpoint in $I$; that is, if $x, y \in V(G) - I$, then $\{x, y\} \in E(\overline{G})$.

Because $G$ is 3-connected, $d(x_i) \geq 3$ for $1 \leq i \leq 4$; order the $x_i$ so that $3 \leq d(x_1) \leq d(x_2) \leq d(x_3) \leq d(x_4)$. Suppose that $d(x_3) \geq 4$. Choose $v_1, v_2 \in N(x_1)$ and $v'_2, v_3 \in N(x_2)$, where it could be that $v'_2 = v_2$ but $v_1 \neq v_3$. Begin a path, $(v_1, x_1, v_2, v'_2, x_2, v_3)$, noting that we will avoid duplication if $v'_2 = v_2$.

Since $d(x_3) \geq 4$, there are vertices $v'_3, v_4 \in N(x_3)$ where it may be that $v'_3 = v_3$, but $v_4 \notin \{v_1, v_2, v_3\}$. Thus, we can lengthen the path to $(v_1, x_1, v_2, v'_2, x_2, v_3, v'_3, x_3, v_4)$, again noting that we will avoid duplication if $v'_3 = v_3$.

Finally, $\sum_{i=1}^{4} d(x_i) = n + 5$ implies that $d(x_4) \geq \lceil (n + 5)/4 \rceil \geq 6$. Suppose that $\{v_1, v_2, v'_3, v_3, v'_4\} \subset N(x_4)$ and assume that at least one of $v_2 \neq v'_2$ or $v_3 \neq v'_3$ is true. Without loss of generality, assume $v_3 \neq v'_3$. Then $(v_1, x_1, v_2, v'_2, x_2, v_3, x_4, v'_3, x_3, v_4, \ldots, v_{n-10}, v_1)$ is a hamiltonian cycle in $G$. On the other hand, if $v_2 = v'_2$ and $v_3 = v'_3$, then there exist distinct vertices $v_5, v_6 \in N(x_4)$ such that $v_5, v_6 \notin \{v_1, v_2, v_3\}$, so $(v_1, x_1, v_2, v_3, x_3, v_4, v_5, x_4, v_6, \ldots, v_{n-10}, v_1)$ is a hamiltonian cycle in $G$. We conclude that if $d(x_3) \geq 4$, $G$ must be hamiltonian.

We now assume $d(x_1) = d(x_2) = d(x_3) = 3$ and $d(x_4) = \sigma_4 - 9 = n - 4$. Note that $d(x_4) = n - 4$ implies that $N(x_4) = G - I$.

Let $M = \min\{|N(x_i) \cap N(x_j)| : 1 \leq i, j \leq 3, i \neq j\}$. We will argue that $G$ must be hamiltonian unless $M = 3$ (in which case, $N(x_1) = N(x_2) = N(x_3)$). Since $d(x_4) = n - 4 \geq 14$, even if the neighborhoods of $x_1, x_2$ and $x_3$ are mutually disjoint, $|N(x_4)| - |N(x_1) \cup N(x_2) \cup N(x_3)| \geq 5$; because $G - I = K_{n-4}$, we can be assured that it will be possible to include $x_4$ in creating a hamiltonian cycle in $G$. Without loss of generality, assume that $M = |N(x_2) \cap N(x_3)|$. We consider the case $N(x_1) = N(x_2) = \{v_1, v_2, v_3\}$. The other cases are handled similarly.

Let $N(x_3) = \{v_4, v_5, v_6\}$. If $M = 0$, then $(v_1, x_1, v_2, x_2, v_3, x_3, v_4, x_4, v_5, x_5, v_6, \ldots, v_n, v_1)$ is a hamiltonian cycle in $G$.

If $M = 1$ or 2, then $N(x_3)$ contains at least one vertex (call it $v_3$) that is not in $N(x_1) = N(x_2)$, and at least one vertex (say, $v_3$) that is in $N(x_1) = N(x_2)$. Then $(v_1, x_1, v_2, x_2, v_3, x_3, v_4, x_4, v_5, \ldots, v_n, v_1)$ is a hamiltonian cycle in $G$.

Thus, we conclude that $M = 3$, and $N(x_1) = N(x_2) = N(x_3) = \{v_1, v_2, v_3\}$. As noted above, $N(x_4) = G - I$; therefore, $G - \{x_1, x_2, x_3\} = K_{n-3}$, and the uniqueness of $G$ has been established.

References